Math 20550 Calculus III Tutorial **Name:** **Solutions**
March 6, 2014

Tutorial Worksheet

Show all your work.

1. Maximize the function $f(x, y, z) = xyz$ subject to the constraint $x^2+2y^2+3z^2 = 9$, assuming that x, y, and z are nonnegative. For full credit, explain why the extremum you find is a maximum.

Solution. The gradient of f is

$$\nabla f = \langle yz, xz, xy\rangle .$$

Let $g = x^2 + 2y^2 + 3z^2$, then $\nabla g = \langle 2x, 4y, 6z\rangle$. The system of equations we get by Lagrange multipliers is thus

$$\begin{cases} yz &= 2\lambda x & \textcircled{1} \\ xz &= 4\lambda y & \textcircled{2} \\ xy &= 6\lambda z & \textcircled{3} \\ x^2+2y^2+3z^2 &= 9 & \textcircled{4} \end{cases}$$

Solving ① for λ gives $\lambda = \dfrac{yz}{2x}$, however, to do this, we have to make sure $x \neq 0$. If $x = 0$, then $f = 0$, but since there are solutions (a, b, c) of $g = 9$ (e.g., $(2, 1, 1)$) such that $f(a, b, c) > 0$, any solution with $x = 0$ will not be a maximum. This same reasoning shows that anything with $y = 0$ or $z = 0$ is not a maximum, so we can assume that $x, y, z > 0$. If we plug λ into ②, we get $xz = 4\dfrac{yz}{2x}y$, and as long as $z \neq 0$ (which is true since $z > 0$) we can divide both sides by z and simplify to get $x^2 = 2y^2$. Plugging λ into ③ we have $xy = 6\dfrac{yz}{2x}z$, and as long as $y \neq 0$ (which is isn't since $y > 0$) we can divide both sides by y and simplify to get $x^2 = 3z^2$. Plugging all this information into ④ gives

$$x^2 + 2y^2 + 3z^2 = 3x^2 = 9 \implies x^2 = 3 \implies x = \pm\sqrt{3}$$

Since $x^2 = 2y^2$ we have $y = \pm\sqrt{\dfrac{3}{2}}$, and since $x^2 = 3z^2$ we have $z = \pm 1$. Now, since we only want the extrema with x, y, and z nonnegative, we get that the only candidate for the maximum is the extreme point $\left(\sqrt{3}, \sqrt{\dfrac{3}{2}}, 1\right)$, and $f\left(\sqrt{3}, \sqrt{\dfrac{3}{2}}, 1\right) = \dfrac{3}{\sqrt{2}}$. To show it is a maximum (we know it is either a maximum or a minimum) we check to see if there is a point on $x^2 + 2y^2 + 3z^2 = 9$ which makes f smaller than $\dfrac{3}{\sqrt{2}}$ (meaning it can't be a minimum). The point $(2, 1, 1)$ lies on $x^2 + 2y^2 + 3z^2 = 9$ and $f(2, 1, 1) = 2$ which is smaller than $\dfrac{3}{\sqrt{2}}$ since $\sqrt{2} < 1.5$. Thus the maximum value of f on $x^2 + 2y^2 + 3z^2 = 9$ is $\dfrac{\sqrt{3}}{2}$ and occurs at $\left(\sqrt{3}, \sqrt{\dfrac{3}{2}}, 1\right)$. □

2. Find the minimum distance from the parabola $y = x^2$ to the point $(0, 9)$.

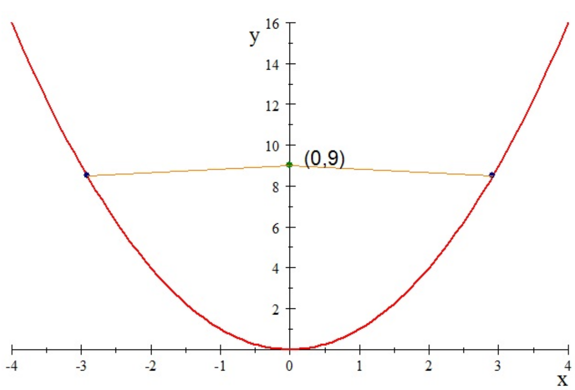
Solution. We want to minimize the function $d(x, y) = \sqrt{x^2 + (y - 9)^2}$ subject to $y = x^2$. Since $f(x, y) = x^2 + (y - 9)^2 \geq 0$ and $d(x, y) \geq 0$, the minimums of d and f occur at the same points, so to make the algebra simpler, let's minimize f subject to $y = x^2$ instead. Let $g = x^2 - y$. So, since $\nabla f = \langle 2x, 2y - 18\rangle$ and $\nabla g = \langle 2x, -1\rangle$, the system we get is

$$\begin{cases} 2x &= 2\lambda x \quad \textcircled{1} \\ 2y - 18 &= -\lambda \quad \textcircled{2} \\ y &= x^2 \quad \textcircled{3} \end{cases}$$

Equation $\textcircled{1}$ has two solutions, either $x \neq 0$ and we can divide by x to find that $\lambda = 1$, or $x = 0$. If $\lambda = 1$, then by $\textcircled{2}$ we have $2y - 18 = -1$, so that $y = \frac{17}{2}$, and plugging this into $\textcircled{3}$ we get that $x = \pm\sqrt{\frac{17}{2}}$. So this case gives us the critical points $\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ and $\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$.
If $x = 0$, then by $\textcircled{3}$, $y = 0$, and so the critical point in this case is $(0, 0)$.
Now we check for the minimums:

point	value
$(0, 0)$	81
$\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$	$\frac{35}{4}$
$\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$	$\frac{35}{4}$

So, the minimum value of f is $\dfrac{35}{4}$, and hence the minimum value of d is $\sqrt{\frac{35}{4}}$, and this occurs at the points $\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ and $\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$. Said another way, the closest points to $(0, 9)$ on the parabola $y = x^2$ are the points $\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ and $\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ which are a distance of $\sqrt{\frac{35}{4}}$ away.



(Note that you could also do this problem by plugging $y = x^2$ into the equation for d or f and using Calculus I methods.) $\square$

3. Minimize the function $f(x, y, z) = x^2+y^2+z^2$ subject to the constraints $x+2z = 6$ and $x+y = 12$, assuming that x, y, and z are nonnegative. To get full credit, explain why the extrema you find is a minimum.

Solution. The gradient of f is

$$\nabla f = \langle 2x, 2y, 2z \rangle .$$

Let $g = x + 2z$ and let $h = x + y$. Then

$$\nabla g = \langle 1, 0, 2 \rangle$$

and

$$\nabla h = \langle 1, 1, 0 \rangle .$$

The system we get is thus

$$\begin{cases} 2x &= \lambda + \mu & \textcircled{1} \\ 2y &= \mu & \textcircled{2} \\ 2z &= 2\lambda & \textcircled{3} \\ x + 2z &= 6 & \textcircled{4} \\ x + y &= 12 & \textcircled{5} \end{cases}$$

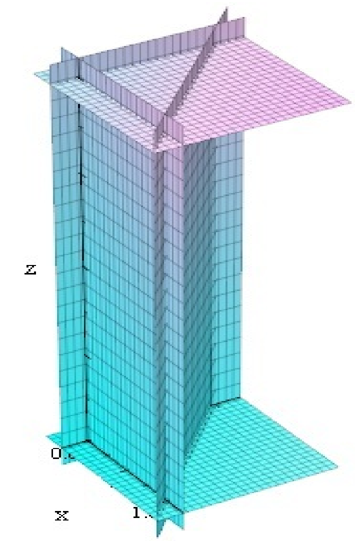
$\textcircled{3}$ gives $z = \lambda$. Plug this into $\textcircled{4}$ to get $x = 6 - 2\lambda$. $\textcircled{2}$ gives $y = \frac{\mu}{2}$. Plug this into $\textcircled{5}$ to get $x = 12 - \frac{\mu}{2}$. So $\lambda = 3 - \frac{1}{2}x$ and $\mu = 24 - 2x$. Plug these into $\textcircled{1}$ to get

$$2x = 3 - \frac{1}{2}x + 24 - 2x \implies \frac{9}{2}x = 27 \implies x = 6.$$

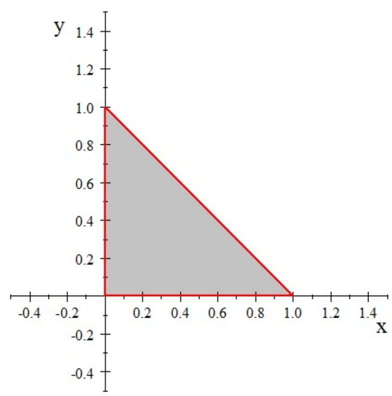
By $\textcircled{5}$, $y = 6$, and by $\textcircled{4}$, $z = 0$. So $(6, 6, 0)$ is our candidate for the minimum, and $f(6, 6, 0) = 72$. As in problem 1, $(6, 6, 0)$ is either a minimum or maximum, and we need to rule out it being a maximum. So we are looking for a point (a, b, c) satisfying $g = 6$ and $h = 12$ with $f(a, b, c) > 72$. Theoretically any such point that is not $(6, 6, 0)$ should work. Consider the point $(0, 12, 3)$ which satisfies $g = 6$ and $h = 12$. Since $f(0, 12, 3) = 0 + 144 + 9 = 153 > 72$, we have that $(6, 6, 0)$ is indeed where the maximum of f subject to $g = 6$ and $h = 12$ occurs. So the minimum value of f subject to $x + 2z = 6$ and $x + y = 12$ is 72 and occurs at the point $(6, 6, 0)$. □

4. Use a double integral to find the volume of triangular prism bounded by the coordinate planes, $y = -x + 1$, and $z = 4$. Check your answer by computing the volume in another way.

Solution. The region we are integrating is the region bounded by these planes:



The projection of the region in the xy-plane is:



The height of the region is given by:

$$\text{Height} = z_{top} - z_{bottom} = 4 - 0 = 4.$$

As such, the volume is given by:

$$\begin{aligned} V &= \int_0^1 \int_0^{-x+1} 4\, dydx = \int_0^1 (4y)\big|_0^{-x+1}\, dx = \int_0^1 [(-4x+4) - 0]\, dx \\ &= \int_0^1 (-4x+4)dx = \left(-2x^2 + 4x\right)\big|_0^1 = (-2+4) - 0 = 2. \end{aligned}$$

Alternatively, the volume of the region is given by the area of the base, times the height. The area of the triangle is $\frac{1}{2}$ (since it is half of the unit square $[0,1] \times [0,1]$) and the height is 4 (as calculated above), so the volume is $V = \frac{1}{2} \cdot 4 = 2$. □