Math 20550 Calculus III Tutorial **Name:** ____________________________
March 26, 2015

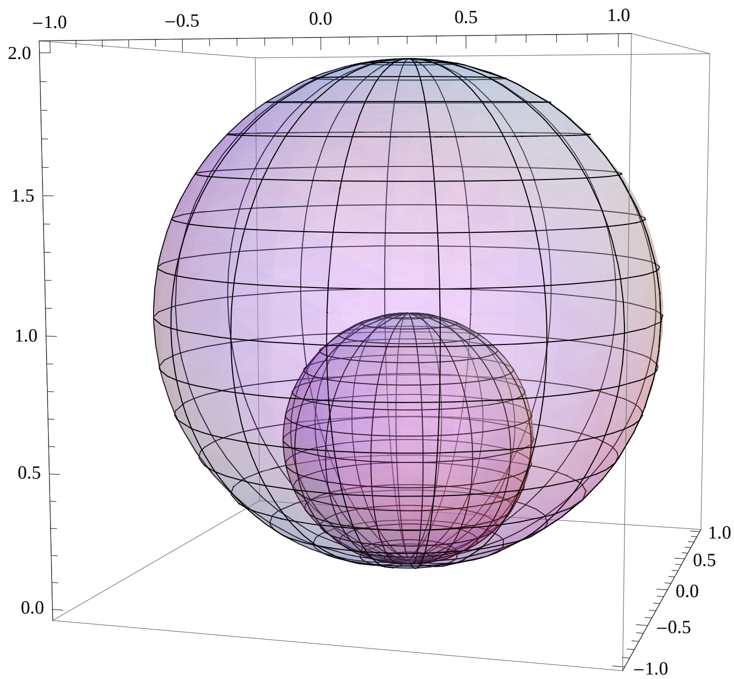
Tutorial Worksheet

Show all your work.
1. Let E be the region between the spheres $x^2 + y^2 + z^2 = z$ and $x^2 + y^2 + z^2 = 2z$. Set up, but do not calculate, the integral $\iiint_E (x^2 + y^2)\, dV$.

Solution: First we graph the spheres. Rearranging and completing the square, we get

Sphere 1: $x^2 + y^2 + (z - 1/2)^2 = 1/4 \implies$ Sphere centered at $(0, 0, 1/2)$ with radius $r = 1/2$
Sphere 2: $x^2 + y^2 + (z - 1)^2 = 1 \implies$ Sphere centered at $(0, 0, 1)$ with radius $r = 1$,

Therefore the spheres look like this:



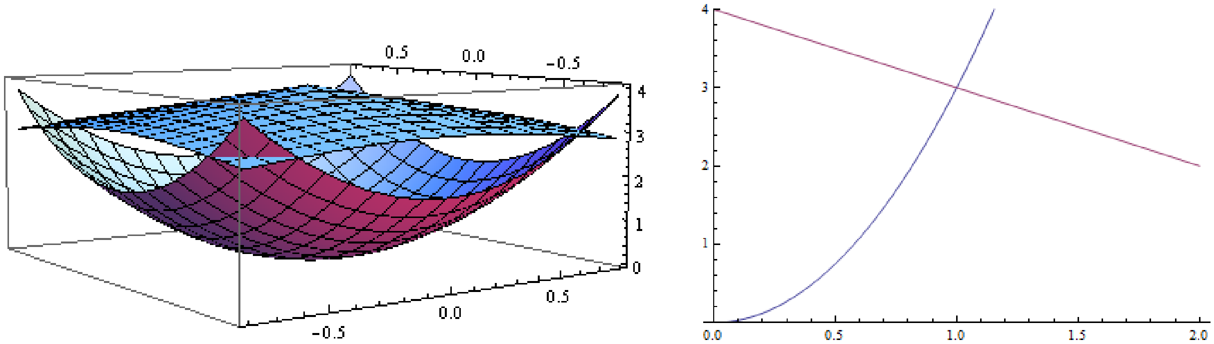
In spherical coordinates, the equation of Sphere 1 becomes $\rho^2 = \rho\cos(\phi)$, or $\rho = \cos(\phi)$. Similarly, the equation of Sphere 2 becomes $\rho^2 = 2\rho\cos(\phi)$, or $\rho = 2\cos(\phi)$. There is no restriction on θ from the spheres, so $0 \le \theta \le 2\pi$. Both spheres lie above the xy-plane, so $0 \le \phi \le \frac{\pi}{2}$. Because Sphere 1 is inside Sphere 2, we have $\cos(\phi) \le \rho \le 2\cos(\phi)$.

Now we convert the integrand to spherical coordinates: $x^2 + y^2 = r^2 = \rho^2 \sin^2(\phi)$. Since $dV = \rho^2 \sin(\phi) d\rho\, d\phi\, d\theta$, we get $(x^2 + y^2)\, dV = \rho^4 \sin^3(\phi)\, d\rho\, d\phi\, d\theta$. Hence the integral is

$$\int_0^{2\pi} \int_0^{\pi/2} \int_{\cos(\phi)}^{2\cos(\phi)} \rho^4 \sin^3(\phi)\, d\rho\, d\phi\, d\theta.$$

2. Set up, but do not solve, the integral that gives the volume of the solid region bounded by the paraboloid $z = 3y^2 + 3x^2$ and the cone $z = 4 - \sqrt{x^2 + y^2}$.

Solution: Because the paraboloid and the cone are both rotationally symmetric, we shall use cylindrical coordinates. Using $r^2 = x^2 + y^2$, the equation for the paraboloid becomes $z = 3r^2$, while the cone becomes $z = 4 - r$. Note that these equations do not depend on θ, so we may graph them in the (r, z)-plane.



Cone opening downward, paraboloid opening upward and
the graphs of $z = 3r^2$ and $z = 4 - r$ in the (r, z) plane.

Therefore, in terms of our limits, we will have $3r^2 \leq z \leq 4 - r$. To find our limits for r, we simply find where the two surfaces intersect. So we have

$$3r^2 = 4 - r \implies r = 1 \text{ } or - 4/3.$$

But a negative r is not possible, so the paraboloid and cone intersect at $r = 1$. So we have $0 \leq r \leq 1$. The bounds on θ are $0 \leq \theta \leq 2\pi$. Finally, since we are computing volume, the integrand is simply $dV = r\,dz\,dr\,d\theta$. Therefore the integral is

$$\int_0^{2\pi} \int_0^1 \int_{3r^2}^{4-r} r\,dz\,dr\,d\theta.$$

3. Let D be the quarter of the disc centered at the origin with radius a with $x \geq 0$ and $y \geq 0$. Suppose that the density at a point on D is proportional to the square of its distance from the origin. Find the center of mass of D. (Hint: $\overline{x} = \overline{y}$ by symmetry.)

Solution: Let $\rho(r, \theta)$ denote the density function in polar coordinates. Saying that ρ is proportional to the square of its distance from the origin is the same as saying that there is a constant number $c > 0$ with

$$\rho(r, \theta) = cr^2.$$

We first compute the mass:

$$\begin{aligned} m &= \iint_D \rho \, dA \\ &= \int_0^{\pi/2} \int_0^a cr^3 \, dr \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{4} ca^4 \, d\theta \\ &= \frac{1}{8} \pi ca^4. \end{aligned}$$

Next we compute the moment about the y-axis:

$$\begin{aligned} M_y &= \iint_D x\rho \, dA \\ &= \int_0^{\pi/2} \int_0^a cr^4 \cos\theta \, dr \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{5} ca^5 \cos\theta \, d\theta \\ &= \frac{1}{5} ca^5. \end{aligned}$$

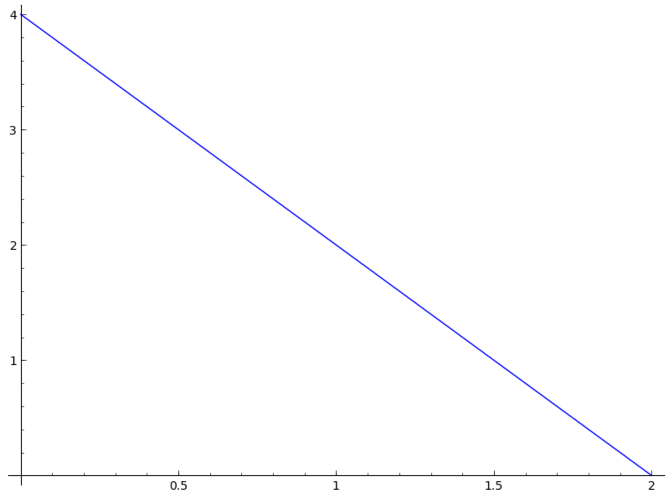
Now

$$\overline{x} = \frac{\frac{1}{5} ca^5}{\frac{1}{8} \pi ca^4} = \frac{8a}{5\pi}.$$

By diagonal symmetry, $\overline{x} = \overline{y}$. Therefore the center of mass is $(\frac{8a}{5\pi}, \frac{8a}{5\pi})$.

4. Use a triple integral to compute the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 4$.

Solution: Let us use a $dz\,dy\,dx$ integral (though other choices are just as good). The bounds on z are $0 \le z \le 4 - 2x - y$. To find the bounds on x and y, we set $z = 0$ (because the tetrahedron is largest at its base). This gives the equation $2x + y = 4$, so $0 \le y \le 4 - 2x$, and $0 \le x \le 2$.



The base of the tetrahedron.

Thus

$$\begin{aligned}
V &= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx \\
&= \int_0^2 \int_0^{4-2x} (4 - 2x - y)\,dy\,dx \\
&= \int_0^2 \left((4-2x)y - \frac{1}{2}y^2\right)\Bigg|_{y=0}^{y=4-2x} dx \\
&= \int_0^2 \left((4-2x)^2 - \frac{1}{2}(4-2x)^2\right) dx \\
&= \int_0^2 \frac{1}{2}(4-2x)^2\,dx \\
&= -\frac{1}{12}(4-2x)^3\Bigg|_{x=0}^{x=2} \\
&= \frac{16}{3}.
\end{aligned}$$