The nature of elliptic sectors in the principal foliations of surface theory

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Abstract

Little is known of the geometry of the principal foliations on a smooth surface in $\mathbb{R}^3$. The unsolved Carathéodory conjecture for such a surface would follow from Bendixson's formula, were it known that the principal foliations never admit an elliptic sector at an isolated umbilic. Evidence in favour of the nonexistence of elliptic sectors in principal foliations is the main result and it offers a first geometric explanation of why the conjecture might be true for smooth surfaces.

1 Introduction

The classical Carathéodory conjecture [1] (for which a proof has recently been given for analytic surfaces in $\mathbb{R}^3$ [3]) posits a special topological behaviour near an isolated umbilic for the principal foliations of a smooth surface, a behaviour not enjoyed by arbitrary smooth foliations near an isolated singularity. The local form of this conjecture states that the index $j$ of the principal foliations at an umbilic is $\leq 1$, and it remains unclear what the geometric source for such an expectation might be.

Bendixson’s formula for the index of an isolated singularity of a smooth foliation says that

$$j = 1 + \frac{e - h}{2},$$

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where \( e \) and \( h \) are the number of elliptic and hyperbolic sectors of the foliation at the singularity [2]. Carathéodory’s conjecture would follow from non-existence of elliptic sectors in the principal foliations at an isolated umbilic and such a conjecture is not only stronger but of genuine geometric interest.

2 The conjecture and the theorem

The theme of this work is then the following:

**The elliptic sector conjecture:** At an isolated umbilic in a smooth surface in \( \mathbb{R}^3 \) the principal foliations cannot have an elliptic sector.

The hypothetical occurrence of such sectors in the principal foliations of surfaces is our interest here. The essence of the theorem is that if such sectors occur, they must be pathological. The first hint of the existence of such a result came in the work of Laurentiu Lazarovici [4], which has a strong convexity assumption on the sector and uses a non-geometric hyperbolic p.d.e argument on Hessian foliations. Our work depends critically on Codazzi’s equation and is carried out on the surface itself.

Let \( f : M \rightarrow \mathbb{R}^3 \) be a smooth immersion of a smooth oriented surface \( M \) in \( \mathbb{R}^3 \). Let \( g \) be the induced metric, \( \xi \) the oriented unit normal field along the immersion \( f \), and \( A \) the second fundamental form of \( \xi \); \( A \) is defined by \( X\xi = -f_*p(A_pX) \) for all vectors \( X \) tangent to \( M \) at \( p \), where \( f_*p \) is the differential of \( f \) at \( p \). The inner product space \( (M_p, g_p) \) has orientation coming from \( M \) and counterclockwise rotation through \( \frac{\pi}{2} \) defines the complex structure \( J_p \). The metric connexion of \( g \) is denoted \( \nabla \) and for our purposes here the important equation is Codazzi’s equation

\[
(\nabla_X A)Y - (\nabla_Y A)X = 0.
\]

On the complement of the umbilic set the eigenvalues of \( A \) are denoted by \( \lambda \) and \( \mu \) and we may assume \( \lambda > \mu \). The eigenspaces of \( A \) give two smooth, possibly non-orientable, foliations on the complement of the umbilic set. These are called the principal foliations of the immersed surface. The foliation determined by \( \lambda \) will be denoted \( \mathcal{F}^+ \) and that determined by \( \mu \) will be denoted \( \mathcal{F}^- \).

Let \( p_0 \) be an isolated umbilic of the immersion \( f \) and \( E \) an elliptic sector of \( \mathcal{F}^+ \) at \( p_0 \). Then \( E \) is bounded by a leaf of \( \mathcal{F}^+ \), born and dying at \( p_0 \), with
no other umbilic within $E$. Hence $\mathcal{F}^+$ is orientable on $E$. We orient $\mathcal{F}^+$ so that it gives the standard orientation to $\partial E$. The orientation of $\mathcal{F}^+$ is given by a unit field $e_1$ on $E$ and $e_2 = Je_1$ orients $\mathcal{F}^-$ on $E$. The curvatures of the oriented leaves of $\mathcal{F}^+$ are given by $g(\nabla e_1, Je_1) = k_1$ and those of $\mathcal{F}^-$ by $g(\nabla e_2, Je_2) = k_2$.

Each $p \in E$ determines a unique oriented leaf segment of $\mathcal{F}^+$ which is born at $p_0$ and ends at $p$; this is denoted $C^+(p)$. Its length $l_1(p)$ is a positive extended real-valued function on $E$. Similarly $C^-(p)$ is the unique oriented leaf of $\mathcal{F}^-$ which begins at $p$ and dies at $p_0$ and its length is denoted $l_2(p)$. The functions $l_1$ and $l_2$ are extended real-valued functions on $E$. The region $L(p)$ with positively oriented boundary $C^+(p) + C^-(p)$ is called the lens determined by $p$.

Codazzi’s equation on $E$ is used to deduce an integral formula with no rectifiability assumption on the curves:

$$-(\lambda - \mu)(p) = \int_{C^+(p)} (\lambda - \mu)k_2 ds + \int_{C^-(p)} (\lambda - \mu)k_1 ds.$$

Since $\lambda - \mu > 0$ on the elliptic sector $E$ there is an immediate contradiction if $k_1$ and $k_2$ are non-negative on some lens $L(p)$ in this elliptic sector, proving Lazarovici’s result (Main Theorem, [4]). A closer analysis of the integral formula gives the following result:

**Theorem 2.1.** Let $p_0$ be an isolated umbilic of a smooth surface smoothly immersed in $\mathbb{R}^3$. If there exists an elliptic sector $E$ in one of the principal foliations at $p_0$, then for every lens $L(p)$ in $E$ either

(a) $\sup_{L(p)} \{l_1, l_2\} = \infty$

or

(b) $\inf_{L(p)} \{k_1, k_2\} = -\infty$.

The ultimate aim is to use the integral Formula to obtain the full conjecture.

**References**

