SOLITON SURFACES IN THE MECHANICAL EQUILIBRIUM OF CLOSED MEMBRANES

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Abstract. For a material membrane in equilibrium in an external force field we inquire of the extent to which the external observables (membrane geometry and force field) determine the internal response of the membrane (the stress tensor $T$). Non-uniqueness of the response is classical. We treat this question when the mean stress $\frac{1}{2} Tr T$ is known.

The answer is decided by membrane geometry. We show uniqueness for all but a class of soliton surfaces — the globally isothermic surfaces (§2); the physical phenomenon exhibited by any closed globally isothermic membrane in equilibrium is that with all observables static there is, in total, a 1-parameter family of responses with the same mean stress — all canonically determined by membrane geometry (Thm. 1, §1). There exist closed embedded globally isothermic surfaces of every genus (Theorem 3, §1).

Isothermic surfaces form a conformally invariant class generalising the constant mean curvature surfaces and, following Cayley [5] in 1872, were of geometric interest into the early 20th century, and most recently for their soliton theory [6]. The recognition here of the role of the global version of these soliton surfaces yields a simple explicit identification of the space of all residual shears in any closed membrane (Thm. 2, §1), where previously no more than a genus-dependent dimension bound was known [13].

§1. Introduction

This work treats the equilibrium of a closed material membrane under the influence of a force field, from the point of view of the geometry of surfaces, identifying connexions between the geometry, the conformal structure and the stress; for exceptional membrane configurations — the surfaces in the title — there are exceptional families of stress responses. These exceptional configurations are stable under the conformal group of $\mathbb{R}^3 \cup \{\infty\}$ and occur in every topological genus.

Given a smooth compact material membrane $M$ with smooth boundary (possibly empty) in space, in equilibrium under a given external force field $F$, it is of interest to consider the extent to which the internal response — that is, the stress tensor $T$ — is determined. The specific question treated here concerns the extent to which

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the *observables* (membrane geometry and force field) and the mean stress function \( \tau = \frac{1}{2} TrT \) determine the full stress tensor \( T \). All surfaces are assumed compact with or without boundary and orientable with a fixed orientation; when non-empty, the boundary \( \partial M \) is given the induced orientation and, when empty, \( M \) is called a closed surface. The orientation and induced metric determine a complex structure on \( M \) (see §2) and, whenever we refer to \( M \) as a Riemann surface, this is the complex structure we have in mind.

Force and moment balance considerations between the force field \( F \) and the response \( T \), applied in the small to \( M \), lead to the equilibrium equation in §3 whose derivation can also be found in Gurtin and Murdoch [7]. This equation may be thought of as an inhomogeneous system of linear first order partial differential equations for the unknown internal response \( T \) into which the observables enter as known quantities. The aim should be to say as much as possible about the response from knowledge of the given observables. I am grateful to Andrew Smyth for a conversation which crystallized for me the significance of this question for closed membranes and led to the identification of the distinguished class of closed soliton membrane surfaces and the extra flexibility in the stress response they allow. I also thank Robert Finn for conversations on classical elasticity when we were both visitors at Max-Planck-Institut für Mathematik in den Wissenschaften in Leipzig.

In the analysis of this question we will see that uniqueness is the rule but an exceptional geometric class of surfaces emerges naturally — the *globally isothermic surfaces* (§2). The occurrence of such a surface as a membrane configuration signals a canonical non-uniqueness in the stress tensor response — that is, there are distinct stress tensors with the same mean stress and any two such stress tensors stand in a canonical relationship to one another determined solely by the geometry of the membrane (Theorem 1, below). The globally isothermic surfaces are defined by a condition linking the second fundamental form and the underlying complex structure; the defining condition is that, on the complement of the umbilic set, the Hopf quadratic differential of the surface (§2) is a real functional multiple of a global holomorphic quadratic differential [17]. The condition is equivalent to the condition that the principal foliations of the surface coincide with the foliations determined by a global holomorphic quadratic differential in the manner of [17] (see §5).

Constant mean curvature surfaces have this property, their Hopf differentials being themselves holomorphic [8]; in particular, by Kapouleas [9], there are compact examples of every genus but none are embedded, by Alexandrov [1], apart from the round sphere. However, the class of globally isothermic surfaces is much larger since the property is conformally invariant, that is, conformal transformations of \( \mathbb{R}^3 \) transform globally isothermic surfaces to globally isothermic surfaces (see §5). Bonnet surfaces (that is, surfaces of non-constant mean curvature admitting a non-trivial 1-parameter family of isometric deformations preserving the mean curvature) form another interesting class of globally isothermic examples but Bonnet surfaces are never compact [3]. Of course compact embedded examples are of greatest interest here, and their existence is in §5, where examples of every genus are given.
Surfaces in $\mathbb{R}^3$ enjoying the above defining condition locally on the complement of the umbilic set are called isothermic surfaces ([2]) and were first studied by Cayley [5] and then by Bianchi, Darboux, Backlund, Christoffel and others (see [4]) in the late nineteenth and early twentieth century; this property is likewise invariant by conformal transformations of $\mathbb{R}^3$ and a soliton theory was developed for these surfaces by Cieśliński-Goldstein-Sym [6] in 1995; the geometric interest in these surfaces in recent years, as well as their earlier history, is chronicled in the lecture notes of Burstall [4]. Because globally isothermic surfaces now emerge as such an exceptional class among membrane configurations, it will be worthwhile contrasting them with the long-studied isothermic examples and giving examples ([5]).

**Theorem 1.** Let $M$ be a smooth closed membrane in equilibrium under an external force field $F$ with mean stress function $\tau$. Either the mean stress uniquely determines the stress tensor or else $M$ is a globally isothermic surface of genus $p \geq 1$.

If $M$ is a globally isothermic surface of genus $p \geq 1$ then the space of solutions of the equilibrium equation with mean stress $\tau$ is in one-to-one correspondence with $\mathbb{R}$ and the correspondence is canonically determined by the geometry of the membrane $M$.

**Remark.** By the geometry of $M$ we mean here by its first and second fundamental forms or, in other words, by its configuration in $\mathbb{R}^3$. The existence of globally isothermic compact embedded surfaces of every genus in $\mathbb{R}^3$ is shown in [5].

A stress tensor $T$ is called a shear if $\tau \equiv 0$ and a surface tension if $T \equiv \tau I$. The equilibrium equation becomes a homogeneous system when $F \equiv 0$ and describing its solution space is important for the general case; these solutions are called the residual stress tensors or static stress tensors. Of itself, this vector space is of some physical interest. A simple model where this problem presents itself is the pericardial sac at end-diastole (see [15]), in which model $F \equiv 0$, and it was proposed to investigate the stresses occurring in this case. For closed strictly convex membranes there are no non-trivial residual stresses, by a result of Cohn-Vossen and Blaschke (see Vekua [18]). It was suspected [13] that this space might always be finite dimensional but we will show, in [16], that there are closed membranes of every genus for which this space is infinite dimensional. It is significant that the space of residual shears for membranes related by a conformal transformation of space are shown in [16] to be canonically isomorphic but no canonical correspondence appears between their spaces of residual stresses.

As Molzon-Man noted (Theorem 5.2, [13]) the space of residual shears is finite-dimensional of real dimension bounded by $6(p - 1)$ if the genus $p$ of the membrane is $p > 1$, 2 if $p = 1$ and 0 if $p = 0$; in particular a topologically spherical membrane admits no residual shears. Our theorem above gives a simple genus-independent explicit classification of the space of residual shears on any closed membrane with the geometry deciding everything. The effectiveness of our approach originates in the recognition of the soliton surfaces in the background of all uniqueness considerations.

Applying Theorem 1 with $\tau \equiv 0$ we obtain the following result.
Theorem 2. If a smooth closed membrane admits a nontrivial residual shear then it must be a globally isothermic surface of genus \( p \geq 1 \) and this shear is then uniquely determined to within a constant multiple by the membrane geometry alone.

Any torus of revolution is globally isothermic (see §5) and so the corollary would apply. In the very special case of circular tori of revolution there are explicitly computed shear solutions of the no load (i.e., \( F \equiv 0 \)) equilibrium equation [13] and from that computation it does emerge that these are unique to within scaling in this one case. Our corollary shows that this is so for all globally isothermic membranes and the underlying holomorphic quadratic differential supplies all of the residual shears in a canonical way (see the tensor field arising in Lemma 3, §2); and of course if the membrane is not globally isothermic there are no nontrivial residual shears.

The fact that globally isothermic surfaces admit no umbilics of positive index (Remark 5, §5) gives at once a simple consequence.

Corollary. If a smooth closed membrane has an isolated umbilic of positive index then it admits no nontrivial residual shear.

Theorem 1 is proved in §4. The argument associates a complex quadratic differential to the responding stress tensor in the way of Hopf [8], and the theory of holomorphic differentials [17] then enters naturally via divergence-trace conditions (see Lemma 2, §2) arising from the equilibrium equation (see §3). The analysis of the index of an isolated umbilic enters into the proof, but the vital algebraic identity comes from the normal component of the equilibrium equation which links the Hopf differential of the membrane surface with the quadratic differential of the stress tensor determined by Lemma 2. The algebraic identity is the key to the appeearance of soliton surfaces in the theory.

The basic existence result is proved in §5.

Theorem 3. There exists closed embedded globally isothermic surfaces in \( \mathbb{R}^3 \) of every genus.

I thank Chuu-Lian Terng for pointing me toward the lecture notes of Burstall [4] on isothermic surfaces; in an earlier preprint on this topic I had used the term ”Hopf surfaces” for what I now call globally isothermic surfaces here.

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§2. Surface theory, quadratic differentials and the soliton surfaces

We first explain some of the basic differential geometry of surfaces, in particular the Codazzi equation, the Hopf differential and the simple geometric characterization of the particular soliton surfaces which dominate the uniqueness considerations in the physical problem considered in this work.
Let $M$ be a smooth oriented surface without boundary and $x : M \rightarrow \mathbb{R}^3$ an immersion of $M$ in $\mathbb{R}^3$. The standard inner product $<,>$ on $\mathbb{R}^3$ induces a Riemannian metric $g$ on $M$ defined by $g_p(X,Y) = < x_*(X), x_*(Y) >$ for any vectors $X,Y \in M_p$; here $M_p$ denotes the tangent space to $M$ at $p$, and $x_*$ denotes the differential of $x$ at $p \in M$. The given orientation on $M$ determines a unique unit normal vector field $\xi$ to $M$ along $x$. The second fundamental form $A$ (with respect to the field $\xi$) of the immersion is defined pointwise by $X\xi = -x_*(A_p X)$ for all $X \in M_p$. The operator $A$ is symmetric with respect to $g$ and depends to within sign on the choice of unit normal field; its characteristic functions on $M$ are the mean curvature function $H = \frac{1}{2} Tr A$ and the Gauss curvature function $K = det A$. The oriented area element $da$ of the metric $g$ is defined by $da(X,Y) = < x_*(X) \times x_*(Y), \xi >$, where $\times$ is the cross product.

If $\nabla$ denotes the Riemannian connexion of the induced metric $g$ on $M$, then the equality of mixed partials of the vector-valued function $x$ on $M$ results in the Codazzi equation for $A$

$$(\nabla_X A)Y - (\nabla_Y A)X = 0,$$

for all vectors $X$ and $Y$ tangent to $M$. For the later correspondence between stress tensors and quadratic differentials we now establish some basic facts on symmetric operators satisfying Codazzi's equation on any Riemannian 2-manifold.

The complex structure $J$ on $M$ is defined pointwise as counterclockwise rotation of the oriented inner product space $(M_p, g_p)$ through the angle $\frac{\pi}{2}$. Local coordinates $(u_1, u_2)$ on $M$ for which $g = e^{2\rho}(du_1^2 + du_2^2)$, where $\rho$ is a smooth real-valued function of these variables, are called local isothermal coordinates for the metric $g$ and if the coordinate frame $\{ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \}$ gives the surface orientation the coordinates are called positive isothermal coordinates; $w = u_1 + iu_2$ is then called a local complex coordinate for the underlying Riemann surface.

The vector space of smooth symmetric tensor fields of type $(1, 1)$ on the Riemannian 2-manifold $(M, g)$ will be denoted $S(M)$. The linear map defined by $S : j(T) = J T J = T - (Tr T) I$ gives an isomorphism of $S(M)$.

The divergence of a smooth symmetric tensor field $T$ on $M$ is defined to be the trace of the vector-valued quadratic form $(X,Y) \rightarrow (\nabla_X T)Y$ and denoted $\text{div} T$. In terms of an orthonormal frame $\{e_1, e_2\}$, $\text{div} T = (\nabla_{e_1} T) e_1 + (\nabla_{e_2} T) e_2$.

Similarly, if $S$ is any smooth symmetric tensor field on $M$ the vector-valued 2-form defined by $(X,Y) \rightarrow (\nabla_X S)Y - (\nabla_Y S)X$ may be written $(\text{Cod} S) da$, where the vector field $\text{Cod} S$ may be calculated in terms of a positive oriented orthonormal frame $\{e_1, e_2\}$ as $\text{Cod} S = (\nabla_{e_1} S) e_2 - (\nabla_{e_2} S) e_1$. If $S = j(T)$ then $\text{J}(\text{Cod} S) = J(\text{J}(\text{Cod} S)) e_2 - (\nabla_{e_2} (J T J) e_1) = - (\nabla_{e_1} T) J e_2 + (\nabla_{e_2} T) J e_1 = \text{div} T$. Thus the space of divergent-free symmetric tensors $\mathcal{D}(M) = \{ T \in S(M) | \text{div} T \equiv 0 \}$ is isomorphic under $j$ to the space of symmetric Codazzi tensors $\mathcal{C}(M) = \{ S \in S(M) | \text{Cod} S \equiv 0 \}$.

The space of complex quadratic differentials on $M$ is denoted $Q(M)$. Any such differential $\Phi$ can be locally represented in terms of a local complex coordinate $w$ (of the underlying Riemann surface structure determined by $g$ and the orientation) as $\Phi = \phi(w) dw^2$. If $\Phi = \psi(z) dz^2$ with respect to another local complex coordinate
$z$ on an overlapping neighbourhood, then the invariant nature of $\Phi$ is expressed by the identity $\phi(w) dw^2 = \psi(z) dz^2$ or $\phi(w(z))(\frac{d w}{d z})^2 = \psi(z)$. There is a natural map $h : S(M) \rightarrow Q(M)$ defined locally by $h(S) = \phi(w) dw^2 = e^{2\rho}(b + i\alpha) dw^2$, where $w = u_1 + iu_2$ is a local complex coordinate with respect to which $g = e^{2\rho}(du_1^2 + du_2^2)$ and the matrix of $S$ with respect to the coordinate frame $\{ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \}$ is

$$
S = \begin{pmatrix}
-\tau + a & b \\
b & -\tau - a
\end{pmatrix}.
$$

Notice that the map $h$ restricted to $\mathcal{C}(M)$ has kernel consisting of all constant multiples of the identity. So $h(S)$ determines the Codazzi tensor $S \in \mathcal{C}(M)$ to within a constant multiple of the identity and in the same way $T = j^{-1}(S) \in \mathcal{D}(M)$ is so determined also.

**Definition.** The Hopf differential of an immersion $x$ is defined to be the complex quadratic differential $h(A)$, where $A$ is the second fundamental form of the immersion with respect to the oriented unit normal $\xi$; it will be denoted $\Omega$. If

$$
A = \begin{pmatrix}
H + \alpha & \beta \\
\beta & H - \alpha
\end{pmatrix}
$$

with respect to the above local coordinates, then the corresponding local representation of the Hopf differential is $\Omega = \omega dw^2 = e^{2\rho}(\beta + i\alpha) dw^2$, or $\Omega = 2ig(A \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_1}) dw^2$, where $\frac{\partial}{\partial w} = \frac{1}{2} \{ \frac{\partial}{\partial u_1} - i \frac{\partial}{\partial u_2} \}$ and $\frac{\partial}{\partial w} = \frac{1}{2} \{ \frac{\partial}{\partial u_1} + i \frac{\partial}{\partial u_2} \}$. From the representation it is seen that $\Omega$ is a well-defined complex quadratic differential on $M$.

**Lemma 1.** Let $S$ be any smooth symmetric tensor field on $M$ given locally in isothermal coordinates $(u_1, u_2)$ with $g = e^{2\rho}(du_1^2 + du_2^2)$ by

$$
S = \begin{pmatrix}
-\tau + a & b \\
b & -\tau - a
\end{pmatrix}.
$$

Then $\text{Cod} S = 2e^{-4\rho} [(\phi_{w\bar{w}} + ie^{2\rho} \tau_w) \frac{\partial}{\partial w} + (\bar{\phi}_{w\bar{w}} - ie^{2\rho} \tau_{\bar{w}}) \frac{\partial}{\partial \bar{w}}]$, where $h(S) = \Phi$ is the global quadratic differential given locally by $\Phi = \phi(w) dw^2 = e^{2\rho}(b + i\alpha) dw^2$.

In particular, $S$ satisfies Codazzi’s equation if and only if $e^{-2\rho} \phi_{\bar{w}} = i\tau_w$. In terms of the global linear differential $\pi(\Phi)$ given locally by $\pi(\Phi) = ie^{-2\rho} \phi_{w\bar{w}} dw$ the condition may be written $\pi(\Phi) = \partial \tau$, where $\partial$ denotes the usual pure complex differential operator.

Proof. The expressions for the covariant derivatives of the connexion of $g$ with respect to the isothermal coordinate frame are easily found to be given by $\nabla_{\frac{\partial}{\partial u_1}} \frac{\partial}{\partial u_1} = -\nabla_{\frac{\partial}{\partial u_2}} \frac{\partial}{\partial u_2} = \rho_{u_1} \frac{\partial}{\partial u_1} - \rho_{u_2} \frac{\partial}{\partial u_2}$ and $\nabla_{\frac{\partial}{\partial u_1}} \frac{\partial}{\partial u_2} = \nabla_{\frac{\partial}{\partial u_2}} \frac{\partial}{\partial u_1} = \rho_{u_2} \frac{\partial}{\partial u_1} + \rho_{u_1} \frac{\partial}{\partial u_2}$. Using the positive orthonormal frame $\{ e^{-\rho} \frac{\partial}{\partial u_1}, e^{-\rho} \frac{\partial}{\partial u_2} \}$ obtained from these coordinates and the equality of mixed partials, it is easy to see that $\text{Cod} S = e^{-2\rho} \{ \nabla_{\frac{\partial}{\partial u_1}} (S \frac{\partial}{\partial u_2}) - \nabla_{\frac{\partial}{\partial u_2}} (S \frac{\partial}{\partial u_1}) \}$. After a further straightforward calculation of the above we obtain

$$
e^{4\rho} \text{Cod} S = \{ (e^{2\rho} b)_{u_1} - (e^{2\rho} a)_{u_2} + e^{2\rho} \tau_{u_2} \} \frac{\partial}{\partial u_1} + \{ (e^{2\rho} a)_{u_1} + (e^{2\rho} b)_{u_2} + e^{2\rho} \tau_{u_1} \} \frac{\partial}{\partial u_2}$$
In complex notation this takes the form

\[ Cods = 2e^{-4\theta} \left[ (\phi_{\bar{w}} + i e^{2\theta} \tau_w) \frac{\partial}{\partial w} + (\phi_w - i e^{2\theta} \tau_{\bar{w}}) \frac{\partial}{\partial \bar{w}} \right], \]

where \( h(S) \) is given locally by \( h(S) = \phi(w)dw^2 = e^{2\theta}(b + ia)dw^2 \).

The vanishing of \( Cods \) is therefore equivalent to the equation \( \phi_{\bar{w}} = -i e^{2\theta} \tau_w \) for the local function \( \phi \) representing the quadratic differential \( \Phi = h(S) \) in these coordinates. Thus \( \pi(\Phi) \), as given in the lemma, is a well-defined complex linear differential and the Codazzi equation for \( S \) is equivalent to the equation \( \pi(\Phi) = \partial \tau \) where \( \tau = -\frac{1}{2}TrS \). This ends the proof of the lemma.

We define the space of Codazzi complex quadratic differentials to be \( Q_{C}(M) = h(C(M)) \). By the lemma, the space of holomorphic quadratic differentials \( Q_{hol}(M) \) is \( \pi^{-1}(\{0\}) \).

The next lemma summarises the properties of the maps introduced above. Notice that the trace zero subspaces \( D_0(M) \) and \( C_0(M) \) of \( D(M) \) and \( C(M) \), respectively, actually coincide and \( j \) is the identity thereon.

**Lemma 2.** Let \( D(M) \) and \( C(M) \) be, respectively, the spaces of smooth symmetric divergence-free and Codazzi tensor fields on the Riemannian 2-fold \((M, g)\) and let \( Q(M) \) and \( \mathcal{L}(M) \) be, respectively, the spaces of smooth complex quadratic and linear differentials on the Riemann surface \( M \). Then the mapping sequence

\[ D(M) \rightarrow C(M) \rightarrow Q(M) \rightarrow \mathcal{L}(M) \]

has the property that \( j \) is an isomorphism, \( h \) has kernel the constant multiples of the identity, \( \text{Im} h = Q_{C}(M) \) is the space of Codazzi quadratic differentials and \( \ker \pi = Q_{hol}(M) \) is the space of holomorphic quadratic differentials.

Codazzi’s equation for \( S \in C(M) \) is

\[ \pi(h(S)) = -\partial \left( \frac{1}{2}TrS \right). \]

The Hopf differential \( \Omega = h(A) \) of the immersion \( \chi \), introduced above, is a global complex quadratic differential on the underlying Riemann surface, and the zeroes of \( \Omega \) are the umbilics of the immersion. If

\[ A = \begin{pmatrix} H + \alpha & \beta \\ \beta & H - \alpha \end{pmatrix} \]

is the matrix of the second fundamental form with respect to the coordinate frame of positive isothermal coordinates \( \{u_1, u_2\} \) for which \( g = e^{2\theta}(du_1^2 + du_2^2) \) then \( \Omega = \omega(w)dw^2 = e^{2\theta}(\beta + i\alpha)dw^2 \), where \( w = u_1 + iu_2 \). The Codazzi equation for \( A \) may be written in terms of the local representation of \( \Omega = \omega dw^2 \) as \( w_{\bar{w}} = ie^{2\theta}H_{\bar{w}} \) and, in terms of the above notation, \( \pi(\Omega) = -\partial H \). Thus \( \Omega \) is holomorphic if and only if the immersion has constant mean curvature.
Definition. An immersion \( x : M \to \mathbb{R}^3 \) is **globally isothermic** if there exists a global holomorphic quadratic differential \( \Phi \) on \( M \) such that \( \Omega = ik\Phi \) on the complement of the umbilic set \( U \) in \( M \), \( k \) being a real-valued function on \( M - U \). The holomorphic differential \( \Phi \) being then uniquely determined by \( \Omega \) — and therefore by the geometry of \( M \) — to within a real constant multiple, it will be called the **underlying holomorphic quadratic differential** of the globally isothermic surface.

The next result follows from Lemma 2.

Lemma 3. If \( M \) is a globally isothermic surface then the underlying holomorphic quadratic differential \( \Phi \) determines a unique smooth, symmetric, trace-zero, divergence-zero Codazzi tensor field \( T_0 \) on \( M \) such that \( h_j(T_0) = h(T_0) = \Phi \). Since the geometry determines \( \Phi \) to within a real constant multiple the same is true of \( T_0 \).

Definition. An immersion \( x : M \to \mathbb{R}^3 \) is **isothermic** if there exist positive isothermal coordinates \( \{u_1, u_2\} \) on a neighbourhood \( V \) of each non-umbilic point and a local holomorphic quadratic differential \( \Phi_V \) in the coordinate \( w = u_1 + iu_2 \) such that the Hopf differential satisfies \( \Omega = ik_V\Phi_V \) on \( V \), \( k_V \) being a real-valued function on \( V \).

Note that if \( A \) diagonalizes with respect to these coordinates then \( \beta \equiv 0 \) and \( \Omega_V = ik_V\Phi_V \), where \( k_V \) is a real function on \( V \) and \( \Phi_V \) is a holomorphic quadratic differential on \( V \). Conversely, if on an isothermal patch \( V \) about a non-umbilic point \( p \), \( \Omega_V = ik_V\Phi_V \), for \( k_V \) and \( \Phi_V \) as above, then \( \Phi_V = f(w)dw^2 \), with \( f \) a holomorphic function of \( w \), and since \( p \) is not an umbilic, \( f(0) \neq 0 \). After effecting a holomorphic change of coordinates we may assume \( f \equiv i \) and, in these new coordinates, \( \beta \equiv 0 \) and so \( A \) diagonalizes. In this form the notion was first introduced by Cayley [5].

Thus being isothermic is equivalent to the existence of isothermal coordinates on a neighbourhood of each non-umbilic point with respect to which the second fundamental form diagonalizes.

Examples and properties of isothermal and globally isothermic surfaces are discussed in §5. The soliton theory of these surfaces is found in [6].

§3. The equilibrium equation

An infinitesimally thin material membrane in equilibrium under the influence of an external force field \( F \) may be considered as a smooth surface \( M \) with smooth boundary all smoothly embedded in \( \mathbb{R}^3 \). The smooth embedding \( x : M \to \mathbb{R}^3 \) will be taken to be smooth at the boundary, in the sense that it extends to a smooth embedding of a larger surface into \( \mathbb{R}^3 \). Although the equilibrium could be formulated for non-orientable surfaces, we keep the discussion to oriented surfaces as in §2 because our interest is in closed membranes and these are automatically orientable. The boundary, if non-empty, is given the canonical orientation, i.e., that for which the oriented unit normal to the boundary points into the interior of \( M \). The force field is a smooth vector-valued function \( F : M \to \mathbb{R}^3 \).

The nature of the responding forces of stress within the membrane (i.e. the stress tensor \( T \)), holding it in equilibrium in response to the applied field \( F \) are explained
below. In our discussion we also assume these are $C^\infty$. Post facto, it will be seen that for the equilibrium equation we need no more than $x$ to be $C^3$, $T$ to be $C^1$ and $F$ to be continuous.

The equilibrium of an isotropic membrane (for which $T \equiv \tau I$, such as happens with a liquid film or bubble) acted on by a smooth force field $F$, was considered first by Young [19] and then by Laplace [10] at the outset of the nineteenth century in their studies of capillary surfaces (see [14]). The more general equation is due to Beltrami [2] and Lecomte [12].

The tangent bundle $T(M) = \{(p, X)| p \in M, X \in M_p\}$ considered with its standard smooth structure has a distinguished submanifold $S(M) = \{(p, X) \in T(M)| g_p(X, X) = 1\}$ determined by the metric $g$ induced on $M$, given in §2. This is the unit tangent bundle of the Riemannian manifold $(M, g)$. The map $\pi : S(M) \rightarrow M$ given by $\pi(p, X) = p$ is the natural projection and the fibre $\pi^{-1}(p)$ is just the unit circle in $M_p$.

If a small slit is cut in the membrane along a geodesic emanating from $p$ in the direction of a unit vector $e \in M_p$, then at points on either side of the cut a restorative force must be applied in opposite directions to maintain the membrane in equilibrium. The limiting value, at the left edge of the cut, of this force (per unit length), as the length of the cut goes to zero, is denoted $r(p, e)$ and called the response at $p$ corresponding to the direction $e$. Clearly $r : S(M) \rightarrow \mathbb{R}^3$.

The equilibrium equation of the membrane is derived in Gurtin-Murdoch ([7], p. 301) from force and moment balance considerations in the small, and in their notation $r(p, e) = -x_{*p}(T_p(J e))$, where $T$ defines a smooth tensor field of type $(1,1)$ on $M$ which is symmetric with respect to the induced metric $g$. The tensor field $T$ is called the stress tensor of the material membrane in equilibrium under the field $F$.

Let $D$ be any compact simply-connected domain in $M$ with oriented boundary a smooth closed curve $\gamma(s), 0 \leq s \leq l$, parametrized by arc length $s$. If this domain be excised from the membrane the only forces keeping it in equilibrium are $F$, acting over the interior, and the forces of response corresponding to the oriented unit tangent field $e$ acting at the boundary $\partial D$ of $D$. The external force field $F$ gives rise to the $\mathbb{R}^3$-valued 2-form $Fda$, where $da$ is the natural area element of the induced metric on $M$ for the fixed choice of orientation. Equilibrium calls for the integral of this 2-form over any domain $D$ with smooth boundary in $M$ to balance the integral of the restorative forces of the response on $D$ along its oriented boundary $\partial D$.

Thus the equilibrium equation for $D$ is

$$\int_D F da + \int_0^l r(\gamma(s), \dot{\gamma}(s)) ds = 0,$$

or alternatively,

$$\int_D F da + \int_{\partial D} \omega = 0,$$

where $\omega(X) = -x_{*}(T(J X))$. This defines an $\mathbb{R}^3$-valued 1-form $\omega$ on $M$, which we call the response 1-form of the membrane and we note that it completely encodes the forces
of stress within the membrane. By Stokes’ theorem, the equilibrium condition on $D$ becomes

$$\int_D (Fda + d\omega) = 0,$$

where $d\omega$ denotes the exterior derivative of $\omega$. The vector-valued 2-form $d\omega$ may be computed from $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$, using the fundamental equations of surface theory, and we obtain $d\omega = [x_* (\text{div}T) + \text{Tr}(AT)\xi] da$. Hence

$$\int_D (F + x_*(\text{div}T) + \text{Tr}(AT)\xi) da = 0$$

and, by continuity of $F$, $T$ and $\text{div}T$,

$$F + x_*(\text{div}T) + \text{Tr}(AT)\xi = 0$$

is the equilibrium equation of the membrane.

**Remark 1.** If $f_t : M \rightarrow \mathbb{R}^3$ is a 1-parameter family of isometric deformations of an oriented surface $M$ with oriented unit normal field $\xi$, and associated second fundamental form $A_t$, then the Gauss equation may be written $J A_t J A_t = K$, where $K$ is the Gauss curvature of the induced metric. Writing $B = (dA_t)/dt$ we have a symmetric tensor field satisfying Codazzi’s equation, so that $T_0 = JB J$ must have zero divergence, $\text{div}T_0 = 0$. Differentiating Gauss' equation above we obtain $T_0A - J AT_0J = 0$, from which it follows that $\text{Tr}AT_0 = 0$. From these results it follows any isometric deformation gives rise to a residual stress $T_0$ (i.e., a solution of the equilibrium equation with $F = 0$). If $H$ is preserved under the deformation then $\text{Tr}T_0 = 0$ and $T_0$ is a residual shear. Similar remarks may be made for infinitesimal isometric deformations.

Let the stress tensor $T$ be given in positive isothermal coordinates $(u_1, u_2)$, with $g = e^{2\rho}(du_1^2 + du_2^2)$, by

$$T = \begin{pmatrix} \tau + a & b \\ b & \tau - a \end{pmatrix}$$

and $\Phi = hj(T)$ be the associated quadratic differential as given in §2. The tangential component $F^T$ of $F$ along the surface is defined by $F = x_*(F^T) + \langle F, \xi \rangle \xi$. Representing $\Phi$ and the tangential component $F^T$ of $F$ in these local coordinates we have

$$\Phi = \phi(w) dw^2 = e^{2\rho}(b + i\sigma) dw^2$$

and

$$F^T = F_1 \frac{\partial}{\partial u_1} + F_2 \frac{\partial}{\partial u_2} = (F_1 - iF_2) \frac{\partial}{\partial \mu_2} + (F_1 + iF_2) \frac{\partial}{\partial \mu_1}.$$ 

From the expression for $\text{div}T = J C \text{od} S$ derived in Lemma 2 of §2, the tangential component of the above equilibrium equation is expressed locally by

$$\phi_{\mu_1} + i e^{2\rho} \tau_\mu + \frac{i e^{2\rho}}{2} (F_1 - iF_2) = 0,$$

or, more invariantly, as the identical vanishing of a globally defined linear differential

$$\{e^{-2\rho} \phi_{\mu_1} + i \tau_\mu + i e^{2\rho} (F_1 - iF_2)\} dw \equiv 0.$$
Note that when the external force field is conservative on $\mathbb{R}^3$ (i.e., $F$ is the gradient of a function $f$ on $\mathbb{R}^3$) that $F_1 - i F_2 = 2e^{-2\rho} f_w$, and the tangential equilibrium equation then takes the form

$$\phi_{\overline{w}} + ie^{2\rho}(\tau + f)_w = 0,$$

and this is equivalent to $T$ being divergent-free on $M$.

The normal component of the equilibrium equation is

$$2H\tau + e^{-4\rho}(\phi_{\overline{\omega}} + \overline{\phi}\omega) + F_3 = 0,$$

where $F_3 = \langle F, \xi \rangle$ and $\Omega = \omega dw^2$ is the local representation of the Hopf differential of the surface.

**Flat Examples.** That there is no uniqueness in the solutions of the equilibrium equation is known classically from the infinite dimensional space of smooth symmetric tensor fields satisfying the equilibrium equation on any planar membrane configuration in the presence of external field $F \equiv 0$ — the *Airy tensors*. Such a tensor field is obtained by taking $T = j^{-1}(H\phi)$, where $\phi$ is any smooth function on the planar configuration $M$ and $H\phi$ is its Hessian operator, defined by $H\phi(X) = \nabla_X(\nabla \phi)$, where $\nabla$ is the connexion of the induced metric. Using the fact that $M$ is planar — in fact vanishing Gauss curvature (flatness) is enough — it is easily checked that $H\phi$ satisfies Codazzi’s equation. Thus $\text{div} T \equiv 0$ and, since $A \equiv 0$ for planar surfaces, referring back to the equilibrium equation we see this is all that is needed for solutions of the equilibrium equation for planar membranes with $F \equiv 0$.

Conversely, suppose that $M$ is a simply-connected compact flat membrane with smooth boundary in equilibrium with $F \equiv 0$. If $T$ is a solution of the equilibrium equation then, since $\text{div} T \equiv 0$, $S = j(T)$ satisfies Codazzi’s equation. Letting $\Delta$ denote the Laplace operator of the flat induced metric on the membrane $M$ we may use the Poisson formula to solve the equation $\Delta \psi = -2\tau$ for $\psi$, where $2\tau = Tr T = -Tr S$. Now $H\psi$ satisfies Codazzi’s equation since $M$ is flat and so also does $S$. Thus, $S_0 = S - H\psi$ is a Codazzi operator with trace zero. Thus the quadratic differential $\Phi = h(S_0)$ must be holomorphic by Lemma 2. Now $(M, g)$ being flat and simply-connected it may be identified with a simply-connected region of the plane endowed with its euclidean metric. Then $\Phi = (b+ia)dw^2$, where $w$ is the complex coordinate on the plane and $b+ia$ is a holomorphic function of $w$. $M$ being simply-connected there exists a holomorphic function $f = k + il$ an $M$ such that $f_{ww} = b+ia$. From the Cauchy-Riemann equations for $f$ this becomes $b+ia = 2ilw$ and this means that $S_0 = H_1$ or $S = H_{\psi+1}$ is Hessian.

In summary, when $M$ is simply connected and flat with smooth boundary the space of solutions of $\text{div} T = 0$ may be identified with the space of Hessians of all smooth functions on $M$. This then is the space of solutions of the equilibrium equation for planar membranes when $F \equiv 0$. When $M$ is flat but not planar, $A$ does not vanish identically and the normal component of the equilibrium equation gives a further restriction on the space of solutions.
§4. The proof of the main result

In this section we prove Theorem 1 for closed membranes. To begin with $M$ will be taken to be compact and orientable with smooth boundary $\partial M$, possibly empty. We consider the tangential and normal equilibrium equations as conditions on the quadratic differentials corresponding to the second fundamental form and the stress tensor. The normal equation will lead to the appearance of globally isothermic surfaces.

If two distinct stress tensors $T$ and $\tilde{T}$ satisfying the equilibrium equation

$$\text{div}T + Tr(AT)\xi + F = 0$$

have the same mean stress then their difference $T_0 = \tilde{T} - T$ is nontrivial and satisfies i) $\text{div}T_0 = 0$, ii) $Tr(AT_0) = 0$ and iii) $TrT_0 = 0$. Now i) and iii) mean that $S_0 = j(T_0)$ is a trace zero solution of the Codazzi equation on $M$, where $j$ is the isomorphism of Lemma 1. Writing $h(S_0) = \Phi \in Q_C(M)$, it follows from Lemma 2 that $\Phi$ is a holomorphic quadratic differential on the Riemann surface $M$. Since $T_0$ is not identically zero, the differential $\Phi$ is nontrivial, and so its zero set $Z(\Phi)$ is a discrete subset of $M$.

Let

$$A = \begin{pmatrix} H + \alpha & \beta \\ \beta & H - \alpha \end{pmatrix}$$

and

$$T_0 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

be the matrices of $A$ and $T_0$ with respect to a positive isothermal coordinate frame $\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\}$ on a neighbourhood $V$ on which $g = e^{2\rho}(du_1^2 + du_2^2)$. Then the corresponding local representation of the associated quadratic differentials, encountered in §2, are $\Omega = \omega dw^2$ and $\Phi = \phi dw^2$, where $\omega = e^{2\rho}(\beta + i\alpha)$ and $\phi = e^{2\rho}(b + ia)$. Since $TrT_0 = 0$, condition ii) above translates into $\omega \bar{\phi}$ being pure imaginary and so $\omega = ik\phi$ on $V - Z(\Phi)$, $k$ being a real function on the latter set. Hence $\Omega = ik\Phi$ on $M - Z(\Phi)$, $k$ being a real function on the latter set.

There remain just two possibilities.

The first is that $k \neq 0$ on $M - Z(\Phi)$ and in this case we show that $M$ must be globally isothermic. If we show that $Z(\Phi)$ consists of umbilics then we will have shown that $\Omega = ik\Phi$ on the complement of the umbilic set, as required. We will prove something more.

Assume $p \in Z(\Phi)$ is not an umbilic. We may take $p$ to be the origin of the local isothermal coordinates. Let $m > 0$ be the order of vanishing of $\Phi$ at $p$. Since $p$ is not an umbilic $\omega(o) \neq 0$ and, we may assume that $\omega$ does not vanish on $V$. From the identity $\omega = ik\phi$ on $V - \{o\}$ it follows that $k$ and $\phi$ do not vanish on $V - \{o\}$.

Any smooth nonvanishing complex-valued function $v = r + is$ on $V - \{o\}$ may be thought of as a vector field on this punctured neighbourhood with an isolated singularity at $o$ and the index $j_v(o)$ of $v$ at $o$ is defined to be the integer
\[ j_\nu(o) = \frac{1}{2\pi} \int_C \frac{rds - sdr}{r^2 + s^2} \]

whose value is independent of the choice of simple closed curve \( C \) circulating within \( V \) once around the singularity \( o \), counterclockwise. Immediately from this definition we see that \( j \) gives a homomorphism from the ring of germs of nonvanishing complex functions on punctured neighbourhoods of \( o \) onto the additive group of integers \( \mathbb{Z} \). Note that \( j_\nu(o) = 0 \) if \( \nu \) is purely real, purely imaginary or if \( \nu \) extends to a non-vanishing continuous complex function on a full neighbourhood of \( o \). Moreover \( j_\nu(o) = m > 0 \) if \( \nu \) is holomorphic with a zero of order \( m > 0 \) at \( o \), as follows at once from the previous remark on representing \( v(z) = z^m u(z) \) with \( u \) holomorphic and nonvanishing on \( V - \{ o \} \). The earlier identity \( \omega = i\kappa \phi \) on \( V - \{ o \} \) therefore gives

\[ j_{\omega}(o) = j_{\delta}(o) + j_{\kappa}(o) + j_{\phi}(o) = m > 0. \]

In particular, \( \omega \) must vanish at \( o \), a contradiction. Hence each \( p \in Z(\Phi) \) must be an umbilic and so we certainly have \( \Omega = i\kappa \Phi \) on the complement \( M - U \) of the umbilic set \( U \). This completes the proof that \( M \) is globally isothermic when \( k \neq 0 \).

In the other event that \( k \equiv 0 \) on \( M - Z(\Phi) \), then \( \Omega \equiv 0 \) on \( M - Z(\Phi) \) and therefore on \( M \). Hence \( M \) is composed of umbilics and, if closed, must be the round sphere. This completes the proof of Theorem 1.

If \( k \equiv 0 \) but \( M \) is not closed then it must be a proper subdomain of a round sphere or plane in \( \mathbb{R}^3 \). We may further note that in the two exceptional cases of surfaces with boundary just mentioned, \( M \) is conformally equivalent to a domain \( D \) in \( \mathbb{C} \) and \( \Phi \) may be represented globally as \( \Phi = \phi(z)dz^2 \), where \( \phi(z) \) is a holomorphic function of the coordinate \( z \) on \( \mathbb{C} \). These form an infinite dimensional vector space over \( \mathbb{C} \) and, conversely, each such \( \Phi \) determines a \( T_0 \) satisfying i) and iii) above — and ii) holds trivially when \( M \) is planar and, in consequence of iii), also when \( M \) is spherical. Thus, for proper subdomains of the round sphere or plane, any solution \( T \) of the equilibrium equation determines an infinite family of solutions \( T + cT_0 \), where \( c \) is any real constant and \( T_0 \) is a tensor field corresponding to any holomorphic quadratic differential via the map introduced in §2; these solutions all have the same mean stress as \( T \).

We summarize these results as follows:

**Theorem 4.** Let \( M \) be a smooth compact orientable membrane with smooth boundary \( \partial M \) (possibly empty) in equilibrium under an external force field \( F \), and let \( T \) be the responding stress tensor on \( M \) and \( \tau = \frac{1}{2} \text{Tr}T \) the mean stress of \( T \).

Either \( T \) is the unique solution of the equilibrium equation with mean stress \( \tau \) or \( M \) is a globally isothermic surface but not the round sphere \( S^2 \).

Conversely, if \( M \) is globally isothermic but not the round sphere, then either

i) \( M \) is a proper subdomain of the round sphere \( S^2 \) or the plane \( \mathbb{R}^2 \) and the space of solutions of the equilibrium equation with mean stress \( \tau \) is infinite dimensional, or

ii) the space of solutions of the equilibrium equation with mean stress \( \tau \) is one-dimensional and canonically determined by the geometry.
Note that the proof did not require any information on the behaviour of the stress
tensor $T$ at the boundary when $\partial M = \emptyset$. If two solutions of the equilibrium equation
also coincide in directions tangential to the boundary then their difference $T_0$ is zero
in these directions. Since $T r T_0 \equiv 0$ on account of the mean stress condition, it follows
that $T_0$ is zero in all directions at each point of the boundary. Hence the holomorphic
differential $h_j(T_0) = \Phi$ associated to $T_0$ in the foregoing vanishes on $\partial M$ and so is
identically zero. Thus if $\partial M \not= \emptyset$ and two solutions $T$ and $\hat{T}$ of the equilibrium equation
with the same mean stress on $M$ coincide in tangential directions along the boundary
then they coincide everywhere on $M$.

**Corollary 3.** If the boundary of $M$ is nonempty then $T$ is uniquely determined by $\tau$
and the knowledge of $T$ tangent to the boundary.

§5. **Remarks on globally isothermic surfaces**

In this study of the equilibrium of material membranes, globally isothermic surfaces
come to the fore with the phenomenon described in Theorem 1, and for this reason we
make some remarks about such surfaces. These serve to distinguish them within the
larger family of isothermic surfaces and we provide examples of both types.

**Definition.** If $\Phi$ is a complex quadratic differential on the Riemann surface $M$ then
at each point of the complement $M - Z(\Phi)$ of its zero set $Z(\Phi)$ there is a unique linear
subspace of the tangent space, defined by $\phi(w)dw^2 > 0$, where $\Phi = \phi(w)dw^2$ is the
representation of $\Phi$ in any local complex coordinate about this point; this then defines
a foliation of $M - Z(\Phi)$ [17]. If we simply look at $\text{Im}(\phi(w)dw^2) = 0$ we obtain the
previous foliation and its orthogonal, and this pair of orthogonal foliations on $M - Z(\Phi)$
we refer to as the foliations of $\Phi$; we will mostly be interested in the case where $\Phi$ is
holomorphic and nontrivial so that $Z(\Phi)$ is discrete.

If $\Omega$ is the Hopf quadratic differential of an immersion $x$ of $M$ then the complement
$M - U$ of the umbilic set $U$ is foliated by the principal foliations and these are described
by $\text{Re}(\omega(w)dw^2) = 0$, where $\Omega = \omega(w)dw^2$ is the representation of $\Omega$ in any local
complex coordinate on $M - U$.

As we saw in §2 if $x$ is globally isothermic then $\Omega = ik\Phi$ on $M - U$, where $\Phi$ is a
holomorphic quadratic differential on $M$ and $k$ is a real function on $M - U$; it follows
that $Z(\Phi)$ is contained in $U$ and, by the definition above, that the principal foliations
of $x$ coincide with the foliations of $\Phi$ on $M - U$; moreover the foliations of $\Phi$ extend
the principal foliations of $x$ onto the set $M - Z(\Phi)$.

Conversely, if the principal foliations of $x$ (which are only defined on the set $M - U$)
coincide with the foliations of a global holomorphic quadratic differential $\Phi$, then $Z(\Phi)$
is contained in $U$ and, from the definition above, $\Omega = ik\Phi$ on $M - Z(\Phi)$ with $k$ some
real function on $M - Z(\Phi)$ and therefore the relation holds on $M - U$ and $x$ is globally
isothermic. Hence

**Remark 2.** An immersion $x$ is globally isothermic if and only if its principal foliations
coincide with the foliations of some global holomorphic quadratic differential.
Of course when $M$ is globally isothermic this holomorphic quadratic differential is unique to within a real constant multiple, and we call it the underlying holomorphic quadratic differential of the globally isothermic surface (because it is unique up to a real constant multiple as we saw earlier).

For a surface of constant mean curvature, Codazzi's equation amounts to the statement that the Hopf differential is itself a holomorphic quadratic differential. Hence such a surface is globally isothermic. From Kapouleas [9] we know that there exist compact globally isothermic surfaces in $\mathbb{R}^3$ of every genus; of these only the round sphere is embedded [1]. The Bonnet surfaces — surfaces not of constant mean curvature but admitting a one-parameter family of isometric deformations preserving the mean curvature function — give another class of globally isothermic surfaces, but none of these can be compact [3].

**Remark 3.** The property of being globally isothermic surfaces is preserved under conformal transformations of $\mathbb{R}^3$ (or rather its compactification $S^3 = \mathbb{R}^3 \cup \infty$).

We see this as follows; let $x : M \rightarrow \mathbb{R}^3$ be a globally isometric immersion with induced metric $g$, second fundamental form $A$, Hopf differential $\Omega$ and $\Omega = i k \Phi$ on the complement of the umbilic set $M - U$ for some global holomorphic quadratic differential $\Phi$. By Liouville, any conformal transformation $\psi$ is a composition of isometries, homotheties and reflexions in 2-spheres or 2-planes. It is straightforward to verify that the induced metric of the new surface $\tilde{x} = \psi \circ x$ obtained on applying $\psi$ is a non-vanishing functional multiple of the original and the second fundamental form of the new surface is a linear combination $pA + qI$ with $p$ and $q$ functions on $M$ and $p$ vanishing nowhere. It follows that the underlying complex structure is the same for both surfaces, the umbilic set $U$ is the same and the Hopf differential $\tilde{\Omega}$ is a nonzero functional multiple of $\Omega$. In particular, $\tilde{\Omega} = i \tilde{k} \Phi$ on $M - U$ and so $\tilde{x}$ is also globally isothermic with the same underlying holomorphic quadratic differential.

Further examples of globally isothermic surfaces are now obtained by applying conformal transformations to any of the constant mean curvature or Bonnet examples above but, again, no new compact embedded examples are obtained in this way.

**Embedded Examples 1.** Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a smooth curve in the upper-half $xy$-plane (i.e., $\gamma_2(t) > 0$), and $f(t, \theta) = (\gamma_1(t), \gamma_2(t) \cos \theta, \gamma_2(t) \sin \theta)$ the surface $M$ obtained by rotating $\gamma$ about the $x$-axis. Let $s(T) = \int^T \sqrt{\gamma_1'^2 + \gamma_2'^2} \, dt$ be the arc length parameter of $\gamma$ with respect to the Poincaré metric on the upper half plane. Then $(s, \theta)$ are isothermal coordinates for the reparametrization $F(s, \theta) = f(t(s), \theta)$ of the given surface. It is easily checked that $\Omega = i k (ds + i d\theta)^2$ with $k$ real. Since $ds + i d\theta$ is a globally defined holomorphic 1-form on $M$, it follows that $M$ is globally isothermic. If $\gamma$ is a smooth simple closed curve in the upper half plane then $M$ is an embedded globally isothermic torus and the image of this surface under a conformal tranformation provides further compact embedded globally isothermic examples, most of which are not surfaces of revolution.

**Remark 4.** Unlike surfaces of constant mean curvature, globally isothermic surfaces
are not necessarily analytic.

To see this one need only take a torus of revolution generated by a smooth non-analytic simple closed curve. The self same example serves to illustrate that Alexandrov’s reflexion principle — and therefore his embedding theorem [1] — do not hold for globally isothermic surfaces.

We now construct globally isothermic compact embedded surfaces of every genus in \( \mathbb{R}^3 \), as announced in §1. The proof uses Lawson’s beautiful construction of closed embedded minimal surfaces in the sphere.

**Proof of Theorem 3** Let \( x : M \rightarrow (S^3, g_0) \) be an immersion of an oriented surface \( M \) into the unit sphere \( S^3 \) in \( \mathbb{R}^4 \) and \( g_0 \) the standard metric on \( S^3 \). The induced metric on \( M \) is denoted \( g \), \( \xi \) will denote the oriented unit normal field along the immersion \( x \) in \( S^3 \) and \( A \) will denote the second fundamental form of the immersion \( x \) with respect to this unit normal field.

Let \( \pi : S^3 - \{(0,0,0,1)\} \rightarrow \mathbb{R}^3 \) be stereographic projection from the north pole \((0,0,0,1)\) onto the equatorial hyperplane \( \mathbb{R}^3 \) in \( \mathbb{R}^4 \) on which the euclidean metric is denoted \( <,> \). For any vector \( Z \) tangent to \( S^3 - \{(0,0,0,1)\} \) at any point \( x \) we have

\[
\pi_{\ast}(Z) = \frac{1}{(1-x_4)^2}((1-x_4)Z + Z_4(x-e_4))
\]

where subscripts are used to label components of corresponding quantities in the \( e_4 \)-direction in \( \mathbb{R}^4 \); in particular, \( \langle \pi_{\ast}(X), \pi_{\ast}(Y) \rangle = \frac{1}{(1-x_4)^2}g_0(Z,W) \). Thus \( \pi \) is conformal and the metrics \( g \) and \( \tilde{g} \) induced on \( M \) by the maps \( x \) and \( \tilde{x} = \pi \circ x \) are related by \( \pi_{\ast}\tilde{g} = \frac{1}{(1-x_4)^2}g \), so that the underlying complex structures of the immersions \( x \) and \( \tilde{x} \) coincide. It also follows that \( N = (1-x_4)\pi_{\ast}(\xi) \) is a unit vector field along the immersion \( \tilde{x} \).

A simple calculation now shows that

\[
XN = -(1-x_4)\tilde{x}_{\ast}(AX) + \xi_4\tilde{x}_{\ast}(X)
\]

from which we obtain the second fundamental form of the immersion \( \tilde{x} \) to be

\[
\tilde{A} = (1-x_4)A - \xi_4I.
\]

Obviously, positive local isothermal coordinates for the metric \( g \) are also such coordinates for the metric \( \tilde{g} \) so that

\[
\tilde{\Omega} = 2i\tilde{g}(\tilde{A}\frac{\partial}{\partial u}, \frac{\partial}{\partial w})dw^2 = \frac{2i}{(1-x_4)^2}\tilde{g}(((1-x_4)A - \xi_4I)\frac{\partial}{\partial u}, \frac{\partial}{\partial w})dw^2 = \frac{1}{(1-x_4)^2}\Omega.
\]

Now if the immersion \( x \) is minimal then \( \Omega \) is itself a holomorphic quadratic differential and hence \( \tilde{\Omega} = ik\Phi \), where \( \Phi \) is the holomorphic differential \( i\Omega \) and \( k = \frac{1}{(1-x_4)} \); thus \( \tilde{x} \) is then a globally isothermic surface in \( \mathbb{R}^3 \). Since, by Lawson’s construction [11], there exist compact embedded minimal surfaces of every genus in \( S^3 \), it follows that:
Remark 5. There exist globally isothermic compact embedded surfaces of every genus in \( \mathbb{R}^3 \).

A distinctive feature of any globally isothermic surface is the character of its isolated umbilics.

Remark 6. On a globally isothermic surface every isolated umbilic has index \( \leq 0 \). On the complement of the set of umbilics of negative index the foliations of the underlying holomorphic quadratic differential extend the principal foliations.

On the complement of the umbilic set \( U \) we have \( \Omega = ik\Phi \) with \( \Phi \) a global holomorphic quadratic differential and since \( \Omega \) does not vanish there neither does \( \Phi \). Hence the zero set \( Z(\Phi) \) of \( \Phi \) must be contained in \( U \). Any isolated umbilic outside \( Z(\Phi) \) has index zero since the foliations of \( \Phi \) extend the principal foliations on \( M - Z(\Phi) \). We now find a formula for the index of any isolated umbilic \( p_0 \). Obviously \( k \) is defined and nowhere zero on a punctured isothermal neighbourhood \( V - \{p_0\} \) of \( p_0 \). Writing \( \Omega = \omega(w)dw^2 \) and \( \Phi = \phi(w)dw^2 \) in the local complex coordinate, \( \phi(w) \) is holomorphic with a zero of order \( m \geq 0 \) at \( w = 0 \). Interpreting the complex-valued functions \( \omega(w) \), \( ik(w) \) and \( \phi(w) \) as vector fields with isolated singularities at \( w = 0 \), the identity \( \Omega = ik\Phi \) on \( V - \{p_0\} \) and the usual property of the index gives the relation \( j_\omega = j_{ik} + j_\phi \) between their indices. Now since \( ik \) is pure imaginary its index is zero. Since \( \phi \) is holomorphic its index is \( -m \), the order of vanishing of \( \Phi \) at \( p_0 \). Hence \( j_\omega = -m \). The umbilic index of \( p_0 \) is easily determined by \( \Omega \) to be \( \frac{1}{2}j_\omega \). Hence the umbilic index of \( p_0 \) is \( -m \leq 0 \).

Finally a few remarks on isothermic surfaces, whose soliton theory is explained in [4] and [6]. It is easy to see that the definition, given in §2, is equivalent to the requirement that, locally on the non-umbilic set, the principal foliations coincide with the foliations of some local holomorphic quadratic differential. It was seen in §2 that being isothermic is equivalent to the requirement that there exist local isothermal coordinates on a neighbourhood of each non-umbilic point with respect to which the second fundamental form diagonalizes. This latter notion of considering the local simultaneous diagonalization of the first and second fundamental forms was Cayley's [5] point of departure in their study. A further equivalent condition can be found in a real 2-form \( \eta(\Omega) \) constructed from the Hopf differential \( \Omega \) on \( M - U \); if \( \Omega = \omega(w)dw^2 \) locally on \( M - U \) we may define locally a single-valued branch \( \theta = \frac{1}{2\pi} \frac{1}{\omega} \) of the argument of \( \omega \). Then \( \eta(\Omega) = i\theta dw \wedge d\bar{\omega} \) gives a well defined real 2-form on all of \( M - U \) and \( M \) is isothermic if and only if \( \eta(\Omega) \equiv 0 \) on \( M - U \).

Clearly globally isothermic surfaces are isothermic, but already Cayley had noted that any ellipsoid is isothermic and it is a simple consequence of the foregoing that it is not globally isothermic — indeed among spherical surfaces only round spheres are globally isothermic. However

Remark 7. There exist closed embedded isothermic surfaces of every genus in \( \mathbb{R}^3 \) which are not globally isothermic.

These can be constructed as follows: in Example 1 choose the generating curve to be convex and containing two congruent straight line segments orthogonal to the axis
of revolution and at the same distance from it. The original curve may be chosen so that no umbilics of the surface occur outside the planar part. The resulting torus of revolution contains two congruent planar annuli. Mark of a small contractible circle in one of these annuli as well as the corresponding circle in the congruent annulus. Take \( p \geq 2 \) copies of this surface (together with its pair of marked circles) stacked along the axis of revolution. We may join the neighbouring planar annuli of revolution by gluing a cylinder of revolution of negative curvature between neighbouring annuli along the circular markings. Then no new umbilics are introduced so that all umbilics of the new genus \( p \) surface are in the interior or boundary of the planar regions and therefore none are isolated. The resulting surface of genus \( p \) is isothermic since it is locally of the type of Example 1, however, since it contains no isolated umbilics, we know from Remark 5 that this surface cannot be globally isothermic.

Let \( M \) be a surface of revolution which intersects its axis of revolution. Then, by the example above the surface made from the complement of the intersection points is globally isothermic. We show that the full surface \( M \) is not globally isothermic unless the surface is a piece of a sphere or a plane. Assume that \( M \) is globally isothermic and that there exist non-umbilic points. Clearly the principal foliations through any non-umbilic point are the orbits of the revolution and the geodesics thereto perpendicular. Since \( M \) is globally isothermic these foliations coincide with the foliations of a global holomorphic quadratic differential \( \Phi \) on \( M \) as we saw in the beginning of this section. However as the singularities of the latter \( \Phi \)-foliations are finite in number, each with negative index (Remark 5, above), they do not have closed trajectories surrounding a singularity. This contradiction means that all points are umbilics and then the surface of revolution is a portion of a sphere or plane.

**Remark 7.** Let \( M \) be an isothermic surface with no umbilics. If it is simply connected then it is globally isothermic.

To see this first note that \( M \) is orientable since it is simply connected. Fixing an orientation of \( M \) we cover it by positive isothermal coordinate patches \( \{ V_\alpha \}_{\alpha \in I} \) with \( \Omega = ik_\alpha \Phi_\alpha \) for some function \( k_\alpha \) and some holomorphic quadratic differential \( \Phi_\alpha \) on each patch \( V_\alpha \). By holomorphicity \( k_{\alpha\beta} = \frac{k_\alpha}{k_\beta} \) is a non-zero real constant on \( V_\alpha \cap V_\beta \) and the cocycle condition \( k_{\alpha\beta}k_{\beta\gamma} = k_{\alpha\gamma} \) is satisfied on each non-empty \( V_\alpha \cap V_\beta \cap V_\gamma \). It now follows that there is a 0-cocycle assigning a non-zero constant \( c_\alpha \) to each \( V_\alpha \) such that \( k_{\alpha\beta} = \frac{c_\alpha}{c_\beta} \) for each non-empty \( V_\alpha \cap V_\beta \) (see, for example, Weil [20], page 85, Lemme 1). Then \( k = \frac{k_\alpha}{c_\alpha} \) defines a global smooth function on \( M \) and \( \Phi = c_\alpha \Phi_\alpha \) defines a global holomorphic quadratic differential on \( M \) and \( \Omega = ik\Phi \) on \( M \).

**Remark 8.** Any isothermic torus with no umbilics is always globally isothermic.

If \( x \) is an isothermic immersion of a 2-torus \( T^2 \) into \( \mathbb{R}^3 \) then the universal cover with the lifted conformal structure is \( \mathbb{C} \), the deck transformations being generated by some real basis \( \{ E_1, E_2 \} \) for \( \mathbb{C} \). Let \( \pi : \mathbb{C} \rightarrow \mathbb{R}^3 \) be the covering map; since the immersion \( \tilde{x} = x \circ \pi \) will be isothermic without umbilics, by Remark 7 we conclude \( \tilde{x} \) is globally isothermic. Thus Hopf differential \( \tilde{\Omega} = \omega(z)dz^2 \) of the immersion \( \tilde{x} \) may be
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written \( \tilde{\Omega} = ik\Phi \), where \( k \) is a non-vanishing real function on \( \mathbb{C} \) and \( \Phi = \phi(z) dz^2 \) is a nonvanishing holomorphic quadratic differential on \( \mathbb{C} \). Since \( \tilde{\Omega} \) is the lift of a quadratic differential on \( T^2 \) it follows that \( \omega \) is a periodic function on \( \mathbb{C}^2 \) with fundamental periods \( E_1 \) and \( E_2 \). The identity \( \tilde{\Omega} = ik\Phi \) now results in the existence of positive real numbers \( c_1 \) and \( c_2 \) such that

\[
\phi(z + m_1 E_1 + m_2 E_2) = c_1^{m_1} c_2^{m_2} \phi(z)
\]

for any \( z \in \mathbb{C} \) and any integers \( m_1 \) and \( m_2 \). It now follows that \( \frac{\phi'}{\phi} \) is a doubly periodic holomorphic function on \( \mathbb{C} \) and therefore a complex constant \( l \). Thus a branch of the logarithm of \( \phi \) is given by \( \log \phi(z) = lz + q \). The above period relations imply that \( l(m_1 E_1 + m_2 E_2) \) is real for all integers \( m_1 \) and \( m_2 \). But this means that either (i) \( l = 0 \) or (ii) \( E_1 \) and \( E_2 \) are multiples of \( \bar{E} \). As \( E_1 \) and \( E_2 \) are linearly independent we must have \( l = 0 \). Hence \( \phi \) is constant. It follows that \( k \) is periodic on \( \mathbb{C} \) and the relation \( \tilde{\Omega} = ik\Phi \) projects to the relation \( \Omega = ike^q dz^2 \) on \( T^2 \), with \( q \) some complex constant. Hence the original immersion is globally isothermic.
REFERENCES


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