Ricci curvature decay
near a real hypersurface singularity.

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Abstract: We give a near universal bound on the decay of the Ricci curvature near a simple singularity in a real hypersurface $\mathbb{R}^{m+1}$, $m \geq 3$ (Theorem 1). The bound is dimension independent, sharp in all dimensions $m \neq 4$ and may be universal. Such exceptions to our bound as might occur must be even-dimensional hypersurfaces with topologically trivial singularities, the profiles of which (i.e., intersections with small spheres centred at the singularity) are odd-dimensional topological spheres admitting, by a canonical construction (Remark 3, §4), Riemannian metrics with positive curvature operator. If, as long conjectured, there are no exotic spheres with positive curvature (or positive curvature operator) then any exception to the bound would have to be differentiably trivial.

§1. Introduction

The only universal curvature bound for complete surfaces in 3-space is sup $K \geq 0$, where $K$ denotes the Gauss curvature; this is Efimov’s pioneering generalization [4] of Hilbert’s theorem [6] that the full Poincaré plane cannot be isometrically immersed in $\mathbb{R}^3$. Smyth-Xavier’s [12] generalization of Efimov’s theorem gives sup $Ric \geq 0$, where $Ric$ denotes the Ricci curvature, for any complete hypersurface in euclidean space whose sectional curvature does not take all real values.

While this phenomenon is global, another simple universal inequality (also dimension independent) begins to emerge when we consider the Ricci curvature of an immersed hypersurface in $\mathbb{R}^{m+1}$ in a neighborhood of an isolated singularity. Roughly speaking, there is no limit to how positively (sectionally) curved a hypersurface might be near a singularity but the essence of our inequality is that there is

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a limit to how negatively (Ricci) curved it may be. The geometry of the singularity is studied by observing the curvature of the hypersurface intersection with small spheres centred at the singularity. While the initial interest was in negative Ricci curvature, there arises a kind of duality which results (after we turn our attention to the connectivity of principal curvature sets) in the question being transformed into a question on manifolds of positive sectional curvature — already solved by Moore [10] via holonomy considerations and Chen’s inequality on the total curvatures [3]. The dichotomy arising in Theorem 1 may yet be as interesting as the inequality itself.

The singularity may be taken to be the origin $o$, and the length $r$ of the position vector (from the origin) and the oriented angle $\phi$ that this vector makes with the (oriented) tangent space to the hypersurface give two smooth functions on the hypersurface. The singularities considered are simple in that as $r \to 0$

a) $\phi \to 0$, 
b) $r \nabla \phi \to 0$, where $\nabla \phi$ is the gradient of $\phi$ in the hypersurface metric, and 
c) $r A$ is bounded, where $A$ is the second fundamental form of the hypersurface.

Cone singularities (obtained by suspending a compact connected immersed hypersurface in the unit sphere $S^m(1)$ over the origin $o$ in $\mathbb{R}^{m+1}$) are the obvious examples; in particular, if $f_0$ is a homogeneous polynomial of degree $k \geq 2$ in $(m + 1)$ variables, which has an isolated critical point at $o$ and changes sign, then the hypersurface $f_0 = 0$ is such an example. If $f$ is any analytic function on a neighborhood of $o$ whose principal part $f_0$ is as above, then $f = 0$ is a smooth non-conical hypersurface with an isolated simple singularity at $o$. More generally, let $V$ be a smooth immersed hypersurface in $\mathbb{R}^{m+1}$ with an isolated singularity at $o$; if $V$ intersects all small spheres $S^m(r)$ transversally, and we assume that the radial derivative of $\phi$ is bounded and that the resulting family of hypersurfaces of $S^m(1)$ — obtained by re-normalization — is $C^2$-convergent as $r \to 0$ to a smooth hypersurface in $S^m(1)$, then $o$ is a simple singularity.

Let $V$ be any smooth immersed hypersurface in $\mathbb{R}^{m+1}$ with an isolated simple singularity at the origin. The tangent vectors to $V$ orthogonal to the position vector are called transverse and the symbol $Ric_T$ stands for the Ricci curvatures of unit vectors $X$ in these angular directions. The singularity is topologically trivial when its profiles (i.e. the intersection of the hypersurface with the small spheres centered at $o$) are topological spheres. Our main result says that $Ric_T$ cannot decay faster than $-\frac{1}{r^2}$ near a simple singularity without an exceptional conjunction in its dimension, topology and geometry.
Theorem 1. Let $V^m$ be a smooth immersed hypersurface in $\mathbb{R}^{m+1}$, $m \geq 3$, with an isolated simple singularity at $o$. Then either

$$\sup_{U} r^2 \text{Ric}_T \geq -1$$

for every neighborhood $U$ of $o$ or else all of the following hold: $m$ is even, the singularity is topologically trivial, the profiles admit a metric with positive curvature operator, and $\lim_{r \to 0} rH$ does not exist, where $H$ is the mean curvature of $V$.

Thus, for simple singularities, the inequality $\sup_{U} r^2 \text{Ric}_T \geq -1$ holds always in odd-dimensional hypersurfaces and in all dimensions when the mean curvature is bounded in a neighborhood of the singularity; in particular, it always holds for minimal or constant mean curvature hypersurfaces.

The inequality is sharp in all dimensions $m \neq 4$.

The inequality of the theorem may be universal and no exceptions are known, so far. However, if $\sup_{U} r^2 \text{Ric}_T < -1$ for a simple singularity then the profiles are odd-dimensional topological spheres; if, further, the hypersurface is embedded the profiles are differentiable spheres by the h-cobordism theorem [9]; the possibility that exotic spheres might occur as profiles is engaging since, by Theorem 1, they would carry canonically constructed Riemannian metrics with positive curvature operators (See Remark 3, §4) and not even a metric of positive sectional curvature is yet known to exist on an exotic sphere.

The inequality of the theorem holds when the mean curvature is bounded in a neighborhood of a simple singularity; even in the best studied case of minimal cones (see [11], for example) the inequality is new.

The inequality of Theorem 1 is sharp $m \neq 4$. In all dimensions $m \neq 4$ there are cones which are not topologically trivial with $r^2 \text{Ric}_T \equiv -1$. When $m > 4$ and we choose integers $p, q \geq 2$ with $p + q = m - 1$, the Clifford cone $X^m$ defined by

$$(q - 1)(x_1^2 + \cdots + x_{p+1}^2) - (p - 1)(y_1^2 + \cdots + y_{q+1}^2) = 0$$

is then such an example with profiles $S^p \times S^q$ (This is the cone over the spherical hypersurface occurring in the remark following Theorem 2). When $m = 3$ the cone

$$x_1^2 + x_2^2 - y_1^2 - y_2^2 = 0$$

is such an example with flat tori as profiles. In dimension $m = 4$ there are no cones satisfying the Ricci identity $r^2 \text{Ric}_T \equiv -1$ and the bound in the theorem may not
be optimal in this dimension. In the case $m = 2$, which we will not be further concerned with here, the singularities are automatically topologically trivial.

The paper is organized as follows: §2 relates the geometry of the hypersurface to that of its profiles near an isolated singularity; in §3 the singularity is assumed simple and we can then effectively compare the Ricci curvatures of the hypersurface with those of its profiles; in §4 we state and prove Theorem 2 on the Ricci curvatures of compact hypersurfaces of the unit sphere; this depends crucially on an inequality of Chen [3] and observations on the connectivity of the principal curvature set for hypersurfaces in the sphere (i.e., the set of real numbers occurring as eigenvalues of the second fundamental form throughout the hypersurface). Theorem 1 follows from Theorem 2 by the remarks at the close of §3. Towards the end of §4 we show that the method can be easily adapted to complete hypersurfaces in the unit sphere to derive the correct form of Efimov's Theorem in the sphere. This is stated and proved in the remarks at the end of §4.

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§2. Calculations near a singularity

Let $V$ be a closed subset of $\mathbb{R}^{m+1}$ containing the origin $o$ such that $(V - \{o\}) \cap B(2\varepsilon)$ is the image of a smooth $m$-manifold $N^m$ under a proper smooth immersion $x : N^m \to \mathbb{R}^{m+1}$; the open ball and sphere of radius $r$ about $o$ in $\mathbb{R}^{m+1}$ are denoted $B(r)$ and $S^m(r)$, respectively. We call $V$ a smooth hypersurface with an isolated singularity at $o$. The considerations which follow may be applied on each component of $N$ so that we may assume $N$ connected. Nothing is lost in assuming $N$ orientable and selecting a unit normal field $\eta$ along the immersion $x$. The second fundamental form of the immersion $x$ is then defined by $X\eta = -x_*(AX)$ for all vectors $X$ tangent to $N$. The metric induced on $N$ from the euclidean metric $<,>$ on $\mathbb{R}^{m+1}$ will also be denoted by the same symbol and its connexion by $\nabla$.

The distance $r(p) = |x(p)|$ may be considered as a smooth function on $N$ and the oriented angle $\phi(p)$ that $x(p)$ makes with the hyperplane $x_*(N_p)$ is another smooth function on $N$ with values in $[-\pi, \pi]$. The decomposition

$$x = x_*(x^T) + <x, \eta> \eta,$$

defines a vector field $x^T$ on $N$, called the tangential component of $x$. Denoting the gradient of a function $f$ in the induced metric by $\nabla f$, we see that $x^T = r\nabla r$. Thus
the above equation may be rewritten, in terms of \( r \) and \( \phi \), as

\[ x = r\{x_*(\nabla r) + \sin \phi \eta\}. \]

From now on we assume transversality of the immersion \( x \) to the spheres \( S^m(r) \), for small \( r \). This is equivalent to saying that \( \nabla r \) is nonvanishing (or \( \phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \)) for such \( r \); this is guaranteed by the first property of the singularity. Writing \( e = \frac{\nabla r}{|\nabla r|} \) the above equation may be rewritten

\[ x = r\{|\nabla r| x_*(e) + \sin \phi \eta\}, \]

from which we conclude that \( \nabla r = \cos \phi e \) and

\[(1) \quad x = r\{\cos \phi x_*(e) + \sin \phi \eta\}. \]

Then the level sets

\[ M^{m-1}(r) = \{ p \in N | |x(p)| = r \} \]

are a compact oriented embedded hypersurfaces in \( N \). By transversality and connectedness of \( N \), these hypersurfaces are connected for sufficiently small \( r \). The unit vector field \( e = \frac{\nabla r}{|\nabla r|} \) on \( N \) is orthogonal to this family of hypersurfaces in \( N \) and the one-parameter family of local diffeomorphisms generated by the vector field \( E = \frac{e}{|\nabla r|} \) determines a diffeomorphism between any two hypersurfaces in the family; note that the derivative of \( r \) in the direction \( E \) is \( E(r) \equiv -1 \). Writing \( M^{m-1} = M^{m-1}(r) \), for \( r \) sufficiently small, and keeping in mind the identifications set up by the flow of \( E \), we obtain a family of profile immersions

\[ x_r = x_{|M^{m-1}(r)} : M^{m-1} \longrightarrow S^m(r). \]

for each small \( r \). If \( \xi_r \) denotes the oriented unit normal field to the immersion \( x_r \) in \( S^m(r) \) and \( B_r \) its second fundamental form then the corresponding normalized profile immersion

\[ x_r^0 = \frac{1}{r} x_r : M^{m-1} \longrightarrow S^m(1) \]

has the same unit normal field \( \xi_r \) and second fundamental form \( B_r^0 = r B_r \).

The distribution \( T = e^\perp \) on \( N^m \) coincides with the tangent space to \( M^{m-1}(r) \) at each point of \( N \) and its elements will be called transverse vectors in \( N \).

The normal field to the profile hypersurfaces

\[ x_r = x_{|M^{m-1}(r)} : M^{m-1} \longrightarrow S^m(r). \]
is given by

\[ (2) \quad \xi = \cos \phi \eta - \sin \phi \, x_\ast(e). \]

Differentiating (1) with respect to any vector \( X \) tangent to \( N \) and taking \( \eta_r \) components yields the equation

\[ (3) \quad \nabla \phi = -A e - \frac{\sin \phi}{r} e, \]

and the tangential components along \( N \) give

\[ (4) \quad \frac{1}{r} X_T = -\sin \phi \, (AX)_T + \cos \phi \, \nabla_X e, \]

where transverse components are denoted by the subscript \( T \). Taking \( X = e \) in this equation we have

\[ (5) \quad \nabla_e e = \tan \phi \, (A e)_T. \]

If now \( X \) is transverse a similar calculation on (2) gives

\[ (6) \quad B_r X = \cos \phi \, (AX)_T + \sin \phi \, \nabla_X e. \]

Using (3) and eliminating between (4) and (6) for transverse vectors \( X \) we have

\[ (7) \quad \cos \phi \, B_r X = (AX)_T + \frac{\sin \phi}{r} X, \]

\[ = AX + X(\phi)e + \frac{\sin \phi}{r} X \]

and similarly

\[ (8) \quad \sin \phi \, B_r X = -\frac{\cos \phi}{r} X + \nabla_X e. \]

§3. The Ricci curvature near an isolated singularity

Throughout this section we will assume that \( N \) meets the spheres \( S^m(r) \) transversally, for small \( r \). By Gauss’ equation, the Ricci curvature of \( N \) in the direction of any unit tangent vector \( X \) is

\[ (9) \quad S(X, X) = \langle [(Tr A) A - A^2] X, X \rangle, \]
where $Tr$ denotes trace. From now on $X$ will be a unit transverse vector at a
distance $r$ from $o$ (i.e. tangent to $M^{m-1}(r)$). Then, by (7),

\begin{equation}
(10) \quad rAX = r \cos \phi \, B_r X - \sin \phi \, X - rX(\phi)e,
\end{equation}

where $B_r$ is the second fundamental form of the immersion $x_r$. By (3) and (10) it
follows that

\begin{equation}
(11) \quad Tr (rA) = r \cos \phi \, Tr \, B_r - m \sin \phi - re(\phi),
\end{equation}

and

\begin{equation}
(12) \quad <rAX, X> = r \cos \phi <B_rX, X> - \sin \phi.
\end{equation}

Furthermore, from (10),

\begin{equation}
(13) \quad <r^2A^2X, X> = <rAX, rAX>
= r^2 \cos^2 \phi <[TrB_r]B_r - B_r^2]X, X>
- r \cos \phi \{(m - 2) \sin \phi + re(\phi)\} <B_rX, X>
- r \sin \phi \cos \phi TrB_r
- r^2X(\phi)^2 + re(\phi) \sin \phi + (m - 1) \sin^2 \phi.
\end{equation}

It now follows from (9)-(13) that for any transverse unit vector $X$ at distance $r$
from $o$ we have

\begin{equation}
(14) \quad r^2S(X, X) = r^2 \cos^2 \phi <[(TrB_r^0)B_r - (B_r^0)^2]X_r^0, X_r^0> - \sin \phi \cos \phi TrB_r^0
- \{(m - 2) \sin \phi + re(\phi)\} \cos \phi g_r^0(B_r^0 X_r^0, X_r^0)
- r^2X(\phi)^2 + re(\phi) \sin \phi + (m - 1) \sin^2 \phi.
\end{equation}

Because of the scale change in replacing the $r$-profile immersion $x_r$ by its
ormalization $x_r^0 = \frac{1}{r}x_r$, the vector $X_r^0 = r \ X$ is a unit vector with respect to the
$x_r^0$-induced metric $g_r^0 = \frac{1}{r^2} <,>$ and the second fundamental forms are related by
$B_r^0 = rB_r$. Thus

\begin{equation}
(15) \quad r^2S(X, X) = \cos^2 \phi g_r^0 [[(TrB_r^0)B_r^0 - (B_r^0)^2] X_r^0, X_r^0] - \sin \phi \cos \phi TrB_r^0
- \{(m - 2) \sin \phi + re(\phi)\} \cos \phi g_r^0(B_r^0 X_r^0, X_r^0)
- r^2X(\phi)^2 + re(\phi) \sin \phi + (m - 1) \sin^2 \phi.
\end{equation}

From Gauss’ equation for hypersurfaces in the unit sphere, the metric expression
in the first term on the right-hand side is $S_r^0(X_r^0, X_r^0) - (m - 2)$ where $S_r^0$ is the
Ricci tensor of the metric $g_r^0$. After a re-arrangement of terms, (15) takes the form

\begin{equation}
(16) \quad r^2S(X, X) + 1 = \cos^2 \phi \{(S_r^0(X_r^0, X_r^0) - (m - 3)\} - \sin \phi \cos \phi TrB_r^0
- \{(m - 2) \sin \phi + re(\phi)\} g_r^0(B_r^0 X_r^0, X_r^0)
- r^2X(\phi)^2 + re(\phi) \sin \phi + m \sin^2 \phi.
\end{equation}
Recalling the definition of a simple singularity in the introduction we note from (3) that, in the presence of (a), condition (b) means $rAc \to 0$ as $r \to 0$. In the presence of (a) and (b), condition (c) is equivalent to $B_r^0 = rB_r$ being bounded independent of $r$, by (10). From now on the singularity is assumed simple and then, from (10), the operators $B_r^0$ have a bound independent of $r$.

Assume that the curvature inequality of Theorem 1 is violated. Then there exists $k > 0$ such that for all unit transverse vectors $X$ tangent to $N$ (in some neighborhood of $o$) the Ricci curvature of $N$ satisfies

$$r^2S(X, X) + 1 < -2k^2.$$  

Then it follows from (16) and the fact that the singularity is simple that, for all $r$ sufficiently small, the normalized profile $x_r^0 : M^{m-1} \to S^m(1)$ has Ricci curvature satisfying

$$(*) \quad Ric < (m - 3) - k^2.$$ 

Thus, by (16), for a simple hypersurface singularity violating the curvature inequality of Theorem 1, the normalized profiles $x_r^0$ in $S^m(1)$ satisfy the inequality $(*)$ for $r$ sufficiently small. Hypersurfaces of the sphere satisfying this condition are the subject of Theorem 2 in §4 and Theorem 1 will follow from Theorem 2 and the following remark on writing $n = m - 1 \geq 2$.

**Remark 1:** Note from (11) that

$$\cos \phi \ Tr B_r^0 = r Tr A + r e(\phi) + m \sin \phi.$$ 

Thus if the singularity is simple and $\lim_{r \to 0} r Tr A = l$ exists — in particular, if $H = Tr A$ is bounded — then the normalized profile $x_r^0$ has mean curvature arbitrarily close to the constant $l$, for $r$ sufficiently small. However, when the inequality of Theorem 1 is violated, we will see in §4 that the normalized profiles have mean curvature of total variation bounded away from zero, with a bound independent of $r$; hence, when the inequality of Theorem 1 is violated, $\lim_{r \to 0} r H$ does not exist, where $H$ is the mean curvature function of the hypersurface $N$.

**§4. A Ricci inequality for hypersurfaces in the sphere.**

In arriving at the extension of Efimov’s Theorem to complete hypersurfaces of euclidean space, Smyth-Xavier [12] introduced the principal curvature set of the hypersurface, i.e., the set of real numbers taken as values of the principal
curvatures. It was their observations on the behaviour of the principal curvature set under parallel deformations together with the Hadamard-Sacksteder-van Heijenoort classification theorem (see Wu [14]) for complete hypersurfaces with non-negative sectional curvature which provided the essential ingredients of this extension.

Let \( f : M^n \rightarrow S^{n+1}(1) \) be a smooth immersion of a smooth compact orientable \( n \)-dimensional manifold \( M^n \) into the unit sphere, \( n \geq 2 \). If \( \xi \) is a unit normal field to \( M^n \) in \( S^{n+1}(1) \) along this immersion and \( B \) is its second fundamental form then

\[
\Lambda = \{ \lambda \in \mathbb{R} | \lambda \text{ is an eigenvalue of } B(p) \text{ for some } p \in M \}
\]

is the principal curvature set of the immersion \( f \).

We will show \( \sup_M Ric \geq n - 2 \) for any compact hypersurface of the sphere — with the exception of at most some odd-dimensional topological spheres — and this inequality is optimal in that there are non-spherical hypersurfaces of every dimension \( n \neq 3 \) with \( Ric \equiv n - 2 \). Such exceptional topological spheres as occur must admit Riemannian metrics of positive sectional curvature, as we will see below, and this construction is natural.

For minimal and constant mean curvature hypersurfaces the exception does not occur and even here the bound seems new.

Our proof involves considerations on the connectivity of \( \Lambda \). The final lemma shows it is connected for convex hypersurfaces in spheres (that is, hypersurfaces with sectional curvature \( K \geq 1 \) in the unit sphere or, equivalently, those hypersurfaces which lie to one side of each tangent great hypersphere [2]). In the course of our proof of Theorem 2, below, we are led to the consideration of hypersurfaces in spheres satisfying the weaker condition of positive sectional curvature, i.e. \( K > 0 \). It must be emphasised that there is no characterization - neither as to diffeomorphism nor extrinsic geometry - of complete (or even compact) positive sectional curvature hypersurfaces in spheres, as there is in euclidean space (that is, the theorem of Hadamard-Sacksteder-van Heijenoort [14]). But, for our purposes here, this lack is a little compensated for by an inequality of Chen [3] on the total curvatures of compact submanifolds \( M^n \) of \( \mathbb{R}^{n+2} \) with \( K > 0 \) or equivalently, with positive curvature operator, by Weinstein’s [13] codimension two result.

**Theorem 2.** Let \( f : M^n \rightarrow S^{n+1}(1) \) be a smooth immersion of a smooth compact orientable \( n \)-manifold in the unit sphere \( S^{n+1}(1) \), \( n \geq 2 \). Then either

\[
\sup_M Ric \geq n - 2
\]
or else each of the following must hold:

(i) $n$ is odd,

(ii) $M^n$ is a topological sphere,

(iii) the Gauss map immerses $M^n$ with positive curvature operator in $S^{n+1}(1)$ with image in no closed hemisphere and disconnected principal curvature set, and

(iv) the total variation of the mean curvature of the immersion exceeds $4c \log c$, where $c^2 = (n - 1) - \sup_{M} \operatorname{Ric} > 1$.

In particular, by (iv), $\sup_{M} \operatorname{Ric} \geq n - 2$ for any compact minimal or constant mean curvature hypersurface in $S^{n+1}(1)$.

**Remark 2:** For each $n \neq 3$ the inequality is sharp. If $p + q = n, p \geq 2, q \geq 2$ then $S^p(\sqrt{\frac{p-1}{n-2}}) \times S^q(\sqrt{\frac{q-1}{n-2}})$ in $S^{n+1}(1)$ has Ricci curvature $\equiv n - 2$; for $n = 2$ the flat torus $S^1(\sqrt{\frac{1}{2}}) \times S^1(\sqrt{\frac{1}{2}})$ in $S^3(1)$ provides an example.

**Proof.** From Gauss’ equation the Ricci curvatures of the induced metric on $M$ are the values of the quadratic form defined by $(\text{Tr} B)B - B^2 + (n - 1)I$ on unit vectors tangent to $M$.

Let us assume

\[(**) \quad \sup_{M} \operatorname{Ric} < n - 2\]

and write $c^2 = (n - 1) - \sup_{M} \operatorname{Ric} > 1$. It follows that at each point $x$ of $M$ the eigenvalues of $B$ lie outside the interval $\left(\frac{H - \sqrt{H^2 + 4c^2}}{2}, \frac{H + \sqrt{H^2 + 4c^2}}{2}\right)$ where $H(x) = \text{Tr } B(x)$ is the mean curvature function evaluated at $x$. Clearly $0 \notin \Lambda$ and, by compactness of $M$, the set $\Lambda$ is closed and so must miss a neighborhood of $o$ in $\mathbb{R}$.

If $\Lambda$ misses $\mathbb{R}^+$ or $\mathbb{R}^-$ all principal curvatures are of one sign and it follows easily from Gauss’ equation that $\operatorname{Ric} > n - 1$, in contradiction to our assumption; hence $\Lambda$ has at least two components, separated by $o$. This is the essential information carried by the Ricci curvature hypothesis (**).

The normal variation $f_t = \cos t \ f + \sin t \ \xi$ of the immersion $f$ is an immersion of $M$ in $S^{n+1}(1)$ when $k = \cot t \notin \Lambda$ and its induced metric is given by the expression $g_t(X, Y) = \sin^2 t < (kI - B)X, (kI - B)Y >$. Its second fundamental form with respect to the unit normal field $\xi_t = \cos t \ \xi - \sin t \ f$ is

\[B_t = \frac{kB + I}{kI - B}\]

where quotient notation is used for inverses. The principal curvature set $\Lambda_t$ of the immersion $f_t$ of $M$ in $S^{n+1}(1)$ is then obtained by applying the obvious Moebius transformation to $\Lambda$, and so has the same connectivity as $\Lambda$. 

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Now the Ricci curvature hypothesis (**) implies that \( \hat{f} = f_\perp \) is an immersion with second fundamental form \( \hat{B} = -B^{-1} \) and therefore the same principal directions as the immersion \( f \). The metric \( \hat{g} \) induced by \( \hat{f} \) has sectional curvature 
\[
1 + \frac{1}{\lambda_i \lambda_j} \quad \text{on the plane spanned by any pair of principal directions } \{ e_i, e_j \} \text{ of the immersion } f,
\]
where \( B e_i = \lambda_i e_i \). When \( \lambda_i \) and \( \lambda_j \) are of the same sign, this sectional curvature is \( > 1 \). When \( \lambda_i \) and \( \lambda_j \) have opposite signs, they lie on either side of the interval \( \left( \frac{-\sqrt{H^2 + 4c^2}}{2}, \frac{\sqrt{H^2 + 4c^2}}{2} \right) \), so that \( \lambda_i \lambda_j \leq -c^2 \) and therefore 
\[
1 + \frac{1}{\lambda_i \lambda_j} \geq 1 - \frac{1}{c^2} > 0.
\]
Hence the curvature operator of the metric \( \hat{g} \) is strictly positive definite and since \( \hat{f} = \xi \) immerses \( M \) in \( \mathbb{R}^{n+2} \) (i.e., codimension 2), by Weinstein [13], this is equivalent to the positivity of all sectional curvatures.

**Lemma 1.** *If the restricted holonomy group of a connected piece of hypersurface \( M \) in \( S^{n+1} \) is not \( SO(n) \) then the hypersurface is a piece of a product of a pair of round hyperspheres lying in complementary orthogonal subspaces and its holonomy splits as a product of orthogonal groups.*

**Proof.** To see this we first note that the holonomy assumption implies that the Lie algebra generated by the curvature transformations \( R_x(X, Y) \) at any point \( x \) of \( M \) is not the full Lie algebra of all skew-symmetric transformations of the tangent space at \( x \). If \( \{ e_1, e_2, \ldots, e_n \} \) is an orthonormal basis of \( M_x \) diagonalising \( B_x \), with \( B_x e_i = \lambda_i e_i \) for each \( i \), then by Gauss’ equation, \( R_{ij} = R(e_i, e_j) = (1 + \lambda_i \lambda_j) E_{ij} \), where \( E_{ij} \) is the skew-symmetric transformation given by 
\[
E_{ij}(e_s) = \delta_{js} e_i - \delta_{is} e_j.
\]

Under the assumption that the curvature transformations do not generate the full Lie algebra of all skew-symmetric transformations of \( M_x \) — and so some \( E_{ij} \) must not occur among the generators — we must have \( 1 + \lambda_i \lambda_j = 0 \) for some distinct pair of principal curvatures. If there is a third \( \lambda_k \) distinct from these, then \( 1 + \lambda_i \lambda_k \neq 0 \) and \( 1 + \lambda_j \lambda_k \neq 0 \) and it is easily computed that 
\[
[R_{ik}, R_{jk}] = -(1 + \lambda_i \lambda_k)(1 + \lambda_j \lambda_k) E_{ij},
\]
so that \( E_{ij} \) is in the holonomy algebra; this contradiction shows that there are precisely two distinct principal curvatures \( \lambda > 0 \) and \( \mu < 0 \). Thus the above holonomy assumption means 
\[
B_x = \lambda I_p + \mu I_q
\]
relative to some orthogonal decomposition of the tangent space into a pair of orthogonal subspaces of dimensions \( p \) and \( q \) with \( p + q = n \). The integers \( p \) and \( q \) are
independent of $x$ and, by Codazzi’s equation for $B$, $\lambda$ and $\mu$ are independent of $x$ and nonzero. It is now easy to see that $M$ is a piece of the standard Clifford product of round spheres $S^p(\frac{1}{\lambda}) \times S^q(\frac{1}{\mu})$ in $\mathbb{R}^{n+2}$ and so has holonomy $SO(p) \times SO(q)$.

Returning now to our hypersurface $\hat{f}$ of the earlier paragraph we see that its holonomy must be $SO(n)$ since it has strictly positive sectional curvature. Since the curvature operator is strictly positive definite, as noted earlier, and the holonomy is $SO(n)$, Meyer’s theorem [8] guarantees that the universal cover of $M$ is a rational homology sphere.

For a compact submanifold $M^n$ immersed in $\mathbb{R}^{n+p}$ the height function corresponding to $a \in S^{n+p-1}(1)$ is a Morse function for almost all $a$ and we denote by $\mu_k(a)$ the number of nondegenerate critical points of index $k$ of each such Morse function. The total curvature of index $k$, first introduced by Kuiper [7], is given by

$$\tau_k = \frac{1}{\text{Vol}S^{n+p-1}(1)} \int_{S^{n+p-1}(1)} \mu_k(a).$$

When $M^n$ has positive sectional curvature (in the induced metric) and codimension $p = 2$ (as for the immersion $\hat{f} = \xi$ in the last paragraph) Chen [3] established the inequality

$$\tau_1 + \tau_2 + \cdots + \tau_{n-1} < \tau_0 + \tau_n$$

among the total curvatures. Moore [10] realised the importance of Chen’s inequality and in conjunction fact that the universal cover of $M$ is a rational homology sphere used it to show that $M^n$ itself must be a homotopy sphere in dimensions $n \geq 5$ and so a topological sphere for $n \neq 3$ and 4. But in these latter dimensions Baldini-Mercalli [1] produce Morse functions with only two critical points so that $M^n$ is a topological sphere in all dimensions, and so (ii) holds.

However, the Ricci curvature hypothesis (***) means $\Lambda$ is not connected at 0, and so it follows that the tangent bundle of $M$ splits into the subbundles on which $B$ is positive definite and negative definite. Since such a splitting is impossible for an even dimensional sphere, (i) is proved.

If a compact immersed hypersurface lies in the closed hemisphere with pole $a \in S^{n+1}(1)$, then computing the Hessian operator of the function $< f, a >$ on $M$ we see that its second fundamental form is semi-definite at points where this height function has an absolute minimum. Recalling that the second fundamental forms of the immersions $f$ and $\hat{f} = \xi$ are, respectively, $B$ and $-B^{-1}$ — and therefore both have principal curvature sets split by 0 — neither of the images $f(M)$ or $\xi(M)$
can be contained in a closed hemisphere. Together with the earlier remarks on the sectional curvature of the immersion $\hat{f} = \xi$ this completes (iii).

Let the range of the function $H$ on $M$ be $[\alpha, \beta]$. From the Ricci curvature hypothesis (**), we see that $\Lambda \cap \left( \frac{-\alpha^2}{\psi(\alpha)}, \psi(\alpha) \right) = \emptyset$, where $\psi$ is the function defined by $\psi(x) = x + \sqrt{x^2 + 4c^2}$. It is easily seen that $\frac{\psi(\beta)}{\psi(\alpha)} < e^{\frac{\beta - \alpha}{\psi(\alpha)}}$ for all $\alpha$ and $\beta$. Assume $\beta - \alpha \leq 4c \log c$. Then it follows that $\frac{\psi(\beta)}{\psi(\alpha)} < c^2$ and from the previous property of $\Lambda$ there exists $k > 0$ such that $\left[ -\frac{1}{k}, k \right]$ does not meet $\Lambda$. Now choose $t > 0$ such that $\cot t = k$ and it follows that $f_t$ is an immersion with $\Lambda_t \subset \mathbb{R}^-$, i.e., $f_t$ has strictly negative definite second fundamental form. By do Carmo and Barbosa [2], $f_t$ immerses $M$ as a strictly convex hypersurface in $S^{n+1}(1)$, i.e., as the boundary of a compact convex set contained in an open hemisphere of $S^{n+1}(1)$. Since $\Lambda$ — and therefore $\Lambda_t$ — is known to be disconnected from (**), as we saw earlier, we obtain a contradiction to $\beta - \alpha \leq 4c \log c$ once we prove the following lemma and (iv) is then proved.

**Lemma 2.** A complete convex hypersurface of $S^{n+1}(1)$ has connected principal curvature set.

**Proof.** Let $F : M^n \longrightarrow S^{n+1}(1)$ be a complete convex hypersurface. Its second fundamental form $B$, with respect to a global unit normal field $\xi$, may be assumed positive semi-definite. From Gauss’ equation the Ricci curvature is positive and bounded away from zero, so that $M$ is compact by Myer’s theorem. Assuming the principal curvature set $\Lambda$ is not connected, we may choose $t > 0$ such that $k = \cot t \notin \Lambda$ and splits $\Lambda$. It is now easy to see that the immersion $F_t$ has a nowhere singular and nowhere definite second fundamental form $B_t$. Since $F$ is convex we may choose a unit vector $a \in \mathbb{R}^{n+2}$ such that the height function $h = < F, a >$ is Morse and has only two critical points. The Hessian operator

$$\text{Hess}_h : X \longrightarrow \nabla_X (\nabla h),$$

where $\nabla h$ is the gradient of $h$ computed with respect to the induced metric $g$ on $M$, is easily found to be

$$\text{Hess}_h = < \xi, a > B - < F, a > I.$$  

From the relations between $F_t, \xi_t$ and $B_t$ and $F, \xi$ and $B$ we easily calculate $\text{Hess}_h$, (where $h_t = < F_t, a >$) with respect to the $F_t$ - induced metric as

$$\text{Hess}_{h_t} = (k^2 + 1) \sin t \cdot \frac{\text{Hess}_h}{(kI - B)}$$

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Now $h_t$ has the same critical set as $h$ and by the previous equation neither of these two points is a maximum or minimum of $h_t$ since $k$ splits $\Lambda$ and, in consequence, $kI - B$ is nowhere definite. This contradiction ends the proof of the lemma, and so, of Theorem 2 also.

**Remark 3:** As remarked in the introduction, when a simple singularity violates the bound in Theorem 1 the profiles admit a metric of positive curvature operator which can be constructed in a canonical way. Indeed, replacing the original immersion $x$ of $N$ by the map $y = r\xi = r\{\cos \phi \eta - \sin \phi \phi_*(e)\}$, one can verify that this is an immersion of $N$ with a simple singularity at $o$ and that the profile hypersurface $M^{n-1}(r)$ in $N$ at distance $r$ from $o$ is the same for both immersions, but the metric induced by

$$y_r = y|_{M^{n-1}(r)}: M^{n-1}(r) \rightarrow S^n(r).$$

on $M^{n-1}(r)$ has positive curvature operator.

A Efimov theorem for complete hypersurfaces of a sphere is now easily developed from the argument of Theorem 2, above.

*Let $f : M^n \rightarrow S^{n+1}(1)$ be a smooth immersion of a smooth complete orientable $n$-manifold into the unit sphere $S^{n+1}(1)$ ($n \geq 3$) with sectional curvature bounded away from $-\infty$. Then either $\sup_M \text{Ric} \geq n-2$ or else $M^n$ is homeomorphic to an odd-dimensional sphere or diffeomorphic to $\mathbb{R}^n$.*

The proof is as follows. Assume that the Ricci curvature of the induced metric fails to satisfy the inequality in the statement. Then $\epsilon^2 = (n-1) - \sup_M \text{Ric} > 1$.

The compact case is already taken care of by Theorem 2 and, if $H$ is bounded, the proof in Theorem 2 works verbatim to show that $M$ must be an odd-dimensional topological sphere. Thus we will be assuming that $H$ is unbounded from here on.

By Gauss’ equation the Ricci condition amounts to saying that at each point $x$ of $M$ the eigenvalues of the second fundamental form $B$ lie outside the interval $\left(\frac{H-\sqrt{\frac{H^2+4\epsilon^2}{2}}, \frac{H+\sqrt{\frac{H^2+4\epsilon^2}{2}}}{2}\right)$ where $H = \text{Tr} B$ is the mean curvature function. In particular $0 \not\in \Lambda$ and the numbers $r$ and $s$ of positive and negative principal curvatures, which satisfy $r + s = n$, are independent of $x$ and neither is zero. Note that since the sectional curvature $K$ is bounded below, the assumption $\sup_M \text{Ric} < n-2$ on the Ricci curvature implies that $K$ must be bounded above as well.

If $\sup H = \infty$ then $r = 1$, since otherwise there is a sequence of points along which sectional curvatures approach $+\infty$; similarly if $\inf H = -\infty$ then $s = 1$. 

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Since \( n \geq 3 \), \( H \) cannot take all real values. Thus, by the preceding remarks, after the appropriate choice of unit normal we may assume \( H \) bounded below, and \( \sup H = +\infty \), so that \( r = 1 \) and \( s \geq 2 \).

It follows easily, from these last two conditions, that \( \inf \Lambda^+ = b > 0 \) and \( \sup \Lambda^+ = +\infty \). Then \( \sup \Lambda^- = 0 \), since otherwise — using Gauss’ equation — we would have \( \inf K = -\infty \). Furthermore \( \inf \Lambda^- = -a \), where \( a \) is finite and positive, since again \( \inf K = -\infty \) otherwise.

Now choosing \( t \) so that \( 0 < k = \cot t < b \) it is easily checked that \( f_t : M^n \to S^{n+1}(1) \) has induced metric \( g_t \geq \sin^2 t (b-k)^2 g \) and so is complete. Furthermore the sectional curvatures of the metric \( g_t \) on the principal planes \( \{e_i, e_j\} \) of the original immersion are all positive. Indeed the sectional curvature of this principal plane in the metric \( g_t \) is \( \frac{(1+k^2)(1+\lambda_i \lambda_j)}{(k-\lambda_i)(k-\lambda_j)} \) and since \( 0 < k = \cot t < b \) this is positive when \( \lambda_i \) and \( \lambda_j \) have the same sign; it is likewise positive when \( \lambda_i \) and \( \lambda_j \) have opposite signs since then \( 1 + \lambda_i \lambda_j < 1 - c^2 < 0 \) by the Ricci condition. Thus \( g_t \) is both complete and has positive definite curvature operator, when \( 0 < k = \cot t < b \). By the theorem of Gromoll-Meyer [5], \( M \) must be diffeomorphic to \( \mathbb{R}^n \).

**Remark 4:** The extension of the above result to \( n = 2 \), that is, whether the Gaussian curvature of a complete surface in the three sphere must satisfy \( \sup K \geq 0 \), remains a question.
REFERENCES