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THE ROLE OF EXOTIC AFFINE SPACES IN THE CLASSIFICATION OF HOMOGENEOUS AFFINE VARIETIES

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The Role Of Exotic Affine Spaces In the Classification Of Homogeneous Affine Varieties

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Let G be a connected linear algebraic group over $\mathbb{C}$ and let H a closed algebraic subgroup. A fundamental problem in the study of homogeneous spaces is to describe, characterize, or classify those quotients G/H that are affine varieties. While cohomological characterizations of affine G/H are possible, there is still no general group-theoretic conditions that imply G/H is affine. In this article, we survey some of the known results about this problem and suggest a way of classifying affine G/H by means of its internal geometric structure as a fiber bundle.

Cohomological characterizations of affine G/H provide useful vanishing theorems and related information if one already knows G/H is affine. Such characterizations cannot be realistically applied to prove that a given homogeneous space G/H is affine. Ideally, one would like to have easily verified group-theoretic conditions on G and H that imply G/H is affine. Very few positive results are known in this direction, the most notable of which is Matsushima's Theorem for reductive groups. For general linear algebraic groups there is a natural generalization of Matsushima's Theorem that provides a necessary condition for G/H to be affine. While this criterion is also sufficient for some special situations, it is not sufficient in general.

In the absence of general group-theoretic conditions for G/H to be affine, it is worthwhile to understand the underlying geometric structure of an affine homogeneous space G/H. Such a space is always isomorphic to a fiber bundle over an orbit of a maximal reductive subgroup of G. The fiber is a smooth affine variety diffeomorphic to an affine space $\mathbb{C}^n$. Here several interesting phenomena seem possible: either the fiber is truly an "exotic" affine space or is in fact isomorphic to $\mathbb{C}^n$. If exotic structures occur, they would also provide counter-examples to the Cancellation Problem for affine spaces. So far, no such exotic examples are known. If such structures are impossible, then an affine homogeneous space G/H would always have the simple description of

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a homogeneous vector bundle. In this case, one can change groups, $G/H = \hat{G}/\hat{H}$, where $\hat{G}$ and $\hat{H}$ are easily classified, giving an indirect group-theoretic characterization for G/H to be affine.

1 Cohomological Characterizations

Recall that a subgroup $H \subset G$ is called *observable* if every finite dimensional rational H-module can be embedded as an H-submodule of a finite dimensional rational G-module. This is equivalent to the condition that for any rational H-module V, the induced module $V\Big|^G = \{s : G \to V \mid s(gh^{-1}) = h \cdot s(g), \forall h \in H, \forall g \in G\}$ surjects onto V under the evaluation map $s \to s(1)$. It is well-known that H is observable in G if and only if G/H is quasi-affine [2]. The subgroup H is call *strongly observable* if, given any rational H module V, V is an H-submodule of a rational G-module W such that $V^H = W^G$. Finally, H is called an *exact* subgroup of G if induction from rational H-modules to rational G-modules preserves short exact sequences.

Theorem 1. [4, 13] *The following are equivalent:*

1. *G/H is affine.*
2. *H is a strongly observable subgroup of G.*
3. *H is an exact subgroup of G.*
4. *$H^1(R_u(H), \mathcal{O}(G)) = 0$ (or, equivalently, $H^1(G/R_u(H), \mathcal{O}) = 0$) where $R_u(H)$ is the unipotent radical of H.*

Such characterizations of affineness are basically "cohomological" in nature. They are primarily used when one already knows that G/H is affine. Verifying the properties themselves may be more difficult than directly proving that G/H is affine.

2 Group-theoretic Conditions

There is a practical need for easily verified conditions on the groups G and H that guarantee the quotient G/H is affine. We shall now investigate some of the known results in this direction.

2.1 Unipotent and Solvable Groups

If G is a unipotent linear algebraic group, then $G/H \cong \mathbb{C}^n$ for any algebraic subgroup H. More generally, if G is a solvable linear algebraic group, then $G/H \cong \mathbb{C}^n \times (\mathbb{C}^*)^m$. The corresponding statements for complex Lie groups are not automatically true. For example if $G = \mathbb{C}^* \times \mathbb{C}^*$ and $H = \{(e^z, e^{iz}) \mid z \in \mathbb{C}\}$, then G/H is a compact complex torus. Nevertheless, some generalizations are possible, see [13].

2.2 Reductive Groups

After these relatively simple cases, the best known group-theoretic criterion for G/H to be affine goes back to Matsushima [8]:

If G is reductive then G/H is affine if and only if H is reductive.

Matsushima's original theorem assumes G is a reductive complex Lie group and characterizes when G/H is Stein. However, a reductive complex Lie group G is in fact biholomorphically isomorphic to an algebraic group [6] and G/H is affine if it is Stein [1]. Matsushima's theorem has been generalized to reductive algebraic groups over algebraically closed fields of positive characteristic, see [11, 3].

2.3 General Linear Algebraic Groups

Any connected linear algebraic group G has a decomposition into a semi-direct product, $G = M \cdot R_u(G)$, where M is a maximal reductive subgroup M and $R_u(G)$ is the unipotent radical of G. A closed algebraic subgroup H has a similar decomposition, $H = L \cdot R_u(H)$ where L is a maximal reductive subgroup of H (not necessarily connected). Since the maximal reductive subgroups of G are conjugate, we may assume $L \subset M$. The group L is not important in determining whether G/H is affine:

G/H is affine if and only if $G/R_u(H)$ is affine.

This follows from the fact that L is reductive and $G/R_u(H) \to G/H$ is a principal L-bundle, see [11]. We therefore focus our attention on $R_u(H)$ and its location in G.

If $R_u(H) \subset R_u(G)$, then, of course, $G/R_u(H) \cong M \times R_u(G)/R_u(H)$ is affine and so G/H is affine. However, $R_u(H) \subset R_u(G)$ is not a necessary condition for G/H to be affine. For example, let G be the semi-direct product $\mathrm{SL}(2, \mathbb{C}) \cdot U$ where U is the standard 2-dimensional representation of $\mathrm{SL}(2, \mathbb{C})$, and let

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \times (0, t) \mid t \in \mathbb{C} \right\}.$$

Then $G/H \cong \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}$.

A necessary condition for G/H to be affine is not hard to discover, see [4, 13].

Lemma 1. *If G/H is affine then the intersection of $R_u(H)$ with any reductive subgroup of G is trivial.*

Proof. Let M be a maximal reductive subgroup of G. We must show that $R_u(H) \cap M^g = 1$ for all $g \in G$. Since G/H affine, so is $G/R_u(H)$, and thus any M-orbit of minimal dimension in $G/R_u(H)$, being automatically closed, is affine, see [10]. By Matsushima's Theorem, the isotropy subgroup in M of such an orbit is reductive. However, it also is of the form $M \cap R_u(H)^g$ for some $g \in G$, and, being a subgroup of a the unipotent algebraic group $R_u(H)^g$, is unipotent. Therefore, $M \cap R_u(H)^g = 1$ and any M-orbit of minimal dimension is isomorphic to M. Hence, all M-orbits have the same dimension. Therefore *every* M orbit in $G/G_u(H)$ is isomorphic to M and $M \cap R_u(H)^g = 1$ for all $g \in G$. □

The above lemma also holds when G/H is Stein [13]. Notice that the lemma is a natural generalization of Matsushima's Theorem in one direction: if G is reductive and G/H is affine, then the lemma implies $R_u(H) = 1$ and H is reductive. It is natural to explore whether the converse to Lemma 1 holds. For convenience, let us say that H satisfies *Matsushima's criterion* in G if the intersection of $R_u(H)$ with any reductive subgroup of G is trivial, or equivalently, if $R_u(H) \cap M^g = 1$ for all $g \in G$.

If H satisfies Matsushima's Criterion, then, as we have seen in the proof of the lemma, a maximal reductive subgroup M of G acts *freely* on $G/R_u(H)$. It is easy to see that this condition is equivalent to $R_u(H)$ acting freely on $M\backslash G$ from the right. Since $M\backslash G \cong R_u(G) \cong \mathbb{C}^n$, we are led to considering free actions of unipotent groups on affine space. In fact, in this situation, the actions are also triangular [14]. However, free unipotent triangular actions on $\mathbb{C}^n$ can be badly behaved in general: the quotient may not be separated and may not be affine even if it is separated [15]. Nevertheless, this point of view does yield positive results in certain simple cases.

Theorem 2. [4, 14] *Let G be a linear algebraic group and let H be a closed subgroup. Assume that* $\dim R_u(H) \leq 1$ *or* $\dim R_u(G) \leq 3$. *Then G/H is affine if and only if H satisfies Matsushima's criterion.*

Matsushima's Criterion is not sufficient in general. The following example was discovered by Winklemann [15]: Let $M = \mathrm{SL}(6, \mathbb{C})$ and let

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ z & A \end{bmatrix} \mid z \in \mathbb{C}^6, A \in M \right\} \cong M \times \mathbb{C}^6. \tag{1}$$

Let

$$H = R_u(H) = \left\{ \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & t & s & 1 & & & \\ 0 & 0 & t & 0 & 1 & & \\ t & t & 0 & 0 & 0 & 1 & \\ s + \frac{1}{2}t^2 & \frac{1}{2}t^2 & 0 & 0 & 0 & t & 1 \end{bmatrix} \mid s, t \in \mathbb{C} \right\}. \tag{2}$$

It is not hard to verify that H satisfies Matsushima's Criterion, and that the quotient $M\backslash G/H$ exists and is a smooth contractible quasi-affine variety. However, direct calculation with invariants shows that $X = G/H$ is not affine. In fact, $M\backslash G/H$ is a smooth four-dimensional affine quadric with a 2-codimensional subspace removed.

3 Geometric Description

We now take a closer look at the underlying geometry of an affine homogeneous space G/H. As before, we let M be a maximal reductive subgroup of G and let $L \subset M$ be a maximal reductive subgroup of H. Let $U = R_u(G)$ and $V = R_u(H)$ be the unipotent radicals of G and H, respectively. If G/H is affine, then we know by Lemma 1 that M acts freely on G/V and, equivalently, that V acts freely on $M\backslash G = U$. Since G/V is affine and M is a reductive group acting freely, Hilbert's theorem on invariants implies that the geometric quotient $M\backslash G/V$ is affine. In fact, the local description of $G/V \to Y$ in terms of "slices," shows that the quotient map is a locally trivial principal bundle with fiber M, see [7, 12]. Consider the following diagram

$$\begin{array}{ccc} G & \xrightarrow{V} & G/V \\ M\big\downarrow & & \big\downarrow M \\ U & \xrightarrow{V} & Y \end{array}$$

where the vertical maps are the quotients by M and the horizontal maps are the quotients by V. Using local sections of $G \to Y$, we see that the fibration $U \to Y$ is also a locally trivial principal V-bundle. Thus, Y is the base of a locally trivial fibration where both the total space $U \cong \mathbb{C}^n$ and the fiber $V \cong \mathbb{C}^m$ are affine spaces.

Proposition 1. *If G/H is affine, then G/V is M-equivariantly isomorphic to $M \times Y$ and U is V-equivariantly isomorphic to $Y \times V$.*

Proof. The principal V-bundle $U \to Y$ is topologically trivial because the structure group is contractible, and this immediately implies that the bundle is holomorphically trivial [5]. To see that it is also algebraically trivial, we proceed by induction on $\dim V$. If $\dim V = 1$, then the triangular action of V on U is equivalent to a translation, see [4, 14] which implies $U \cong Y \times V$. If $\dim V > 1$, then there is a normal subgroup $V_1 \subset V$ such that $\dim V/V_1 = 1$. The bundle $G/V_1 \to G/V$ with fiber $V/V_1 \cong \mathbb{C}$ is trivial because $H^1(G/V, \mathcal{O}) = 0$. In particular, G/V_1 is affine, and hence the quotient $Y_1 = M\backslash G/V_1$ exists and is affine. Since $H^1(Y, \mathcal{O}) = 0$, the principal V/V_1-bundle $Y_1 \to Y$ is also trivial. By induction, $U \cong Y_1 \times V_1$, and therefore, $U \cong Y \times V/V_1 \times V_1 \cong Y \times V$. Finally, composing a global section $Y \to U$ with the projection $G \to G/V$ gives a global section $Y \to G/V$, and this implies G/V is M-equivariantly isomorphic to $M \times Y$. □

3.1 Cancellation Problem

The Cancellation Problem is the following: if $\mathbb{C}^n \cong Y \times \mathbb{C}^m$, is $Y \cong \mathbb{C}^{n-m}$? Since $U \cong \mathbb{C}^n$ and $V \cong \mathbb{C}^m$, the isomorphism $U \cong Y \times V$ of Proposition 1 gives an example of the Cancellation Problem. This problem remains unsolved in general, but has a positive answer if $\dim Y \leq 2$ [9].

Obviously, Y is a smooth contractible affine variety. If $\dim Y \geq 3$, then Y is in fact diffeomorphic to $\mathbb{C}^{n-m}$, [16]. If Y is not algebraically isomorphic to $\mathbb{C}^{n-m}$ then Y is called an *exotic* affine space. Exotic affine spaces are known to exist, although no examples are known in the context of the Cancellation Problem, [16]. The relative simplicity of the subgroup H in (2) leads one to believe that it may indeed be possible to create exotic affine spaces of the form $Y = M\backslash G/V = U/V$ which would provide a negative answer to the Cancellation Problem at the same time.

3.2 Homogeneous Bundle Structure

The isomorphisms of Proposition 1 provide a natural bundle structure on an affine homogeneous space G/H.

Theorem 3. *Let L be a maximal reductive subgroup of H and let M be a maximal reductive subgroup of G containing L. If $X = G/H$ is affine then X is isomorphic to a homogeneous bundle over M/L, $X \cong M \times_L Y \to M/L$, with fiber Y a smooth contractible affine variety.*

Proof. The reductive group L acts by conjugation on both $U = R_u(G)$ and $V = R_u(H)$ and these actions are isomorphic to a linear representations. By Proposition 1, the V-equivariant isomorphism $U \cong Y \times V$ yields a V-equivariant map $s : U \to V$ satisfying $s(uv) = s(u)v$ for all $u \in U$, $v \in V$.

If we average s over a maximal compact subgroup K of L,

$$\hat{s}(u) = \int_{k \in K} k^{-1} s(kuk^{-1}) k \, dk, \quad u \in U$$

(where dk is some invariant measure on K), then $\hat{s}$ is still V-equivariant. Moreover, since K is Zariski-dense in L, $\hat{s}(lul^{-1}) = l\hat{s}(u)l^{-1}$ for all $l \in L$, $u \in U$. If we identify Y with the L-invariant subvariety $\hat{s}^{-1}(1) \subset U$, we obtain a natural action of L on Y and the isomorphism $U \cong Y \times V$ is L-equivariant. Moreover, the right L action on G/V preserves the decomposition of Proposition 1, $G/V \cong M \times Y$, so that G/H is isomorphic to the homogeneous bundle $M \times_L Y = M \times Y/\sim$ where $(m, y) \sim (ml^{-1}, l \cdot y)$, for all $m \in M$, $y \in Y$, $l \in L$. □

If the homogeneous bundle of Theorem 3 is a homogeneous vector bundle, then it is possible to "change" the groups G and H so that their maximal reductive subgroups and unipotent radicals are aligned.

Theorem 4. *Let $X = G/H$ be affine and let $X = M \times_L Y \to M/L$ be the homogeneous bundle of Theorem 3. If Y is isomorphic to a linear representation of L, then there exist linear algebraic groups $\hat{G} = M \cdot R_u(\hat{G})$ and $\hat{H} = L \cdot R_u(\hat{H})$ such that $G/H \cong \hat{G}/\hat{H}$ and $R_u(\hat{H}) \subset R_u(\hat{G})$.*

Proof. By Theorem 1, the sections $H^0(M/L, X) \cong Y\big|^M$ generate the bundle. Therefore, there exists a finite dimensional M-submodule $U \subset H^0(M/L, X)$ that spans the vector space fiber Y over the identity coset $z_0 \in M/L$. The semi-direct product $\hat{G} = M \cdot U$ then acts on $X = M \times_L Y$ by $(m, u) \cdot [m', y] = [mm', y + u(m')]$ for all $m, m' \in M$, $y \in Y$, and $u \in U$. (Recall that the section u is an L-equivariant map $u : M \to Y$, $u(ml^{-1}) = lu(m)$ for all $m \in M$, $l \in L$.) This action is clearly transitive, because the sections U span Y over the identity coset $z_0 \in M/L$. The isotropy subgroup of the point $[1, 0] \in M \times_L Y$ is easily computed to be the semi-direct product $\hat{H} = L \cdot V$ where $V = \{u \in U \mid u(L) = 0\}$. □

References

1. Barth, W., Otte, M.: Invariante holomorphe Functionen auf reduktiven Liegruppen. Math. Ann. **201** 97–112 (1973)
2. Białynicki-Birula, A., Hochschild, G. Mostow, G.D.: Extensions of representations of algebraic linear groups. Am. J. Math. **85** 131–144 (1963)
3. Borel, A.: On affine algebraic homogeneous spaces. Arch. Math. **45** 74–78 (1985)
4. Cline, E., Parshall B., Scott L.: Induced modules and affine quotients. Math. Ann. **230** 1–14 (1977)
5. Grauert, H.: Analytische Faserungen über holomorph-vollständingen Räumen. Math. Ann. **135** 263–273 (1958)
6. Hochschild, G. Mostow, G.: Representations and representative functions of Lie groups, III. Ann. Math. **70** 85–100 (1959)
7. Luna, D.: Slices étales. Bull. Soc. Math. France, Mémoire **33** 81–105 (1973)
8. Matsushima, Y.: Espaces homogènes de Stein des groupes de Lie complexes. Nagoya Math. J. **18** 153–164 (1961)
9. Miyanishi, M., Sugie, T.: Affine surfaces containing cylinderlike open sets. J. Math. Kyoto Univ. **20** 11–42 (1980)
10. Mumford, D., Fogarty, J.: *Geometric Invariant Theory*, 2nd edn. (Springer, Berlin, Heidelberg, New York 1982)
11. Richardson, R.: Affine coset spaces of affine algebraic groups. Bull. London. Math. Soc. **91** 38–41 (1977)
12. Snow, D.M.: Reductive group actions on Stein spaces. Math. Ann. **259** 79–97 (1982)
13. Snow, D.M.: Stein quotients of connected complex Lie groups. Manuscripta Math. **50** 185–214 (1985)
14. Snow, D.M.: Triangular actions on $\mathbf{C}^3$. Manuscripta Math. **60** 407–415 (1988)
15. Winkelmann, J.: On Stein homogeneous manifolds and free holomorphic $\mathbf{C}$ actions on $\mathbf{C}^n$. Math. Ann. **186** 593–612 (1990)
16. Zaidenberg, M.G.: On exotic algebraic structures on affine spaces. St. Petersburg Math. J. **11** 703–760 (2000)