Prob 2.3.1

a) $\sqrt{z^2+1}$ 2 roots for each $z \neq \pm\sqrt{-1}$

holomorphic (i.e., complex analytic) for either of the 2 values in a open set around z except when $z = \pm i$ so

branch points $= \pm i$.

$z^2+1 = (z-i)(z+i)$ $\sqrt{z+i}$ is complex analytic in ~~and~~ an open neighborhood of i for either root. Ditto around $-i$ for $\sqrt{z-i}$. Thus around i, $\sqrt{z^2+1}$ behaves like $\sqrt{z-i}$. Going around i (or $-i$) interchange two roots.

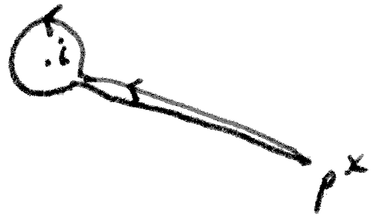
$\cdot\, i$

$\cdot\, p^*$ some fixed point not $= i, -i$

$-i$

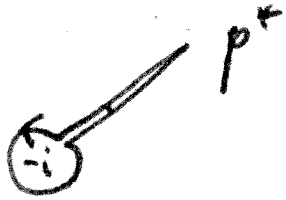
2 roots at p^*, call them $a, b = -a$.

Going around
the loop



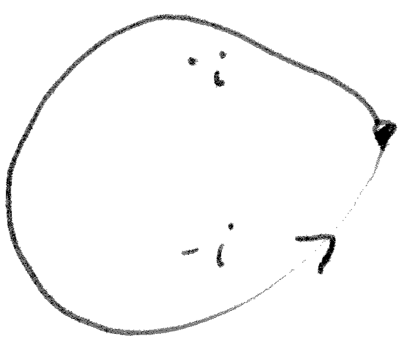
sends $\begin{matrix} a \\ b \end{matrix} \longmapsto \begin{matrix} b \\ a \end{matrix}$

Going around the loop sends



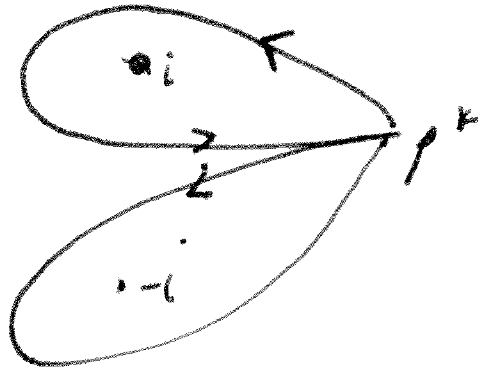
sends $\begin{matrix} a \\ b \end{matrix} \longmapsto \begin{matrix} b \\ a \end{matrix}$

The loop



p^* has the same

effect as



as $\begin{matrix} a & \to & b & \to & a \\ b & & a & & b \end{matrix}$

another way of seeing this is to note at ∞ we have, so ∞ is not a

$$\frac{1}{\sqrt{1+\frac{1}{w^2}}} = \frac{\pm w}{\sqrt{w^2+1}}$$

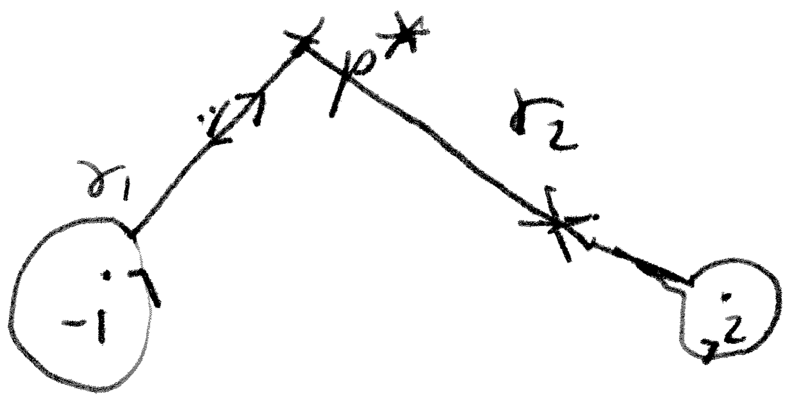
branch point.

2.3.1 b

$$\sqrt[3]{(z+1)(z-2)}$$

as before the points -1, 2 are branch points and around -1, $\sqrt[3]{(z+1)(z-2)}$ behaves like $\sqrt[3]{z-2}$. Ditto for around 2.

Fix a point $p^* \neq -1, 2$



Let's see how the three roots $a, b = e^{2\pi i/3} a, c = e^{4\pi i/3} a$ behave going around the loops.

$$\begin{array}{ccc} a & & b \\ b & \xrightarrow{\text{under } \gamma_1} & c \\ c & & a \end{array}$$

$$\begin{array}{ccc} a & & b \\ b & \xrightarrow{\text{under } \gamma_2} & c \\ c & & a \end{array}$$

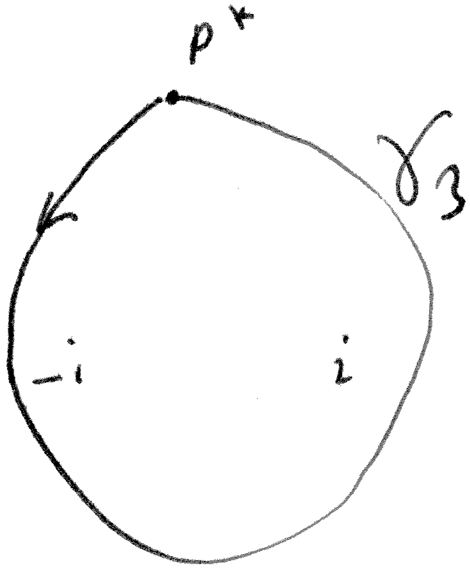
To see this let $z = 2 + \varepsilon e^{i\theta}$

going around ~~[illegible]~~ the loop γ_2 everything happens on the circle $\partial \Delta_\varepsilon(2)$. $\sqrt[3]{\varepsilon e^{i\theta}} \cdot \sqrt[3]{3 + \varepsilon e^{i\theta}}$

starts at $\sqrt[3]{\varepsilon}\,\sqrt[3]{3+\varepsilon}$ ends at

~~[illegible]~~ $\sqrt[3]{\varepsilon}\, e^{2\pi i/3} \sqrt[3]{3+\varepsilon}$

Note going around the loop



is the same

as γ_1 followed by γ_2

$$\begin{array}{ccccc} a & & b & & c \\ b & \mapsto & c & \mapsto & a \\ c & & a & & b \end{array}$$

so ∞ must be a branch point. Let's check.

$$\frac{1}{\sqrt[3]{\left(\frac{1}{w}+1\right)\left(\frac{1}{w}-2\right)}} = \frac{\sqrt[3]{w^2}}{\sqrt[3]{(1+w)(1-2w)}}$$

since the denominator is not 0 at 0

the branching ~~the~~ behaves like

$\sqrt[3]{w^2}$. Going around γ_3 is

the same as going around a large circle (Radius > 2). ~~$z = e^{i\theta}$~~ ~~$z = Re^{i\theta}$~~

$z = Re^{i\theta}$ $\quad 0 \leq \theta \leq \pi$ is the same

as

$w = \frac{1}{z} = \frac{e^{-i\theta}}{R}$. So going around

the large ~~circle~~ circle is the same as

$w = \varepsilon e^{i\theta}$ $\quad \theta$ from 2π to 0

$\sqrt[3]{w^2} = \sqrt[3]{\varepsilon}\, e^{2/3\,\theta i}$ ends up at $\sqrt[3]{\varepsilon}\; e^{2 \cdot 2\frac{\pi}{3}\theta i}$

which agrees with what we already computed.

2.3.2

$\ln (z-1)(z-2)$

ln is singular at 0, ∞ so

$z = 1, 2, \infty$ are the branch points

There are infinitely many solutions g of

$\ln(z-1)(z-2) = g$ at $z = p^* \neq 1, 2, \infty$

For example take $p^* = R > 3$ on the real axis.

$$e^{g} = (R-1)(R-2)$$

$$g = \ln(R-1)(R-2) + 2\pi i N$$

↑ pos. the real log

$N = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

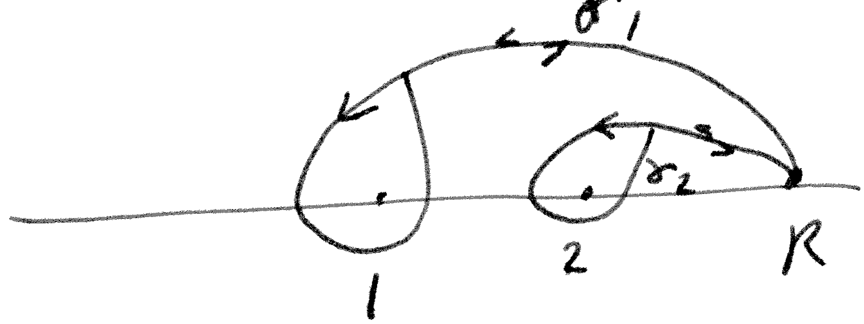
call them

$a_{-2}, a_{-1}, a_0, a_1, a_2 \ldots$

Going around γ_1 (around 1)

sends $a_i \to a_{i+1}$

same for γ_2 around 2



Going around both is the same as doing one and then the other

or $a_i \to a_{i+2}$.

We can check this in another way going around a large circle

$$\ln (z-1)(z-2)$$

$$= \ln z^2\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)$$

if the radius of the circle is $R>2$

the loops $1-\frac{1}{z}$ $\quad z = Re^{i\theta}$

$\theta \in [0, 2\pi]$

and $1-\frac{2}{z}$ . . .

don't enclose 0. So the behavior of

$\ln(z-1)(z-2)$ is the same as

that of $\ln(z^2)$

$\ln R^2 + 4\pi i n$

are the values on

$z = R^2$.

2.3.3

$\ln\left(z-(z^2+1)^{\frac{1}{2}}\right)$

(11)

ln is analytic away from 0

~~z~~ So solutions of $z = \sqrt{z^2+1}$

plus the branch points of

$\sqrt{z^2+1}$ are ~~f~~ branch points.

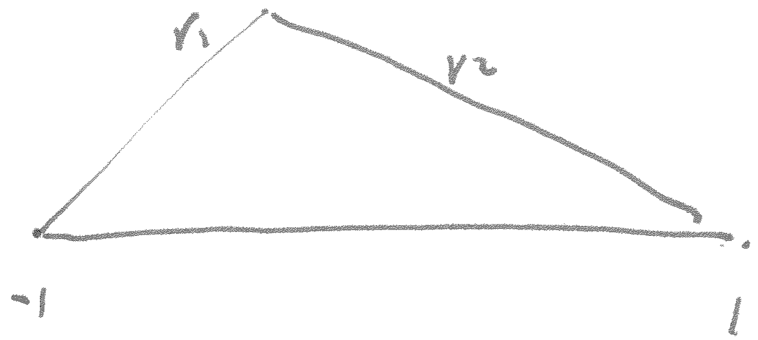
~~$\sqrt{z^2+1}$~~ Since $z = \sqrt{z^2+1}$ has no

solutions, we ~~see~~ the branch points

are at $\pm i$. We will discuss

the branch structure in class.

2.3.6 Laplace equation will be done in class



$$z = -1 + r_1 e^{i\theta_1}$$

$$z = 1 + r_2 e^{i\theta_2}$$

$$\frac{1}{z + \sqrt{z^2 - 1}}$$

$$\frac{1}{z + \sqrt{r_1 r_2 e^{i(\theta_1+\theta_2)}}} = \frac{1}{z + \sqrt{r_1 r_2}\, e^{i\frac{\theta_1+\theta_2}{2}}}$$

$$\lim_{y \downarrow 0} \left(\quad \right) = \frac{1}{x + \sqrt{r_1 r_2}\, e^{i\frac{\pi}{2}}}$$

$$= \frac{1}{x + i\sqrt{r_1 r_2}}$$

$$= \frac{x - i\sqrt{r_1 r_2}}{x^2 + r_1 r_2} = \frac{x - i\sqrt{1-x^2}}{1}$$

$\shortparallel$

$\theta_1 \downarrow 0$

$\theta_2 \uparrow \pi$

$$\lim_{\substack{y \uparrow 0 \\ \Updownarrow \\ \theta_1 \uparrow 2\pi \\ \theta_2 \uparrow \pi}} \left(\quad \right) = \frac{1}{x + \sqrt{r_1 r_2}\, e^{\frac{3}{2}\pi}}$$

$$= \frac{x + i\sqrt{r_1 r_2}}{x^2 + r_1 r_2} = x + i\sqrt{1-x^2}$$

~~Dy~~ $y = 0 \quad x \geq 1$

$$\Updownarrow$$

$$\theta_1 = 0$$

$$\theta_2 = 0$$

$$\frac{1}{z + \sqrt{z^2 - 1}} = \frac{1}{z + \sqrt{r_1 r_2}}$$

mit $y = 0$

$$\frac{1}{\;\;} \quad \mathrm{Im} = 0$$
