Prof 2.3.1

a) \[ \sqrt{z^2 + 1} \]

2 roots for each \( z \neq \pm i \)

Holomorphic (i.e., complex analytic) for either of the 2 values in any open set around \( z \) except when \( z = \pm i \) so branch points = \( \pm i \).

\[ z^2 + 1 = (z - i)(z + i) \]

\[ \sqrt{z^2 + 1} \] is complex analytic in an open neighborhood of \( i \) for either root. Ditto around \(-i\) for \( \sqrt{z^2 - i} \). Thus around \( i, \sqrt{z^2 + 1} \) behave like \( \sqrt{z^2 - i} \). Going around \( i (\neq -i) \) interchanging two roots.

\( \bullet \) \( p^* \) some fixed point

\(-i\)

\( not = i, -i \)

2 roots at \( p^* \), call them \( a, b = -a \).
Going around the loop:

\[ p^x \]

send \( a \rightarrow b \)

\[ b \rightarrow a \]

Going around the loop sends:

\[ p^x \]

send \( a \rightarrow b \)

\[ b \rightarrow a \]

The loop \( p^x \) has the same


\[ a \rightarrow b \rightarrow a \]
\[ b \quad a \quad b \]

another way of seeing this is to note at \( \infty \) we have, so \( a \) is not a

\[ \frac{1}{\sqrt{1 + \frac{1}{w^2}}} = \frac{\pm w}{\sqrt{w^2 + 1}} \]

branch point.
2.3.1 \( b \)

\[
\sqrt[3]{(z^2 + 1)(z - 2)}
\]

as before the points \(-1, 2\) are branch points and around

\(-1, \sqrt[3]{(z+1)(z-2)}\) behaves like

\[
\sqrt[3]{z-2}
\]

Ditto for around 2.

\textbf{Exercise} Fix a point \( p^* \neq -1, 2 \).
Let's see how the three roots $a, b = e^{2\pi i/3}a, c = e^{4\pi i/3}a$ behave going around the loop.

To see this, let $z = 2 + \varepsilon e^{i\theta}$ going around the loop $D_{\varepsilon}(2)$. The loop does everything happens on the circle $D_{\varepsilon}(2)$.

The loop starts at $\sqrt[3]{2}$, $\sqrt[3]{3+\varepsilon}$ and ends at $\sqrt[3]{2} e^{2\pi i/3} \sqrt[3]{3+\varepsilon}$. 

\[ a \underbrace{\text{under } \gamma_1} \to b \]
\[ b \to c \]
\[ c \to a \]

\[ a \underbrace{\text{under } \gamma_2} \to b \]
\[ b \to c \]
\[ c \to a \]
Note going around the loop

in the same

as \( \delta \), followed by \( \delta_2 \)

\[
\begin{align*}
& a \quad b \quad c \\
& b \quad c \quad a \\
& c \quad a \quad b
\end{align*}
\]

so \( \infty \) must be a branch point. Let us check.

\[
\frac{1}{\sqrt{\frac{1}{w} + 1} \left( \frac{1}{w} - 2 \right)} = \frac{\sqrt{3w}}{\sqrt{(1+w)(1-2w)}}
\]

since the denominator is not 0 at 0.
the branching ratio behaves like

$$\sqrt[3]{W^2}$$.

Going around $R$ is the same as going around a large circle (radius $>2$).

$$z = Re^{i\theta}, \quad 0 < \theta < \pi$$ in the same sense.

$$W = \frac{1}{z} = \frac{e^{-i\theta}}{R}.$$ So going around the large circle is the same as

$$w = \xi e^{i\theta}, \quad 0 \text{ from } 2\pi \text{ to } 0$$

$$\sqrt[3]{w^2} = \sqrt[3]{\xi} e^{\frac{5}{3}i}$$ and it ends up at

$$\sqrt[3]{\xi} e^{\frac{\pi}{3} i}$$

which agrees with what we already computed.
2.3.2

\[\ln (z-1)(z-2)\]

\(\ln\) is singular at 0, \(\infty\) so

\(z = 1, 2, \infty\) are the branch points.

There are infinitely many solutions of

\[\ln (z-1)(z-2) = y\] at \(z = p^k \pm 1, 2, \infty\)

For example, take \(p^k = R > 3\) on the
real axis.

\[e^y = (R-1)(R-2)\]

\[y = \ln(R-1)(R-2) + 2\pi i N\]

\[N = \ldots, -3, -2, 0, 1, 2, 3, \ldots\]

call them

\[a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\]
Going around $T_1$ (around 1)

send $a_i \rightarrow a_{i+1}$

same for $T_2$ around 2

Going around both in the same as doing one and then the other

or $a_i \rightarrow a_{i+2}$. By definition

We can check this in another way going around a large curve
\[ \ln (z - 1)(z - 2) \]

\[ = \ln z^2 \left( 1 - \frac{1}{z} \right) \left( 1 - \frac{2}{z} \right) \]

if the radius of the circle is \( R > 2 \)

the loops \( 1 - \frac{1}{z} \quad z = Re^{iu} \quad 0 \leq u \leq \pi \)

and \( 1 - \frac{2}{z} \)

don't enclose \( 0 \). So the behavior of \( \ln (z - 1)(z - 2) \) is the same as that of \( \ln(z^2) \)

\[ \ln R^2 + \frac{4\pi i m}{n} \]

are the values on \( z = Re^{i\theta} \).
\[ \ln\left(\frac{z - (z^2 + 1)^{\frac{1}{2}}}{z}\right) \]

\ln \text{ is analytic away from 0}

\( z = \sqrt{z^2 + 1} \)

For solutions of \( z = \sqrt{z^2 + 1} \)

\( \sqrt{z^2 + 1} \) are \( 2 \) branch points.

Since \( z = \sqrt{z^2 + 1} \) has no solutions, we see the branch points are at \( + i \). We will discuss the branch structure in class.
2.3.6 Laplace equation will be done in class.

\[
Z = x - i + r_1 e^{i \theta_1} \\
Z = 1 + r_2 e^{i \theta_2}
\]

\[
\frac{1}{Z + \sqrt{z^2 - 1}} = \frac{1}{Z + \sqrt{r_1 r_2} e^{i \frac{\theta_1 + \theta_2}{2}}}
\]

\[
\lim_{y \to 0} \frac{1}{Z + \sqrt{r_1 r_2} e^{i \frac{\theta_1 + \theta_2}{2}}} = \frac{1}{x + i \sqrt{r_1 r_2}}
\]

\[
= \frac{x - i \sqrt{r_1 r_2}}{x^2 + r_1 r_2} = \frac{x - i \sqrt{1 - x^2}}{1 + x^2}
\]
\[ \lim_{y \to 0} (x) = \frac{1}{x + \sqrt{\pi \cdot r_2}} e^{\frac{x^2}{2}} = \frac{x + i \sqrt{\pi \cdot r_2}}{x^2 + \pi \cdot r_2} = x + i \sqrt{1-x^2} \]

\[ y = 0 \quad x \geq 1 \]
\[ C_1 = 0 \]
\[ C_2 = 0 \]

\[ \frac{1}{x^2 + \sqrt{\pi \cdot r_2}} = \frac{1}{x^2 + \sqrt{\pi \cdot r_2}} \]

\[ \text{with } y = 0 \]
\[ \frac{1}{x^2 + \sqrt{\pi \cdot r_2}} \]
\[ \Rightarrow \Delta \omega = 0 \]