3.1.3

$$\lim_{n\to\infty} \int_0^1 n z^{n-1}\, dz$$

$$= \lim_{n\to\infty} z^n \Big|_0^1 = \lim_{n\to\infty} 1 = 1$$

$$\lim_{n\to\infty} n z^{n-1} dz = \begin{cases} 0 & z \neq 1 \\ 1 & z = 1 \end{cases}$$

$$\text{So} \quad \int_0^1 \lim (\quad) dz = 0$$

Note given any measure $d\mu$ (e.g., dx on the line, $dx\,dy$ on the plane ...)

$|f_n| \le g$ on D

$f_n \to f$ pointwise

$\int g\, d\mu$ exists $\Longrightarrow$

$$\lim_{n\to\infty} \int_D f_n\, d\mu = \int_D \lim_{n\to\infty} f_n\, d\mu$$ ← Lesbesgue dominated convergence

if not bounded by such a g

$$\lim_n \int_D f_n\, d\mu \ge \int_D \lim_n f_n\, d\mu$$ Fatou Lemma

3.4.3

Show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n n^z}$ is entire

This is equivalent to showing ~~absolute~~ uniform convergence for any $|z| \leq R$ (for any $R > 0$).

<u>Use Weierstrass M-test</u>

For fixed $R > 0$ show

$$\left| \frac{e^n n^x}{e^{n+1} (n+1)^x} \right| \leq a < 1$$

for all n sufficiently large

$$= \frac{1}{e\left(1+\frac{1}{n}\right)^x} \leq \frac{\left(1+\frac{1}{n}\right)^R}{e} < 1$$ for all suff. large n.

3.5.2 a, b, c

C unit circle

(a) $\frac{1}{2\pi i}\int_C f(z)\,dz = 0$ if f ~~[illegible]~~ analytic on $\Delta_1(0)$

so $\frac{1}{2\pi i}\int_C \frac{g(z)}{z-w}\,dz = \begin{cases} 0 & |w|>1 \\ g(w) & |w|<1 \end{cases}$

(b) $\frac{1}{2\pi i}\int_C \frac{z}{z^2-w^2}\,dz = \frac{1}{2\pi i}\int_C \frac{1}{2}\left(\frac{1}{z-w}+\frac{1}{z+w}\right)dz$

$= \begin{cases} \frac{1}{2}+\frac{1}{2} = 1 & \text{if } |w|<1 \\ 0 & |w|>1 \end{cases}$

(c) $\displaystyle \frac{1}{2\pi i}\int_C z\, e^{\frac{1}{z^2}}\, dz$

$\displaystyle = \frac{1}{2\pi i}\int_C \left(z + \frac{1}{z} + \frac{1}{2!z^3} + \cdots \right) dz$

$= 1$

3.6.1

(a) $\displaystyle \sum_{m=1}^{N} \ln(1+z^m) = \sum_{1}^{N} \ln|1+z^m| + i\sum \arg(1+z^m)$

$|z| > 1$ then diverges

$|z| = 1 \Rightarrow \{z, z^2, \ldots\}$ dense in circle or periodic ~~dense or periodic~~

so only hope if $|z| < 1$ in this

can ~~[illegible]~~ since

$\sum |z|^m$ converges we have $\overline{\prod (1+z^m)}$ i defined ditto for $\prod_{n=1}^{\infty}(1-z^n)$

<u>Note</u>

$$\frac{1}{\prod_{n=1}^{\infty}(1-z^n)} = \sum_{0}^{\infty} p(n)\, z^n$$

where $p(0) = 1$, $p(n)$ is the partition function $=$ # ways n may be decomposed into pos. integers; i.e.,

$$\begin{aligned} 5 &= 1+1+1+1+1 \\ &= 1+1+1+2 \\ &= 1+1+3 \\ &= 1+4 \\ &= 2+3 \\ &= 2+2+1 \\ &= 5 \end{aligned}$$

So $p(5) = \cancel{6}\,7$

There is no closed expression for $p(n)$, but Hardy + Littlewood found an [redacted] series expansion (using their the circle method) for $p(n)$. Rademacher improved it so that $p(n)$ is now very quick to compute. Note these methods are all based on clever clever residue calculations.

(b) $\prod \left(1+\frac{z^n}{n!}\right)$

$\sum \frac{|z|^n}{n!}$ $\underbrace{\text{converges}}_{}$ uniformly all $|z| \le R$

so entire

(c) $\prod \left(1+\frac{2z}{n}\right)$

$\sum \ln\left(1+\frac{2z}{n}\right)$

for $|z| \le \varepsilon < 1$

now $\left|\ln\left(1+\frac{2z}{n}\right) - \frac{2z}{n}\right| \le C\frac{|z|^2}{n^2}$

for all large enough n

C independent of ε.

So $\left|\sum_{N'}^{N} \ln\left(1+\frac{2z}{n}\right) - \sum_{N'}^{N} \frac{2z}{n}\right| \le \sum_{N'}^{N} C\frac{|z|^2}{n^2}$

converges

$\lim_{N\to\infty} \sum_{N'}^{N} \frac{2z}{n} \to \infty$

so diverges all $z \ne 0$

(d) $\prod_{1}^{\infty}\left(1+\left(\frac{2z}{n}\right)^2\right)$

$|z|^2 \sum_{1}^{\infty} \frac{4}{n^2}$ converges so is

entire