1

2.6.10

$|z| < 1$

$\left|\frac{1}{z}\right| > 1$

so $\dfrac{f(\zeta)}{\zeta - \frac{1}{z}}$ holomorphic (= complex analytic) for $|\zeta| < \left|\frac{1}{z}\right|$

$$\Rightarrow \quad 0 = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(\zeta)}{\zeta - \frac{1}{z}} \, d\zeta$$

letting $\zeta = e^{i\theta}$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)\,\zeta}{\zeta - \frac{1}{z}} \, d\theta = \frac{-1}{2\pi} \int \frac{f(\zeta)\,\bar{z}}{\bar{\zeta} - \bar{z}} \, d\theta$$

$$f(z) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)\,\zeta}{\zeta - z} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{f(\zeta)\,\zeta}{\zeta - z} \pm \frac{f(\zeta)\,\bar{z}}{\bar{\zeta} - \bar{z}} \right) d\theta$$

using + $\quad f(z) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \frac{f(\zeta)}{|\zeta - z|^2} \left(\zeta\bar{\zeta} - \zeta\bar{z} + \zeta\bar{z} - z\bar{z} \right) d\theta =$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(\frac{1-|z|^2}{|\zeta-z|^2}\right)d\theta$$

So taking the real part $u(r,\phi)$ of $f(z)$
(let $u(1,\phi)=U(\phi)$)

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\right)d\theta$$

Note $|\zeta - z|^2 = |e^{i\theta} - re^{i\phi}|^2$

$$= (e^{i\theta}-re^{i\phi})(e^{-i\theta}-re^{-i\phi})$$

$$= 1+r^2 - r\left(e^{i(\phi-\theta)}+e^{-i(\phi-\theta)}\right)$$

$$= 1+r^2-2r\cos(\phi-\theta)$$

Here is what comes out of all this

(A) (1) any reasonable real valued function g on $D=\{|z|=1\}$ (or even the boundary of a simply connected open domain with "nice" boundary) gives rise to an harmonic function u on D with boundary values g on ∂D.

Note $\ln|z|$ and 0 both have the same values on $|z|=1$. Which is the unique extension? (Ans. 0 since $\ln(z)$ is not defined at $0 = z$) ③

(B) up to a constant C, there is a unique v on D with $u + iv$ analytic

(C) if $f = u + i\tilde{v}$ on ∂D with $\tilde{v} - v$ not constant on ∂D, then $f|_{\partial D}$ cannot have an analytic extension to D.

Of course, sometimes f might be $f_1 + f_2$ where f_1 is analytic on $|z| < 1 + \varepsilon$ and f_2 is analytic on $|z| > 1 - \varepsilon$,

e.g. $\frac{1}{z} + \sin(z)$

4.2.3 $\int_0^{2\pi} \cos^{2m}(\theta)\,d\theta =$

$$\int_0^{2\pi} \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^{2m} d\theta$$

$$= i\int_0^{2\pi} \frac{\left(z+\frac{1}{z}\right)^{2m}}{\cancel{i}\; z\; 2^{2m}}\, dz$$

$$= 2\pi \cancel{i}\, \mathrm{Res}_0\, \frac{\left(z+\frac{1}{z}\right)^{2m}}{z\, 2^{2m}}$$

$$= \frac{2\pi}{2^{2m}}\binom{2m}{m}$$

note $\binom{2m}{m} = \frac{(2m)!}{m!\,m!} = \frac{2^m\cdot(2m-1)(2m-3)\cdots}{m!}$

(I don't know why the book doesn't just have $\frac{2\pi}{2^m}\binom{2m}{m}$

$\int_0^{2\pi} \sin^{2m}(\theta)\,d\theta$ [redacted] is done by the same argument.

4.3.1 Show.

$$\int_0^\infty \frac{\cos(kx) - \cos(mx)}{x^2}\,dx$$

$$= \frac{\pi}{2}\left(|m| - |k|\right)$$

Note. $\cos(kx) - \cos(mx) =$

$$\frac{m^2 - k^2}{2} x^2 + \cdots$$

so $\frac{\cos(kx) - \cos(mx)}{x^2}$ analytic

in a neighborhood of 0. Also

since $\left|\frac{\cos(kx) - \cos(mx)}{x^2}\right| \leq \frac{2}{|x|^2}$

for $|x| > 0$ $\lim_{R \to \infty} \int_\varepsilon^R \text{---}\,dx$ exists.

Moreover the function is even so $\int_0^\infty = \frac{1}{2}\int_{-\infty}^\infty$

$\frac{\cos(kx)-\cos(mx)}{x^2}$ is the real

part of

$\frac{e^{ikx}-e^{imz}}{z^2}$ so suffices to

compute $\int_{-\infty}^{\infty} \frac{e^{ikz}-e^{imz}}{z^2}\,dz$

Note $e^{ikz}-e^{imz} = i(k-m)z + \cdots$

So the integrand has a simple pole.
We note that since cos is even we can assume k, m are ≥ 0.
We have (assume $k, m > 0$, so we can use Jordan's lemma)

$$\int_{-\infty}^{\infty} \text{——}\, dz = \pi i \operatorname{Res}_0\left(\frac{e^{ikz}-e^{imz}}{z^2}\right)$$

$$+ \sum \operatorname{Res}_{\text{upper half plane}}$$

since the function is analytic in the upper $\frac{1}{2}$ plane we have

$$\int_{-\infty}^{\infty} = \pi i \cdot i(k-m) = \pi(m-k)$$

so $m \neq 0$, $k \neq 0$

$$\int_0^\infty \frac{\cos(kx) - \cos(mx)}{x^2}\, dx$$

$$= \frac{\pi}{2}\left(|m| - |k|\right).$$

Now as $k \to 0$ the integrand is bounded by ~~a~~ a pos. integrable function $2m^2$ near 0 and $\frac{1}{x^2}$ for $|x| > \varepsilon$ for some ε. So by dominated ~~conv~~ convergence the formula holds true ~~as k~~ for $k = 0$. Ditto for $m = 0$.

4.3.10 This will be done in class as part of § 4.5

4.4.5 Straightforward