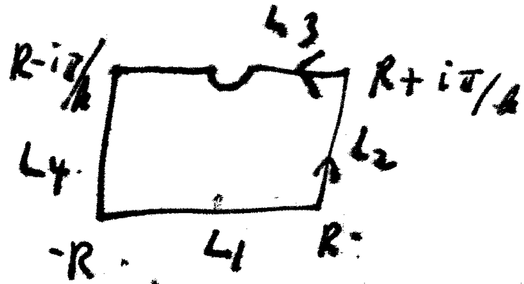
4.3.4 (1)



$$\int = 0 \qquad \int \frac{z\,dz}{\sinh(kz)}\,dz$$

$$\left| \int_{L_2} \right| \le \int \frac{2R\,dy}{e^R} \to 0$$

$$\left| \int_{L_4} \right| \le 0$$

$$\int_{L_1} \to 2\int_0^\infty \frac{x\,dx}{\sinh(kx)}$$

$$\int_{L_3} \frac{z}{\sinh z}\,dz = \int_{R+i\pi/k}^{\varepsilon+i\pi/k} + \int_{-\varepsilon}^{-R+i\pi/k} + \int_{2\pi,\ |z|=\varepsilon}^{\pi} \frac{z\,dz}{\sinh(kz)}$$

$$= \int_R^{\varepsilon} \frac{(x + i\pi/k)}{\sinh(kx+i\pi)}\,dx + \int_{-\varepsilon}^{-R} \frac{x+i\pi/k}{\sinh(kx+i\pi)}\,dx$$

$$+ \int_{2\pi}^{\pi} \frac{(i\pi/k + \varepsilon e^{i\theta})\varepsilon e^{i\theta} \cdot i\,d\theta}{\sinh(i\pi + \varepsilon k e^{i\theta})}$$

$$\to \int_{-R}^{R} \frac{x\,dx}{\sinh(kx)} + \frac{i\pi}{k} \int_{-R}^{R} \frac{dx}{\sinh(kx)} \leftarrow \text{odd}$$

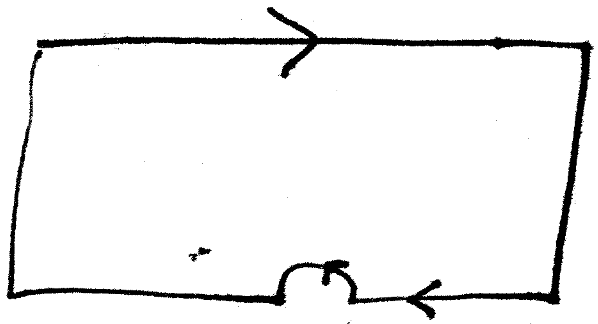
$$+ -\frac{1}{2}\, 2\pi i \left(\text{Res}\, \frac{z}{\sinh(kz)} \text{ at } \pi i/k\right)$$

$$\text{So } 4\int_0^{\infty} \frac{x}{\sinh(kx)}\,dx + 0$$

$$\to \pi i \cdot \frac{\pi i}{k} \frac{1}{k \cdot \cosh(\pi i)} = \frac{-\pi^2}{k^2}$$

$$\text{so } \int_0^{\infty} \frac{x}{\sinh(kx)}\,dx = \frac{\pi^2}{4k^2} \quad \text{this was for } k>0$$

When $k < 0$



so the residue is $\frac{\pi^2}{k^2}$ at $\frac{i\pi}{k}$

k neg.

So

$$\int_0^\infty \frac{x}{\sinh(kx)}\,dx = \frac{\pi^2}{4k|k|}$$

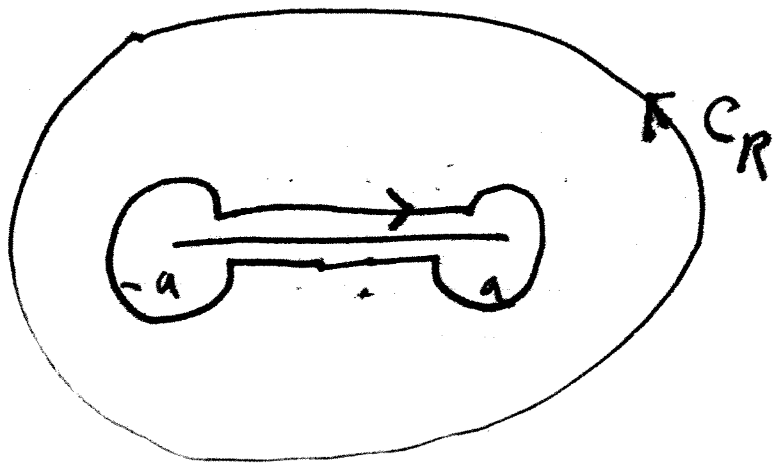
4.3.10 (a)

$$-\frac{1}{2\pi i}\int_{C_{\frac{1}{R}}} \frac{dw}{-w^2\left(\frac{1}{w^2}-a^2\right)^{\frac{1}{2}}} = \frac{1}{2\pi i}\int_{C_{\frac{1}{R}}} \frac{dw}{w\left(1-a^2w^2\right)^{\frac{1}{2}}}$$

$=$!

4.3.11 b

assume $a > 0$



$$\int_{C_R} + \int = 0$$

$\boxed{\text{Note } \int_{\rho}, \int_{c} \to 0}$

$z = a + R_2 e^{i\theta_2}, \; z = -a + R_1 e^{i\theta_1}$

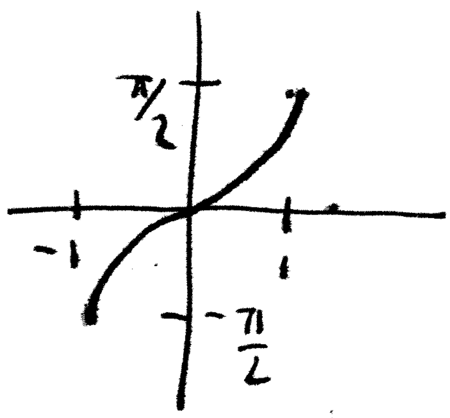
$$\lim_{\substack{z \downarrow 0 \\ -a \le x \le a}} \frac{1}{\sqrt{z^2 - a^2}} = \lim_{\substack{\theta_1 \to \pi \\ \theta_2 \to 0}} \frac{1}{\sqrt{R_1 R_2}\, i} = \frac{1}{i\sqrt{a^2 - x^2}}$$

$$\lim_{\substack{z \uparrow 0 \\ -a \le x \le a}} = \lim_{\substack{\theta_1 \to -\pi \\ \theta_2 \to 0}} = \frac{1}{-i\sqrt{a^2 - x^2}}$$

$\cancel{\oint_R \ldots S_0 - 2i\, I_R = 2\pi i = 0}$

5. $$1 + \frac{2}{2\pi i\cdot i}\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}} = 0$$

or $$\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{\pi}\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}$$



$$\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}1 - \sin^{-1}-1 = \pi$$

4.5.11 f

$$\mathcal{L}^{-1}\left(\frac{e^{sz}}{s^2(s^2+w^2)}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{sz}\,ds}{s^2(s^2+w^2)}$$

$$= \mathrm{Res}_{iw} + \mathrm{Res}_{-iw} + \mathrm{Res}_0$$

$$= \frac{e^{iwz}}{(iw)^2\,2(iw)} + \frac{e^{-iwz}}{(-iw)^2\,2(-iw)} + \mathrm{Res}_0$$

$$= \frac{-\sin(wz)}{w^3} + \text{coef of } -1 \text{ term of} = -\frac{\sin(wz)}{w^3} + \frac{z}{w^2}$$

$$\left(\frac{1}{w^2}\left(1 + sz + \cdots\right)\frac{1}{s^2}\left(1 - \frac{s^2}{w^2} + \cdots\right)\right)$$

$$= \frac{+z}{w^2}$$

4.5.14

$$\frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty}\frac{e^{sz}\,e^{-as^{\frac{1}{2}}}}{s}\,ds$$

<u>Note letting</u> $s = A + i\sigma$
$A > 0$

we have

$$\frac{e^{Az}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\sigma z}}{A+i\sigma}\,e^{-a\sqrt{A+i\sigma}}\,d\sigma$$

$z = x$ ~~[illegible]~~ then $|e^{i\sigma z}| = 1$
real

$$e^{-a\sqrt{}} = e^{-a\sqrt{A^2+\sigma^2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ so

$$\left| e^{-\sqrt{}} \right|^2 = e^{-a\sqrt{A^2+\sigma^2}\,\cos\left(\frac{\theta}{2}\right)}$$

~~since $0 < \cos\left(\frac{\theta}{2}\right) < \frac{\sqrt{2}}{2}$ in the range~~

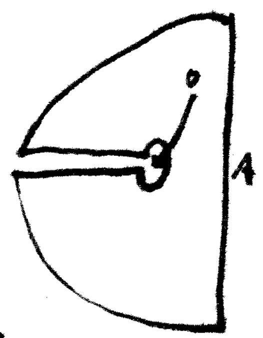
since $\frac{\sqrt{2}}{2} \le \cos\left(\frac{\alpha}{2}\right) \le 1$ for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ 8

we have

$$\left| \frac{e^{i\sigma x}}{A+i\sigma} e^{-a\sqrt{A^2+\sigma^2}\cos\frac{\alpha}{2}} \right|$$

$$\le \frac{1}{\sqrt{A^2+\sigma^2}} e^{-a\frac{\sqrt{2}}{2}\sigma}$$

so the integral converges for any $A > 0$.



use the keyhole contour

$\sqrt{1} = 1$ as usual

$$\frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} + \frac{1}{2\pi i}\int_{\frown} + \frac{1}{2\pi i}\int_{\smile} + \int_{\circ} + \int_{\rightleftarrows}$$

integral around $|s|=\varepsilon$

$$\frac{1}{2\pi i}\int_{\pi}^{-\pi} \frac{e^{\varepsilon e^{i\theta} z}}{\varepsilon e^{i\theta}} e^{\sqrt{\varepsilon}\, e^{i\theta/2}}\, d\theta\, \varepsilon e^{i\theta} i$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} \underbrace{e^{\varepsilon e^{i\theta} z} e^{\sqrt{\varepsilon}\, e^{i\theta/2}}}_{\downarrow \; e^{0}}\, d\theta$$

$$\text{so } \int \to 1$$

$= 0$ By Jordan Lemma

we have $\int \to 0$ and $\int \to 0$

So we have

$$\frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} \frac{e^{sz}\, e^{-a\sqrt{s}}}{s}\, ds$$

$$= 1 - \frac{1}{2\pi i}\int_{\rightarrow} - \int_{\leftarrow}$$

$e^{-a\sqrt{s}}$ $\qquad s = \tau + i\sigma$

$= e^{-a\tau i}$ on $\rightarrow$

$= e^{a\tau i}$ $\leftarrow$

for

$$\frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty}\frac{e^{sz}}{s}\,e^{-a\sqrt{s}}\,ds$$

$$=1-\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{e^{(\tau\ )z}}{\tau}\,e^{-ia\sqrt{-\tau}}\,d\tau$$

$$+ -\frac{1}{2\pi i}\int_{0}^{-\infty}\frac{e^{(\tau\ )z}}{\tau}\,e^{ia\sqrt{-\tau}}\,d\tau$$

$$=1-\frac{1}{2\pi i}\int_{0}^{-\infty}\frac{e^{\tau z}}{\tau}\left(e^{ia\sqrt{-\tau}}-e^{-ia\sqrt{-\tau}}\right)d\tau$$

$$=1+\frac{1}{2\pi i}\int_{0}^{\infty}\frac{e^{-\tau z}}{\tau}\left(e^{-ia\sqrt{\tau}}-e^{ia\sqrt{\tau}}\right)d\tau$$

$$=1-\frac{1}{\pi}\int_{0}^{\infty}\frac{e^{-\tau z}}{\tau}\sin\left(a\sqrt{\tau}\right)$$

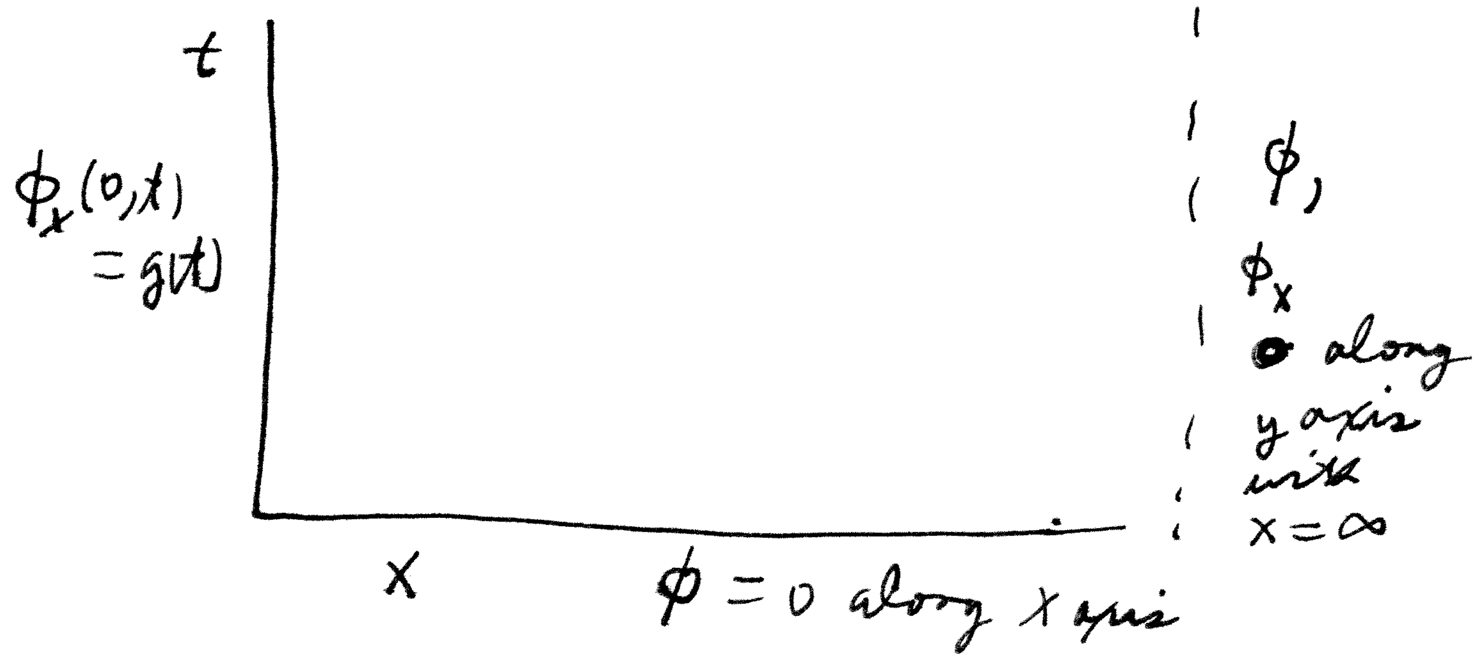
Prob 4.6.6b

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

(a) $\phi(x,0) = 0$

(b) $\frac{\partial \phi}{\partial x}(0,t) = g(t)$

(c) $\lim_{x\to\infty} \phi(\cancel{x},t) = \lim_{x\to\infty} \frac{\partial \phi}{\partial x} = 0$



We can treat $\phi(x,t)$ as 0

for $t < 0$

$$\int_0^\infty \phi(x,t)\, e^{-kt}\, dt = \hat{\phi}$$

$$\int_0^\infty \frac{\partial \phi}{\partial t}\, e^{-kt}\, dt = \int_0^\infty \frac{\partial^2 \phi}{\partial x^2}\, e^{-kt}\, dt$$

assume ϕ is ((well enough behaved such that))

$$\phi\, e^{-kt}\Big|_{t=0}^{\infty} - \int_0^\infty \phi\, k e^{-kt}\, dt \qquad \frac{\partial^2}{\partial x^2} \hat{\phi}(x,k)$$

used (a)

$$= -\phi(x, 0) + k\int_0^\infty \phi\, e^{-kt}\, dt$$

So $\hat{\phi}_{xx} = k\,\hat{\phi}$

$$\hat{\phi} = C(k)\,e^{\sqrt{k}\,x} + \tilde{C}(k)\,e^{-\sqrt{k}}$$

$$\frac{\partial \hat{\phi}}{\partial x}(0,k) = \hat{g}(k) \quad \text{used } b$$

$$\lim_{x\to\infty} \int_0^\infty e^{-kt}\,\phi(x,t)\,dt = 0$$

assume $\lim_{x\to\infty} \phi(x,t) = 0$ in a nice manner

say ϕ in L^p or even L^∞

So $\hat{\phi} = \tilde{C}(k)\,e^{-\sqrt{k}\,x}$ used (c)

$$\frac{\partial \hat{\phi}}{\partial x}(0,k) = -\tilde{C}(k)\sqrt{k}\,\cancel{e^{-\sqrt{k}\,x}}$$

$$\parallel$$

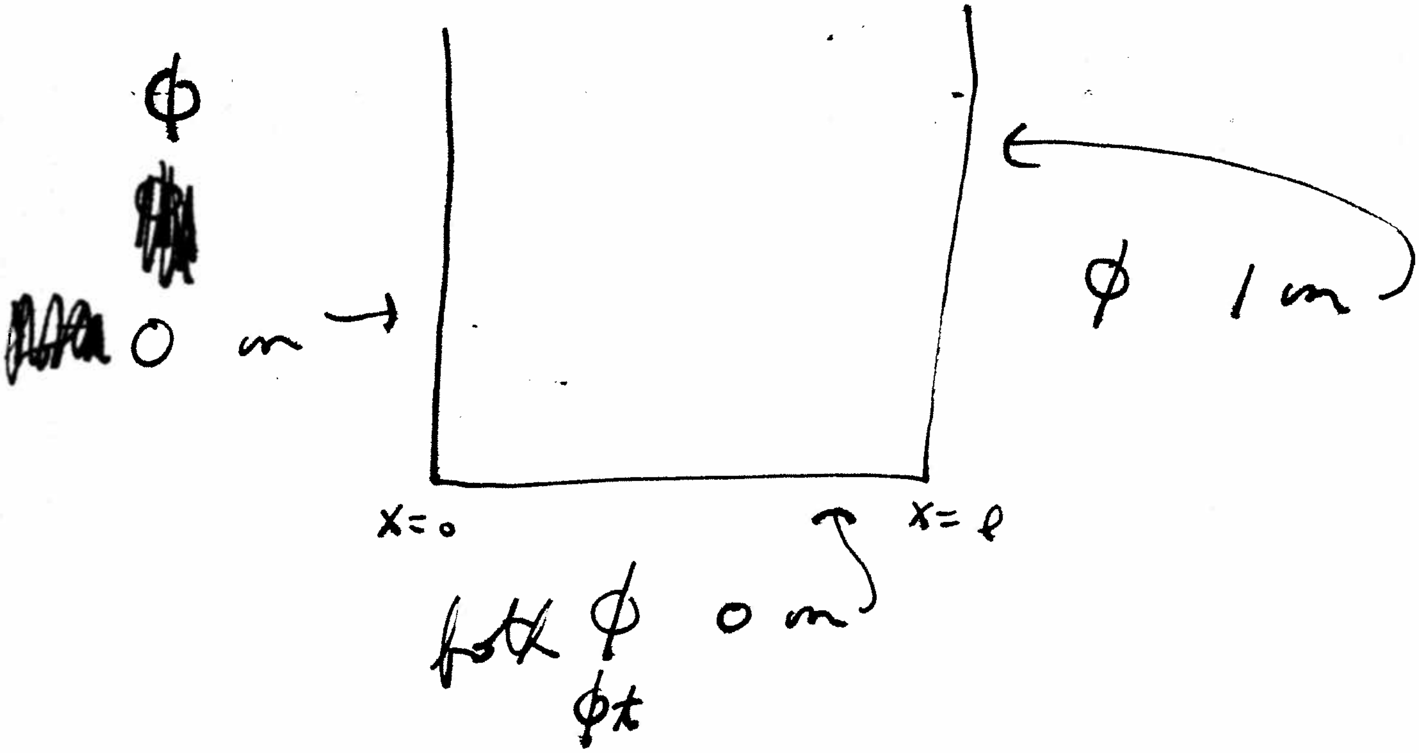
$$\hat{g}$$

$$\tilde{C}(k) = -\frac{\hat{g}(k)}{\sqrt{k}}$$

So

$$\phi(x,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{kx}\,\frac{\hat{g}(k)}{\sqrt{k}}\,e^{-\sqrt{k}\,x}\,dk$$

Prob. 4.6.7



$$\mathcal{L}(\phi_{tt}) = \int_0^\infty \phi_{tt}\, e^{-kt}\, dt$$

$$= -\phi_t(x,0) - k\,\phi(x,0) + k^2\int_0^\infty \phi e^{-kt}\, dt$$

$$= k^2\,\mathcal{L}(\phi)$$

using the two x-axis boundary conditions

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So $k^2 \hat{\phi} = \hat{\phi}_{xx}$

$$\hat{\phi} = C(k) e^{kx} + \tilde{C}(x) e^{-kx}$$

$$\hat{\phi}(0, k) = \int_0^\infty e^{-h \cdot 0} \cdot 0 \, dh = 0$$

$$\hat{\phi}(l, h) = \int_0^\infty e^{-hl} \cdot 1 \, dh$$

$$= \left. \frac{e^{-hl}}{-l} \right|_0^\infty$$

$$= \frac{1}{l}$$

So ~~$C + \tilde{C} = 0$~~

~~$C(l)-$~~

$$C + \tilde{C} = 0$$

$$C e^{hl} + \tilde{C} e^{-hl} = \frac{1}{l}$$

So

$$C\left(e^{kl} - e^{-kl}\right) = \frac{1}{l}$$

$$C = \frac{2}{l \sinh(kl)}$$

$$\tilde{C}\left(e^{kl} - e^{-kl}\right) = -\frac{1}{l}$$

or $\tilde{C} = \dfrac{-2}{l \sinh(kl)}$

$$\hat{\phi}(x, k) = \frac{\sinh(kx)}{l \sinh(kl)}$$

Part a done

Part b same as $\frac{1}{2\pi i}\int_{C_N} \pi \cot(\pi z) f(z)\, dz$