Mathematics 40485: Test I

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February 14, 2012

Name: ....

This test is due at the start of class on Tuesday, February 19, 2013.

This test is being conducted under the honor code. From the time you receive this exam *until the time it is collected you may not communicate about course material with anyone (in the class or not)*, except me. This includes comments on which problems are easy or hard. *You may not use anyone else's notes or programs on the test.* You are allowed and encouraged to use your notes, Maple, any books, or material on websites (at your own risk of course) while doing the tests.

If you find some problem on the test which you think is unclear (or wrong), please contact me immediately—my e-mail address is

`sommese@nd.edu`.

If necessary I will send a correction or clarification to the class by e-mail. It is your responsibility to check your e-mail regularly between now and Tuesday, February 19, 2013.

There are a total of 111 points below. Letting T denote the total number of points you get, your test grade will be $\min\{T, 100\}$.

In the following you must show your work. You should also include your Maple worksheets used on the problems. You may use the differentiation and algebraic simplification commands (including `taylor`, which computes Taylor series), but you must show all work as discussed in class.

Problems

Problem 1 (16 points) *Given*

$$f(z) = \frac{z}{(z-1)},$$

expand $f(z)$ in a Laurent series in powers of z in the regions

1. $|z| < 1$; *and*
2. $1 < |z|$.

Problem 2 (12 points) *Letting C denote the unit circle traversed counter-clockwise, compute*

$$\frac{1}{2\pi i}\int_C \frac{1}{\cos(2z)}\mathrm{d}z$$

using residues.

Problem 3 (24 points) *One way of defining the n-th Bessel function, $J_n(t)$, is as the coefficient of z^n in the Laurent expansion*

$$e^{\frac{t}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n.$$

1. *Using the relation between residues and Laurent series show that*

$$J_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin(\theta))}\, \mathrm{d}\theta.$$

2. *Using the relation in 1) show that*

$$J_n(t) = \frac{1}{\pi} \int_{0}^{\pi} \cos\left(n\theta - t\sin(\theta)\right)\, \mathrm{d}\theta.$$

Problem 4 (16 points) *Compute*

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1} \mathrm{d}x$$

using residues.

Problem 5 (16 points) *Compute*

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

by using residues, i.e., using

$$\frac{1}{2\pi i} \int_{C_N} \frac{\pi \cot(\pi z)}{z^4} \mathrm{d}z = 2 \sum_{n=1}^{\infty} \frac{1}{n^4} + \mathrm{Res}_0 \left(\frac{\pi \cot(\pi z)}{z^4} \right)$$

where C_N *denotes the square with vertices*

$$\pm \left(N + \frac{1}{2} \right) + \pm \left(N + \frac{1}{2} \right) i.$$

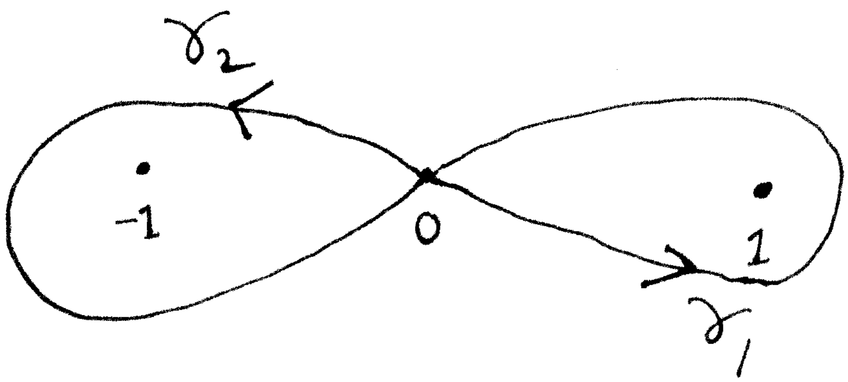
Hint: I strongly recommend appropriate use of `taylor` in this problem.

Problem 6 *Let*

$$f(z) = \left(1 - z^2 \right)^{\frac{1}{4}}.$$

7 points *Verify that the branch points of* $f(z)$ *are* 1, -1, *and* ∞.

20 points *Compute the monodromy action of* T_1, T_2, $T_1 T_2$, *and* $T_2 T_1$ *on the solutions* $\{1, i, -1, -i\}$ *for* $z = 0$ *of* $z^2 + w^4 = 1$ *where* T_1 *and* T_2 *correspond to the paths* γ_1 *and* γ_2 *below.*



① $$\frac{z}{z-1} = -\frac{z}{1-z} \quad \text{around } 0$$

$$= -z - z^2 - z^3 \ldots\ldots$$

$$\frac{z}{z-1} = \frac{1}{1-\frac{1}{z}}$$

around ∞

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots$$

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② $$\frac{1}{2\pi i}\int_{|z|=1} \frac{dz}{\cos(2z)}$$

$$\|$$

$$\operatorname{Res}_{\frac{\pi}{4}} \frac{1}{\cos(2z)} + \operatorname{Res}_{-\frac{\pi}{4}} \frac{1}{\cos(2z)}$$

the zeroes of $\cos(2z)$ are of first order at $\frac{\pi}{4} + n\frac{\pi}{2}$ n integral

within unit circle $= \pm\frac{\pi}{4}$

$$= \frac{1}{-2\sin\left(\frac{\pi}{2}\right)} + \operatorname{Res}_{-\frac{\pi}{4}} \frac{1}{-2\sin\left(-\frac{\pi}{2}\right)} = 0$$

(3) $\frac{1}{2\pi i}\int_{|z|=1}\left(\sum_{k=-\infty}^{\infty} J_k(t)\, z^k\right)\Big/ z^{n+1}\, dz = \mathrm{Res}_0$ —

$= \text{coeff of } \frac{1}{z} = J_n(t)$

$$\frac{1}{2\pi i}\int_{|z|=1}\left(\sum_{k=-\infty}^{\infty} J_k(t)\right)\Big/ z^{n+1}\, dz =$$

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{e^{\frac{t}{2}\left(z-\frac{1}{z}\right)}}{z^{n+1}}\, dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{t i \sin\theta}}{e^{n i\theta}}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin(\theta))}\, d\theta$$ (end of part (1))

"

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(-(n\theta - t\sin\theta)) + i\sin(-(n\theta - t\sin(\theta)))\right] d$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n\theta - t\sin(\theta))\,d\theta$$

$$- \frac{i}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sin}_{\text{odd}}\underbrace{(n\theta - t\sin\theta)}_{\text{odd}}\,d\theta$$

odd odd so integral = 0

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\cos(n\theta - t\sin(\theta))}_{\text{even func of odd func}}\,d\theta$$

even func of odd func
even

$$\| $$

$$\frac{2}{2\pi}\int_{0}^{\pi} \cos(n\theta - t\sin(\theta))\,d\theta$$

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\, dx$$

$= \text{sum of residues in upper } \tfrac{1}{2} \text{ plane}$

$$= \operatorname{Res}_{\frac{1+i}{\sqrt{2}}} \left(\frac{x^2+1}{x^4+1} \right) + \operatorname{Res}_{\frac{(-1+i)}{\sqrt{2}}} \frac{x^2+1}{x^4+1}$$

$$= \frac{\left(\frac{1+i}{\sqrt{2}}\right)^2+1}{4\left(\frac{1+i}{\sqrt{2}}\right)^3} + \frac{\left(\frac{-1+i}{\sqrt{2}}\right)^2+1}{4\left(\frac{-1+i}{\sqrt{2}}\right)^3}$$

using Maple for algebra (see worksheet)

$$= \pi\sqrt{2}$$

(5)

(5)

$$\frac{1}{2\pi i}\int_{C_N} \frac{\pi\cot(\pi z)}{z^4} = \sum_{n=-\infty}^{\infty}{}' \frac{1}{n^4} + \mathrm{Res}_0\left(\frac{\pi\cot(\pi z)}{z^4}\right)$$

$= 0$

using series command (see worksheet)

$$= \sum_{n=-\infty}^{\infty}{}' \frac{1}{n^4} - \frac{\pi^4}{45}$$

$$\text{or} \quad \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

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(6)

$$T_1 = \begin{pmatrix} i & -1 & -i & 1 \\ -1 & -i & 1 & i \end{pmatrix}$$

$$T_2 = \begin{pmatrix} i & -1 & -i & 1 \\ -1 & -i & 1 & i \end{pmatrix}$$

$T_1 T_2$

$=$

$$\begin{pmatrix} i & -1 & -i \\ -i & 1 & i- \end{pmatrix}$$

$$T_2 T_1 = T_1 T_2$$

Problem 4

```
> omega:= exp(Pi*I/4); expand(simplify(omega^4));
```

$$\omega := \frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}$$

$$-1 \quad (1)$$

```
> 2*Pi*I*((omega^2+1)/(4*omega^3)+((omega^3)^2+1)/(4*(omega^3)^3));
```

$$2 I \pi \left(\frac{1}{4} \frac{\left(\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}\right)^2 + 1}{\left(\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}\right)^3} + \frac{1}{4} \frac{\left(\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}\right)^6 + 1}{\left(\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}\right)^9} \right) \quad (2)$$

```
> simplify(expand(%));
```

$$\pi\sqrt{2} \quad (3)$$

Let's check this

```
> int((x^2+1)/(x^4+1),x=-infinity..infinity);
```

$$\pi\sqrt{2} \quad (4)$$

problem 5

```
> series(Pi*cot(Pi*z)/z^4,z=0);
```

$$z^{-5} - \frac{\frac{1}{3}\pi^2}{z^3} - \frac{\frac{1}{45}\pi^4}{z} - \frac{2}{945}\pi^6 z + O(z^2) \quad (5)$$

Problem 6

```
> restart;
> p := (z,w) -> z^2+w^4-1;
```

$$p := (z, w) \rightarrow z^2 + w^4 - 1 \quad (6)$$

```
> omega1:=-1.0;
  omega2:=exp(Pi*I/2);
  N:=100;
  h:= evalf(2*Pi/N):
```

$$\omega 1 := -1.0$$

$$\omega 2 := I$$

$$N := 100 \quad (7)$$

```
> for k from 1 to 2 do
  f := unapply(-diff(p(omega1^k+exp(I*t),w),t)/diff(p(omega1^k+exp(I*t),w),
  w),t,w);
  for m from 1 to 4 do
  u:=omega2^m;
  if k=1  then tt:= 0
  elif k=2 then tt:=  -evalf(Pi)
  end if;
  for j from 1 to N do
  u:= u + f(tt,u)*h:
  tt:=tt+h:
  od:
  print(omega1^k,omega2^m,u):
  od:
  od:
```

$$-1.0, I, -1.000008438 + 0.0007253433818 I$$

$$-1.0, -1, -0.0007253433818 - 1.000008438 I$$

$$
\begin{array}{c}
-1.0,\ -I,\ 1.000008438 - 0.0007253433818\ I \\
-1.0,\ 1,\ 0.0007253433818 + 1.000008438\ I \\
1.00,\ I,\ -1.000008437 + 0.0007253433961\ I \\
1.00,\ -1,\ -0.0007253433961 - 1.000008437\ I \\
1.00,\ -I,\ 1.000008437 - 0.0007253433961\ I \\
1.00,\ 1,\ 0.0007253433961 + 1.000008437\ I
\end{array}
\quad (8
$$