Mathematics 40485: Test II

Instructor: Andrew Sommese

March 21, 2013

Name: ....

This test is due at the start of class on Tuesday, March 26, 2013.

This test is being conducted under the honor code. From the time you receive this exam *until the time it is collected you may not communicate about course material with anyone (in the class or not)*, except me. This includes comments on which problems are easy or hard. *You may not use anyone else's notes or programs on the test.* You are allowed and encouraged to use your notes, Maple, any books, or material on websites (at your own risk of course) while doing the tests.

If you find some problem on the test which you think is unclear (or wrong), please contact me immediately—my e-mail address is `sommese@nd.edu`.

If necessary I will send a correction or clarification to the class by e-mail. It is your responsibility to check your e-mail regularly between now and Tuesday, March 26, 2013.

There are a total of 112 points below. Letting T denote the total number of points you get, your test grade will be $\min\{T, 100\}$.

In the following you must show your work and include your reasoning. You should also include your Maple work sheets used on the problems. You may use the differentiation and algebraic simplification commands (including `taylor`, which computes Taylor series and `series`, which computes Laurent series), but you must show all work as discussed in class.

Please write neatly!

Problems

In all problems involving inverse Laplace transforms, you must use the Bromwich contour.

Problem 1 (16 points) *Compute*

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x,$$

where b, a, k *are real and satisfy* $b > a > 0$ *and* $k > 0$.

Problem 2 (16 points) *Show that*

$$\int_{-\infty}^{\infty} \frac{(\cos(x)-1)}{x^2(x^2+1)} \mathrm{d}x = -\frac{\pi}{e}.$$

.

Problem 3 (20 points) *Let* $f(z) = z^4 + e^z$.

1. *Show that* $|e^z| \leq e^R$ *when* $|z| \leq R$ *for* $R > 0$.
2. *Show that* $|e^z| < |z|^4$ *when* $|z| = 2$.
3. *Explain how Rouché's Theorem lets you conclude how many roots (counting multiplicities)* $f(z) = 0$ *has when* $|z| \leq 2$.
4. *How many roots (counting multiplicities) does* $f(z) = 0$ *have when* $|z| \leq 2$.

Problem 4 (20 points) *Assuming that $\omega > 0$, obtain the inverse Laplace transform of*

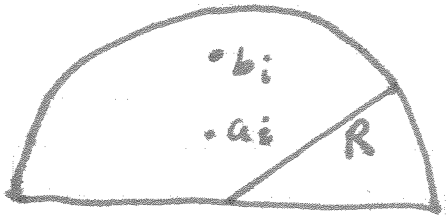
$$\frac{1}{s^2(s^2+\omega^2)}.$$

Problem 5 (20 points) *Do problem 16 of §4.5.*

Problem 6 (20 points) *Do problem 3 of §4.6. You can leave the inverse Laplace transform as a sum of residues: the simplification in the book from the sum of residues is tricky and in fact slightly wrong.*

1) $$\int_{-\infty}^{\infty} \frac{\cos(kx)\,dx}{(x^2+a^2)(x^2+b^2)} = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ikz}\,dz}{(z^2+a^2)(z^2+b^2)}\right)$$

$k>0$, $b>a>0$ all real

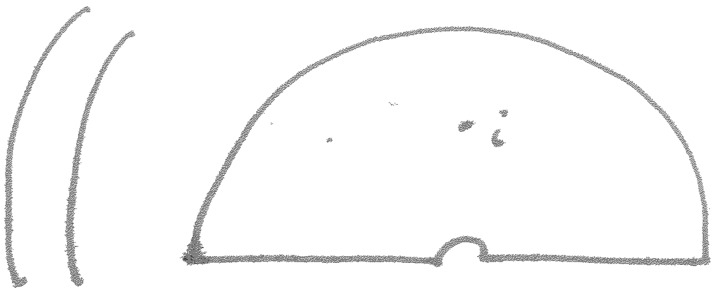


$$\int_{\frown} \boxed{}\,dx \to 0$$ by Jordan's Lemma

So $$\int_{-\infty}^{\infty} = 2\pi i\left[\mathrm{Res}_{ai} \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)} + \mathrm{Res}_{bi} \text{———}\right]$$

$$= 2\pi i\left(\frac{e^{-ka}}{2ai(b^2-a^2)} + \frac{e^{-bk}}{2bi(a^2-b^2)}\right) = \frac{\pi}{b^2-a^2}\left(\frac{e^{-ka}}{a} - \frac{e^{-kb}}{b}\right)$$

2 $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)-1}{x^2(x^2+1)}\,dx$ Note integrand has no singularity at $x=0$



$$\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x^2(x^2+1)}\,dx$$

$\int \longrightarrow 0$

by Jordan's lemma for $\dfrac{e^{iz}}{z^2(z^2+1)}$

$$\int_{-\infty}^{\infty} \frac{e^{iz}-1}{z^2(z^2+1)}\,dz$$

$$= \frac{2\pi i}{2}\,\mathrm{Res}_0\,\frac{e^{iz}-1}{z^2(z^2+1)} + 2\pi i\,\mathrm{Res}_i\,\frac{e^{iz}-1}{z^2(z^2+1)}$$

$$= \pi i \cdot i + 2\pi i\,\frac{e^{-1}-1}{i^2\,2i}$$

$$= -\pi + \pi(1-e^{-1}) = -\frac{\pi}{e}$$

③

③

ⓐ $|e^z| = |e^x e^{iy}| = e^x$

$x^2 + y^2 = R^2 \Rightarrow |x| \leq R$

So $|e^z| \leq e^R$

ⓑ $|e^z| \leq e^2 \leq 9 < 2^4$

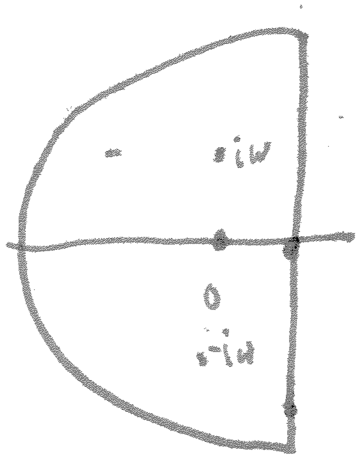
ⓒ $|z^4| > |e^z|$ on $|z| = 2$, so ≥ 0 by Rouché

z^4, $z^4 + e^z$ have the same

number of zeroes on $|z| = 2$
counting multiplicities

ⓓ z^4 has 4 zeroes so
$z^4 + e^z$ has 4 zeroes

$$\frac{1}{2\pi i}\int_{a+-i\infty}^{a+i\infty} \frac{e^{st}}{s^2\,(s^2+w^2)}\,ds$$



$\int \to 0$ by Jordan Lemma

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} = Res_0 + Res_{iw} + Res_{-iw}$$

$$\frac{e^{st}}{s^2(s^2+w^2)} = \frac{1}{w^2}\left(\frac{1}{s^2}-\frac{1}{w^2}+\frac{s^2}{w^4}\dots\right)\left(1+st+\frac{s^2t^2}{2!}\dots\right)$$

$$= \frac{1}{w^2}\cdot\frac{t}{s} + H.O.T.$$

so $Res_0\ \dfrac{t}{w^2}$

$$\mathrm{Res}_{i\omega} = \frac{e^{t\omega i}}{(2i\omega)(-\omega^2)}$$

$$\mathrm{Res}_{-i\omega} = \frac{e^{-t\omega i}}{(-2i\omega)(-\omega^2)}$$

so $\mathcal{L}^{-1}\left(\quad \right) =$

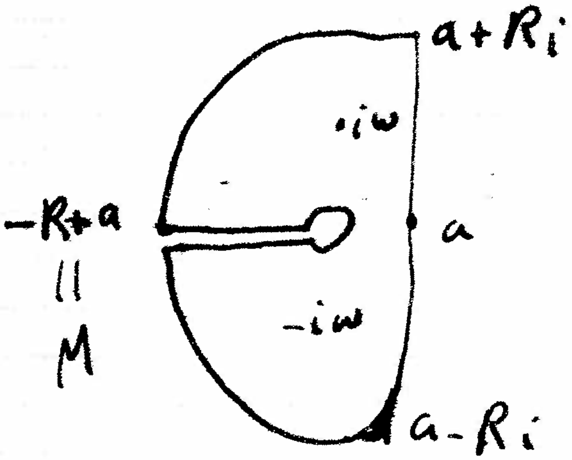
$$\frac{1}{\omega^2}\left(t + \frac{e^{i\omega t} - e^{-i\omega t}}{-2i\omega}\right)$$

$$= \frac{1}{\omega^2} t - \frac{1}{\omega^3}\sin(\omega t)$$

⑤ $\omega > 0$, $a > 0$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{sx}\ln s \, ds}{s^2+w^2}$$

Choose branch of log
$\ln 1 = 0$



$R >> a$
$R > w$

$$\int_{|s|=\varepsilon} \frac{\overbrace{|e^{sx}|}^{\to 1}\,|\ln \varepsilon + i\theta|}{\underbrace{|s^2+w^2|}_{\to w^2}} \, |\overset{\varepsilon}{\overset{\|}{}}\!\varepsilon e^{i\theta}| \, d\theta$$

$$\le C \, \varepsilon\left(\ln\frac{1}{\varepsilon} + \tilde{C}\right) \to 0$$

$s = \sigma + i\nu$ ⑦

$$\lim_{\substack{s \to \sigma^+ \\ \sigma < 0}} \frac{e^{sx} \ln s}{s^2+w^2} = \frac{e^{\sigma x}}{\sigma^2+w^2}\left(\ln(-\sigma) + i\pi\right)$$

$$\lim_{\substack{s \to \sigma^- \\ \sigma < 0}} \frac{e^{sx} \ln s}{s^2+w^2} = \frac{e^{\sigma x}}{\sigma^2+w^2}\left(\ln(-\sigma) - i\pi\right)$$

So

$$\int_{-M}^{-\varepsilon} + \int_{-\varepsilon}^{M} =$$

$$\int_{-M}^{-\varepsilon} \frac{2\pi i \, e^{\sigma x}}{\sigma^2+w^2} \, d\sigma$$

$$\underset{\substack{\text{let} \\ \sigma \to -\sigma}}{=} -\int_{M}^{\varepsilon} \frac{2\pi i}{\sigma^2+w^2} e^{-\sigma x} \, d\sigma = \int_{\varepsilon}^{M} \frac{2\pi i \, e^{-\sigma x}}{\sigma^2+w^2} \, d\sigma$$

So

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{sx}\ln s \, ds}{s^2+w^2}$$

$$+ \int_0^\infty \frac{e^{-\sigma x}}{\sigma^2+w^2}\, d\sigma$$

$$= \operatorname{Res}_{iw} + \operatorname{Res}_{-iw}$$

$$= \frac{e^{iwx}\left(\ln w + i\frac{\pi}{2}\right)}{2iw}$$

$$+ \frac{e^{-iwx}\left(\ln w - i\frac{\pi}{2}\right)}{-2iw}$$

$$= \frac{\ln w}{w}\sin(wx) + \frac{\pi}{2w}\cos(wx)$$

Problem 6

(a) $\mathcal{L}(g(t)) = \hat{G}(s)$ for any function $g(t)$

So letting $\mathcal{L}(y(t)) = \hat{Y}(s)$

given $y''' + w_0^3 y = f(t)$

$w_0 > 0$

$y(0) = y'(0) = y''(0) = 0$

$$\mathcal{L}(y''') = s^3 \mathcal{L}(y) - s^2 y(0) - s y'(0) - y(0)$$

$$= s^3 \mathcal{L}(y)$$

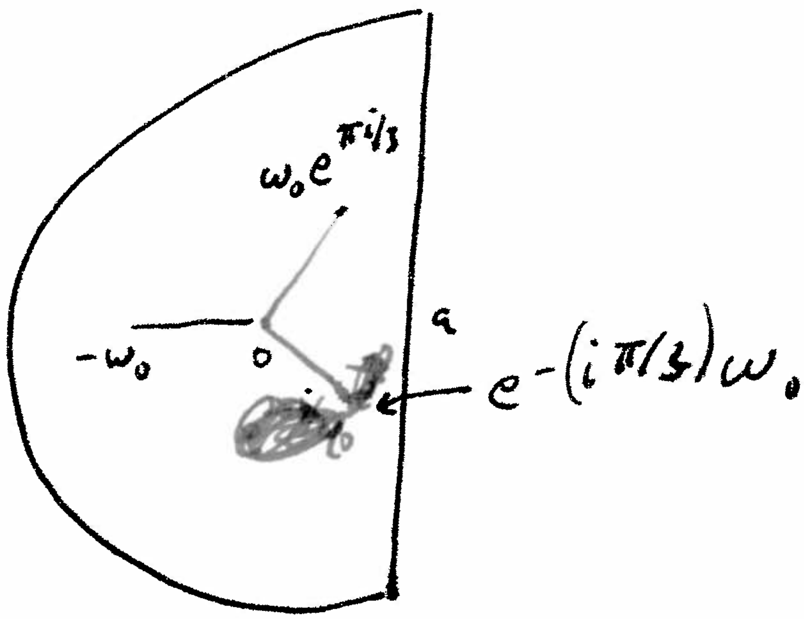
$$(s^3 + w_0^3)\,\mathcal{L}(y) = \hat{F}(s)$$

or $$\hat{Y}(s) = \frac{\hat{F}(s)}{s^3 + w_0^3}$$

Part b

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}}{s^3+\omega_0^3}\,ds$$

$=\ ?$



$\int \to 0$ Jordan lemma

So

$$\mathcal{L}^{-1}\left(\frac{e^{st}}{s^3+\omega^3}\right) =$$

$$\mathrm{Res}_{-\omega_0} + \mathrm{Res}_{\omega_0 e^{\pi i/3}} + \mathrm{Res}_{\omega e^{-\pi i/3}}$$

$$= \frac{e^{-\omega_0 t}}{3\omega_0^2} + \frac{e^{\omega_0 e^{\pi i/3} t}}{3\omega_0^2 e^{2\pi i/3}}$$

$$+ \frac{e^{\omega_0 e^{-\pi i/3} t}}{3\omega_0^2 e^{-2\pi i/3}}$$

$e^{\pi i/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $\qquad$ $e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$e^{-\pi i/3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ $\qquad$ $e^{-2\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$\mathcal{L}(h(t)) = \frac{1}{s^3+\omega^3}$$

$$\mathcal{L}(y) = \hat{Y}(s)$$

$$\mathcal{L}(f) = \hat{F}(s)$$

so $$\mathcal{L}(f*h) = \hat{F}(s)\cdot\frac{1}{s^3+\omega^3} = \hat{Y}(s)$$

$$\Rightarrow y(t) = f*h(t)$$

$$= \int_0^t f(t')h(t-t')\,dt'$$