Homework 10 (Due Monday, December 8)
pg 544 Problems 1 and 5 from the Exercises for Section 13.8
(You may use Ito’s Formula).
pg 547 Problems 2, 3, 4, and 5 from the Exercises for Section 13.9.
pg 561 Problem 1ab from the Exercises for Section 13.12.
§13.8  pg 544

\[ d(tW_t) = W_t \, dt + t \, dw_t + \frac{1}{2} \int_0^t \langle W_r \rangle \, dr \]

\[ tw_t - ow_0 = \int_0^t \omega_t \, dw_t + \int_0^t \omega_t \, dt \]

\[ \frac{d}{3} \left( \frac{W_t^3}{3} \right) = \frac{1}{3} W_t^2 \, dw_t + W_t \, dt \]

So,
\[ \int_0^t W_t^2 \, dw_t + \int_0^t \omega_t \, dt = \frac{W_t^3}{3} \]

\[ E(\left( \int_0^t \omega_t \, dw_t \right)^2) = E(\int_0^t \omega_t^2 \, dt) \]

It r. Isom.
\[ \int_0^t E(\omega_t^2) \, dt = \int_0^t \omega_t^2 \, dt = \frac{t^2}{2} \]
5. Using an integrating factor

\[ d(e^{\beta X_t}) = \beta X_t e^{\beta X_t} dt + e^{\beta X_t} dW_t \]

So,

\[ d(e^{\beta X_t}) = e^{\beta X_t} dW_t \]

\[ = d(e^{\beta X_t}) - \beta e^{\beta X_t} W_t dt \]

Integrating from 0 to t

\[ e^{\beta X_t} = \int_0^t e^{\beta s} \, dW_s = e^{\beta X_t} - \beta \int_0^t e^{\beta s} W_s \, ds \]

\[ \Delta X_t = W_t - \beta \int_0^t e^{-\beta (t-s)} W_s \, ds \]
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2. \[ dY_t = 4 W_t^3 dW_t + 6 W_t^2 \, dt \]
\[ dE(Y_t) = 6 E(W_t^2) \, dt \]
\[ \Rightarrow E(W_t^4) = E(Y_t) = 3 t^2 \]

3. \[ dY_t = W_t \, dt + t \, dW_t = \frac{Y_t}{t} \, dt + t \, dW_t \]
(and in particular \( Y_t \) is an \( \mathcal{F}_t \)-process)

4. Note \( \xi_1 = \cos(W_t) \) and \( \xi_2 = \sin(W_t) \). Now use 5c below.

5. a) \[ dX_t = \frac{1}{1+t} \, dW_t - \frac{X_t}{1+t} \, dt \]
   b) \[ dX_t = \cos(W_t) \, dW_t + \frac{-\sin(W_t)}{2} \, dt \]
   \[ = \sqrt{1-X_t^2} \, dW_t - \frac{X_t}{2} \, dt \]
\[ d\xi_t = -a \sin(W_t) dW_t \]
\[ + a \cos(W_t) dt \]
\[ d\eta_t = b \cos(W_t) dW_t - \frac{b}{2} \sin(W_t) dt \]

So
\[ d\xi_t = -\frac{a}{b} \eta_t dW_t - \frac{a}{2} \xi_t dt \]
\[ d\eta_t = \frac{b}{a} \xi_t dW_t - \frac{b}{2} \eta_t dt \]

Ex. 13.12

1. \( W(t|x) \) is a standard Wiener process (aka Brownian motion)

2. \( W(0|x) = 0 \)

3. \( W(t|x) \) is continuous, real-valued, and \( W(t|x) \) for each \( w \in \mathbb{R} \)

4. \( \xi + \eta \) is independent of \( \xi \): \( \mathbb{P}(\xi \in A) \) for any \( A \subseteq \mathbb{R} \)

5. \( \xi + \eta \) is independent of \( \eta \): \( \mathbb{P}(\eta \in B) \) for any \( B \subseteq \mathbb{R} \)
\[ E(\alpha W(t/\alpha)) = \alpha E(W(t/\alpha)) = 0 \]
\[ E(\alpha^2 W(t/\alpha)^2) = \alpha^2 E(W(t/\alpha)^2) = \alpha^2 \frac{t}{\alpha^2} = t \]

\( b \)
\[ \beta(t) = W(t+\alpha) - W(\alpha) \]

1. \( \beta(0) = 0 \)
2. \( W(t) \) continuous for each \( w \in \mathbb{R} \)
3. \( \beta(t)(w) = W(t+\alpha)(w) - W(\alpha)(w) \) is continuous

\( \beta(t+h) - \beta(t) = W(t+\alpha + h) - W(\alpha) - (W(t+\alpha) - W(\alpha)) \]
\[ = W(t+\alpha + h) - W(t+\alpha) \]

is independent of \( \sigma \left( W_n \left| n \leq t+h \right. \right) \) and similarly \( \sigma \left( W_n+h \left| n \leq t \right. \right) \)
\[ = \sigma \left( \beta(n) \left| n \leq t \right. \right) \]