Lecture 24  Dec. 3, 2014

What does \( \int_a^b f(t, B_t) \, dB_t \) mean?

For each \( a \leq t_n \leq b \) we have

\[
\frac{1}{N} \sum_{i=0}^{N-1} f(t_i, B_{t_i}(w))(B_{t_{i+1}}(w) - B_{t_i}(w))
\]

This is a measurable function on \( \Omega \).

(a) as \( \Delta = \max(t_{i+1} - t_i) \to 0 \), \( \frac{1}{N} f_{i, \pi} \to \int_a^b f(t, B_t) \, dB_t \)

in an \( L^2 \) sense

(b) for a set \( U \subseteq \Omega \) of prob. 1

\[
\frac{1}{N} f_{i, \pi}(w) \quad \text{converges}
\]

We have convergence of the \( f_{i, \pi} \) on a "big" set of \( \Omega \) by Egoroff's THM, but the set is sequence dependent.
Classically (pre-δt), f was known to cause problems!

$$\int f(t, B_t) \, dt$$ was every

but at the level of w+R,

$$\int f(t, B_t(w)) \, dB_t(w)$$ makes little sense.

Classically, people made sense when they could use the chain rule to reduce to

Since dB typically comes as a noise term, we only care about the process, so the fact that on the w+R level, matter makes little sense is irrelevant!

Before we turn to Martingales and stopping times, let's give one example of how bad Brownian...
Some cases of the Ito formulation of products rule

\[ d(f(B_t)g(B_t)) = \left( \frac{df}{dx} g + f \frac{dg}{dx} \right) dB + \left( \frac{1}{2} \frac{d^2f}{dx^2} g + \frac{df}{dx} \frac{dg}{dx} + \frac{1}{2} \frac{dg}{dx} \frac{d^2g}{dx^2} \right) dt \]

\[ d(g(t)B_t) = g(t) dB_t + g(t) dt \]
How bad is $B_t(w)$ for $w \in \mathcal{S}$?

$B_t(w)$ is nowhere differentiable for a set of probability 1.

Fix $t$.

Let us compute the measure of the set where

$$\lim_{k \to 0} \frac{B(t+h) - B(t)(w)}{h}$$

does not exist.

$$\bigcap_{k=1}^{\infty} \left\{ w \in \mathcal{S} \mid \left| \frac{B(t+\frac{1}{N}) - B(t)}{\frac{1}{N}} \right| \geq k \right\} \text{ for infinitely many } N$$

is a set where the limit does not exist.

Since $Z_1 \geq Z_2 \geq \ldots$

we conclude $P(\bigcap Z_i) = \lim_{i \to \infty} P(Z_i)$

Let us show $P(Z_i) = 1$. 

\[ Z_k = \bigcap_{m=1}^{\infty} \bigcup_{N=m}^{\infty} \{ \omega \mid \frac{|B_{x+N+\frac{1}{m}} - B_x|}{\sqrt{N}} \geq k \} \]

Since \( Y_m > Y_{m+1} \) \( \ldots \)

it suffices to show for any fixed \( m \)

\[ P(Y_m) = 1. \]

If we show \( P\left( \left\{ \omega \mid \frac{|B_{x+N+\frac{1}{m}} - B_x|}{\sqrt{N}} \geq k \right\} \right) \rightarrow 1 \) as \( m \to \infty \)

we have \( P(Y_m) = 1. \)

\[ \frac{B_{x+\frac{1}{m}} - B_x}{\sqrt{\frac{1}{m}}} = \frac{B_{Y_m}}{\sqrt{\frac{1}{m}}} = \frac{\overline{X}}{\sqrt{\frac{1}{m}}} \]

where \( \overline{X} \in N(0,1) \)

so \( \# ) \) is \( \lim_{m \to \infty} P\left( |\overline{X}| \geq k \sqrt{\frac{1}{m}} \right) \rightarrow 1 \)

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Fact: For \( w \in \mathbb{R} \) in a prob. 1 set, \( B_x(w) \) is not of bounded variation, e.g., \( \sum |B_{w+\frac{1}{m}}(w) - B_x(w)| \to \infty \).

$X_n$ a martingale with $X_0 = x \quad a < x < b$

$T$ stop if either $a$ or $b$ is reached

$p(T<\infty) = 1$.

$X_{m+T}(w) \to X_T(w)$

$|X_{m+T}| \leq \max\{1a, 1b\}$

$\Rightarrow \lim_{m \to \infty} E(X_{m+T}) = E(\lim_{m \to \infty} X_{m+T}) = E(X_T)$.

$\Rightarrow \lim_{m \to \infty} E(x) = E(x)$

$\Rightarrow \lim_{m \to \infty} x = x$

True in continuous time.

Let us apply it to $B_t$, $B_0 = x \quad a < x < b$.

To do so we need $p(T<\infty) = 1$.

Useful Fact: If $B_t$ with $B_0 = 0$ is a Brownian motion, then

$\tilde{B}_t = \frac{B_{t/c^2}}{c} \quad \text{is a Brownian motion for any } c \neq 0$. 

Assume $c > 0$ for simplicity. $\tilde{B}_0 = 0$, $\tilde{B}_t$ has continuous paths. 

$E(\tilde{B}_{t/c}^2) = c^2 \frac{t}{c^2} = t \quad \text{so variance of } \tilde{B}_t \text{ is } t. \tilde{B}_t \text{ is thus }\mathcal{N}(0, t)$. Finally...
It is easy to check $B_t - B_s$ in $N(0, t-s)$ and independent of $F_s$. Now onto $P(T < \infty) = 1$

I follow Rogers+Williams Vol. 1 on the following argument for the unboundedness of Brownian motion.

Recall that given a probability space $(\Omega, \mathcal{F}, P)$, a Brownian motion is a family of processes $B_t$ for $t \geq 0$ such that:

(a) $B_0(w) = 0$ all $w \in \Omega$ (this is often relaxed)

(b) $t \rightarrow B_t(w)$ is continuous in $[0, \infty)$ for all $w \in \Omega$,

(c) for all $t, \Delta t \geq 0$, $B_{t+\Delta t} - B_t$ is independent of $\mathcal{F}_t = \sigma(B_r | r \leq t)$ and is distributed $N(0, \Delta t)$.

There are several useful invariance properties. If $B_t$ is a Brownian motion, then:

(a) $-B_t$ is a Brownian motion (e.g.)

(b) for any fixed $a \geq 0$, $B_{t+a} - B_a$ is a Brownian motion (straightforward, use (a)) — these and (c) are listed among homework problems.

takes some work...
To see how these may be used

Claim $P\left(\sup_t B_t = +\infty, \inf_t B_t = -\infty\right) = 1$

This implies for any $a \in \mathbb{R}$, $\{t \mid B_t = a\}$ is not bounded above with probability 1.

Thus $B_t$ will repeatedly get arbitrarily positive and the arbitrarily negative.

Proof

Let $Z = \sup_{t \geq 0} B_t$

Since for $c > 0$

$cB_t / c$ is a Brownian motion

$cZ$ has the same distribution as $Z$.

So $\forall k > 0$ $P(Z \leq k) = P(Z \leq 0)$

and in fact $P(k < Z \leq k) = P(Z = 0)$

So $Z$ has only 1 value with prob 1, either 0 or $\infty$. Let $p = P(Z = 0)$.

We show $1 - p = 0$.

$P(Z = 0) \leq P(B_1 \leq 0$ and $B_u \leq 0$ all $u \geq 1) = P(B_1 \leq 0$ and $B_{1+u} - B_1 \leq c^{\mathbb{R}}$ for $u > 0$)
Since $B_{1+t} - B_t$ is a Brownian motion and $\sup B_{1+t} - B_t \leq c < \infty$ we conclude $\sup B_{1+t} - B_t = 0$

So $p = P(B_1 \leq 0), P(Z \leq 0) = \frac{1}{2}p$

$\Rightarrow p = 0$

The inequalities above the martingale coming from $e^{\omega B_t - \frac{\omega^2}{2}t}$ combined with optional stopping yield many precise results on distributions and times until events happen.
Let $B_t$ be a Brownian motion with $B_0 = x$. Assume $a < x < b$. We know that $T$ is time until $B_t$ reaches $a$ or $b$ is finite. What is $E(T)$?

The usual optional stopping theorem tells us $E(B_T) = x$ but this also equals $bp_{xb} + a p_{xa}$ where

$$p_{xb} = \text{probability } B_t = b \text{ before } a$$
$$p_{xa} = \text{probability } B_t = a \text{ before } b$$
$$1 - p_{xb}$$

So

$$p_{xb} = \frac{x-a}{b-a} \quad p_{xa} = \frac{b-x}{b-a}$$

$B^2_T - T$ is a martingale, so

$$x^2 = E(B_T^2 - T) = E(B_T^2) - E(T)$$
$$= a^2 \frac{b-x}{b-a} + b^2 \frac{x-a}{b-a} - E(T)$$
So

\[ x^2 + E(T) = \frac{a^2 b - b^2 a}{b - a} + \frac{b^2 a^2}{b - a} x \]

\[ = -ab + (b+a)x \]

\[ E(T) + (x-a)(x-b) = 0 \]

\[ E(T) = (b-x)(x-a) \]

just like in the usual random walk.
Recall if $g$ is a function of $x$ and $y$ and harmonic

$$\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0$$

then

$$d_g B_1(x) = \frac{\partial g}{\partial x} dB_1 + \frac{\partial g}{\partial y} dB_2$$

Ex $e^x \cos y$ is harmonic

$$d(e^{B_1 \cos(B_2)} dB_1 - e^{B_1 \sin(B_2)} dB_2)$$

$$d(B_1^2 + B_2^2)$$

$$= 2B_1 dB_1 + 2B_2 dB_2 + 2dt$$
Kakutani showed results along the following lines.

Let \( u \) be a real valued, non-negative harmonic function on the unit disk \( D \). Assume \( |u| \leq M \). Let \( (B_1(t), B_2(t)) \) be a Brownian motion starting at \( (x^*, y^*) \in \Delta = \{(x, y) | x^2 + y^2 = 1\} \). Let \( T(t) \) be the first value of \( t \) where

\[ B_1^2 + B_2^2 = 1. \]

Then \( u(x^*, y^*) = \lim_{{N \to \infty}} \sum_{j=1}^{N} u(B_1(T(w_j)), B_2(T(w_j))) \)

where \( w_1, \ldots, w_N \) are random elements of \( \mathbb{R} \).

Otherwise said, repeatedly run a Brownian motion starting at \( (x^*, y^*) \). Average \( u \) evaluated at the first meetings on \( \Delta \), that the Brownian motion reaches...
The limit of the average approaches $U(x^*, y^*)$. A virtue of this is that it works in very bad situations (not like the disk).

Pr. $U(B_1(t), B_2(t))$ is a martingale. Indeed $dU(B_1(t), B_2(t)) = \frac{\partial U}{\partial x}(B_1(t), B_2(t)) dB_1 + \frac{\partial U}{\partial y}(B_1(t), B_2(t)) dB_2$ (since $\frac{2}{2x^2} + \frac{2}{2y^2} = 0$)

So $U(B_1(t), B_2(t)) = U(x^*, y^*) +$

$$\int_0^t \frac{\partial U}{\partial x}(B_1, B_2) dB_1 + \int_0^t \frac{\partial U}{\partial y}(B_1, B_2) dB_2$$

$|U| \leq M$, and $B_1, B_2$ eventually hit the boundary. So,

$U(x^*, y^*) = E(U(B_1(t), U(B_2(t))$. By the law of large numbers we are done.