Lecture 25, Monday December 8: Discussion of
You can read about diffusion processes and
the forward-backward Chapman-Kolmogorov
equations in §13.3. Focus on the connection between
Brownian motion and the C-K equations is slight.
It underlies the connection needed in the to calculate
with the study of processes (master, diffusion)
and if we were going further, it would play
a more important role. In physics, the process
is second to the C-K equations (often called
the master equation following a 1940 article of
Nordheim, Lamb, + Uhlenbeck). The very word diffusion
makes clear the physics origin of much of this theory.
A well-known book approaching this material from the
point of view is
Van Kampen, Stochastic Processes in Physics
and Chemistry

For simplicity, we work with a Brownian motion \( B_t \)
and the filtration \( \mathcal{F}_t = \sigma (B_z | z \leq t) \).

The density of \( B_t - B_z \) with \( B_z = y \), i.e., \( P(B_t - B_z = z | B_z = y) \),
is \( \Psi(z-y) \) where \( \Psi(z) \sim N(0, t-z) \). Similarly, for \( B_z - B_0 \)
with \( B_0 = x \), the density is \( \Psi(y-x) \) where \( \Psi(y) \sim N(0, t) \).
Since \( B_t - B_s \) and independent of \( B_s - B_0 \), we have \( B_t - B_0 \) with \( B_0 = x \) has the density
\[
\int_{-\infty}^{\infty} y(z-y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy
\]

Letting \( p_x(u,v) \) be the density of \( B_{t+s} - B_u \) with \( B_0 = u \), we have from \( \bigcirc \)
\[
p_{t+s}(x,z) = \int_{-\infty}^{\infty} p_s(x,y) p_t(y,z) \, dy
\]

the C-K equation.

To work with this, usually we define a semigroup acting on the Borel functions:

\[
P_t(f) = E(f(B_{t+s}) | \mathcal{F}_s)
\]

Though defined on \( \mathbb{R} \), we have by the Markov property that the process at time \( t+s \) only depends on \( B_s \), so we may (as undergrads do) think of the conditional expectation \( E(\cdot | \mathcal{F}_s) \) as a function of the value of \( B_s \), e.g., \( B_0 = x \) giving
\[
P_t(f)(x) = \int_{-\infty}^{\infty} f(z) p_t(x,z) \, dz
\]
\[
P_{t+h}(f) = P_t(P_{h}(f)) = P_{h}(P_{t}(f)),
\]

Indeed,

\[
P_t(P_h(f)) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(z) P_h(y,z) \, dz \right) P_t(x,y) \, dy
\]

\[
= \int_{-\infty}^{\infty} f(z) \left( \int_{-\infty}^{\infty} P_h(y,z) P_t(x,y) \, dy \right) \, dz
\]

\[
= \int_{-\infty}^{\infty} f(z) P_{t+h}(x,z) \, dz \quad \text{using \# on}
\]

the preceding page.  \( P_0(f) = E(f(B_0) \mid B_0 = x) = f(x), \) i.e. \( P_0 = I \)

Let's derive the forward equation

\[
\lim_{h \to 0} \frac{P_{t+h}(f) - P_t(f)}{h} = P_t \left( \lim_{n \to \infty} \frac{P_h - I}{n} \right) = P_t(G(f))
\]

so we need to compute \( G(f) \).

It equals the limit as \( h \to 0 \) of
\[
\frac{P_h(f) - f}{h} = \frac{\int_{-\infty}^{\infty} f(z) p_h(x,z) \, dz - \int_{-\infty}^{\infty} f(x) p_h(x,z) \, dz}{h}
\]

Here we have used \( \int_{-\infty}^{\infty} f(x) \, p_h(x,z) \, dz \)

\[
= f(x) \int_{-\infty}^{\infty} p(x,z) \, dz = f(x)
\]

Letting \( z = x + \sqrt{h} \, u \), we have

\[
\int_{-\infty}^{\infty} \frac{f(x + \sqrt{h} \, u) - f(x)}{\sqrt{h}} \, p_h(x, x + \sqrt{h} \, u) \, du
\]

\[
= \int_{-\infty}^{\infty} \frac{f(x + \sqrt{h} \, u) - f(x)}{\sqrt{h}} \, e^{-\frac{u^2}{2}} \, du =
\]
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{f(x+\sqrt{n}u) - f(x)}{u} e^{-u^2/2} du \]

Assuming \( f \) is twice continuously differentiable with compact support, we have

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( f(x) u + \frac{u^2 f''(x) + O(\sqrt{n})}{\sqrt{n}} \right) e^{-u^2/2} du \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{u^2 f''(x) e^{-u^2/2} du}{2} \]

\[ \to \frac{f''(x)}{2 \sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2/2} du \]

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2/2} du = \frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du \]

\[ = 1 \]

So \( \delta f = \frac{1}{2} f''(x) \).
Thus we have (for twice diff. $f$)

\[
\frac{\partial}{\partial x} P_x(t) = - \frac{1}{2} P_x(f'') \quad \text{and} \quad P_x G(f) = \frac{1}{2} P_x(f''')
\]

\[
= \frac{1}{2} \left[ \int_{-\infty}^{\infty} P_x(x,y) \frac{\partial^2 f(y)}{\partial y^2} dy \right]_{-\infty}^{\infty}
\]

integrating by parts twice

So

\[
\int_{-\infty}^{\infty} \left( \frac{\partial}{\partial x} P_x(x,y) - \frac{1}{2} \frac{\partial^2}{\partial y^2} P_x(x,y) \right) f(y) dy = 0
\]

for all twice continuously diff. $f$

\[
\Rightarrow \quad \frac{\partial}{\partial x} P_x(x,y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} P_x(x,y)
\]

The heat equation (the simplest diffusion equation)

Note the backward equation is

\[
\frac{\partial}{\partial t} P_x = \frac{1}{2} \frac{\partial^2}{\partial x^2} P_x(x,y)
\]
If we had Brownian motion in the plane, we would have the infinitesimal generator

\[ \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \]

Now on to Black–Scholes.
we follow "The Mathematics of Financial Derivatives" (Wilmott, Howison, Dewynne, Cambridge) European, American, Asian, barrier, lookback, + so on

We have a stock price $S$

European call on the stock

You buy the right on a certain day in the future to buy a share of the stock many times at a certain price called the strike price.

Let's say the stock had $S = 100$ at the expiration date. If the strike price was $\geq 100$, you would definitely be foolish to exercise or use the call. If the strike price was $< S$, you would make money by exercising the call.

(Similarly a put is the right to sell a share...
of stock at a certain price.

Let \( C(t) \) be the price of a call at time \( t \). We assume the expiration date is \( T \). Similarly with \( P(t) \) for a put with same strike price and expiration date. At expiration \( C(S, T) = \max (S - E, 0) \) and \( P(S, T) = \max (E - S, 0) \). Often we suppress the dependence of \( S \) and write \( \tilde{C}(t), \tilde{P}(t) \).

We assume that there is an interest rate \( r \) so money may be invested at safely. So A dollars will be worth \( e^{rt} \) at \( t \) units of time in the future. Similarly A dollars at time \( t \) in the future is worth \( A e^{-rt} \) now.

Let you had a portfolio

\( 1 \) share of the stock - worth \( S \)

\( 1 \) put

The portfolio has

\[-1 \text{ call} \]

The value \( \Pi(t) = \tilde{S}(t) + \tilde{P}(t) - C(t) \)

at \( t = T \), \( \Pi(T) = S(T) + \max (E - S, 0) - \max (S - E, 0) = E \)
No matter what happens 
\[ \pi(T) = E. \]

So we conclude 
\[ \pi(t) = E e^{-r(T-t)} \]

The relation 
\[ S(t) - C(S, t) + B(S, t) = e^{-r(T-t)} \]
is called "put-call parity."

What should \( C(S, t) \) be?

The SDE for \( S(t) \) is 
\[
\frac{dS}{S} = \mu dt + \sigma dB_t
\]

(\text{so called \textit{lognormal random walk}})

Assume \( S(0) = 1 \)
\[ \frac{d\ln S}{S} = \frac{dS}{S} - \frac{(dS)^2}{S^2} = \frac{ds}{s} - \sigma^2 dt = (\mu - \sigma^2) dt + \sigma dB_t \]

\[ \ln S = (\mu - \sigma^2) t + \sigma B_t \]

\[ S = e^{(\mu - \sigma^2) t + \sigma B_t} \]
\[ dC = \frac{dc}{ds} ds + \frac{dc}{dt} dt + \frac{1}{2} \frac{d^2c}{ds^2} (ds)^2 \]

\[ = S \frac{dc}{ds} u dt + S \frac{dc}{s} 0 dB_t + \frac{1}{2} \left[ \frac{d^2c}{ds^2} \right] So^2 dt \]

\[ + \frac{dc}{dt} dt \]

\[ = \left( u S \frac{dc}{ds} + \frac{1}{2} \frac{d^2c}{ds^2} So^2 + \frac{dc}{dt} \right) dt + So \frac{dc}{ds} dB_t \]

This was all standard before Black and Scholes. Their idea:

Construct a make believe portfolio consisting of \(-\Delta\) share of Stock and I call

\[ \Pi = C - \Delta S \]

Make believe \(\Delta \) weak

\[ d\Pi = dC - \Delta ds \]

If we take \( \Delta = \frac{dc}{ds} \)

we get
We get
\[
\frac{d}{dt}\Pi(t) = \left( \frac{dC}{dt} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt
\]

so \( \Pi(t) \) is risk-free

so \( \Pi(t) = \Pi(0) e^{rt} \)

\( n \ d\Pi = n \Pi(t) dt \)

so
\[
\frac{dC}{dt} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - r(C - \frac{S_0}{e^{rt}}) = 0
\]

This is the famous

Black-Scholes differential equation.

\( C \) may be replaced with any option or financial instrument.