Problem 1 (12 points) A family decides to keep having children until they have a girl. Assume
\[ P(B) = P(G) = \frac{1}{2}, \]
and that the genders of the different children are independent. Let \( N \) be the number of children the family will have had when they have their first girl.

1. What is the \( E(N) \), the expectation of \( N \)?

2. What is the Variance of \( N \)?

Problem 2 (12 points) You have a six-sided die. Assume that the number \( N \) of dots on the side facing up after a toss has probability
\[ P(N = n) = C \ln(n + 1), \]
for \( 1 \leq n \leq 6 \) and some constant \( C \).

1. What is \( C \)?

2. To two decimal points, what is \( E(N) \)?

Problem 3 (12 points) A continuous nonnegative random variable \( X \) has distribution function \( F(x) \) and probability density function \( f(x) \).

1. In terms of \( F \), what is the distribution function of \( \sqrt{X} \)?

2. In terms of \( f(x) \), which you may assume continuous, what is the density function of \( \sqrt{X} \)?
Problem 4 (11 points) $X$ and $Y$ are independent exponential random variables with parameters $\lambda > 0$ and $\mu > 0$ respectively. Compute

$$P(X < Y).$$

Problem 5 (24 points) Let $\mathbb{D}$ denote the diamond

$$\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}.$$ We take $S = \mathbb{D}$ and $\mathcal{F}$ the usual space of Borel subsets of $\mathbb{D}$. For the probability of a set $E \in \mathcal{F}$, we take $P(E)$ equal to the area of $E$ divided by 2, i.e.,

$$P(E) = \frac{1}{2} \int_E \text{d}x\text{d}y.$$ Let $X$ be the random variable given by $(x, y) \to x$ and $Y$ the random variable given by $(x, y) \to y$.

1. Verify that the joint density of $X$ and $Y$ is $\frac{1}{2}I_{\mathbb{D}}(x, y)$, where $I_E$ is the indicator function of a set $E$.

2. What are the density functions of $X$ and of $Y$?

3. Are $X$ and $Y$ independent?

4. Compute $E(Y^2 | X)$.

Problem 6 (12 points) Let $\mathbb{D}$, $X$, $Y$, $\mathcal{F}$, and $P$ be as in Problem 5. Let $U = X + Y$ and $V = X - Y$.

1. What are the density functions of $U$ and of $V$?

2. Are $U$ and $V$ independent?
Problem 7 (12 points) Let \( G(S) = E(s^X) \) be the probability generating function of a random variable \( X \). Assume that \( X \) takes nonnegative integer values, i.e., 0, 1, 2, \ldots, with
\[
P(X = k) = \frac{1}{e \cdot k!}.
\]

1. What is \( G(s) \) in closed form?
2. Using \( G'(1), G''(1), \) and \( G'''(1) \) compute \( E(X^3) \).

Problem 8 (15 points) Let \( X_1, X_2, X_3 \) be three independent exponential random variables with parameter \( \lambda = 1 \) for each integer \( i = 1, 2, 3 \).

1. What is \( E(X_3|X_1, X_2, X_3) \)?
2. What is \( E(X_3|X_1, X_2) \)?
3. What is \( E(X_2X_3|X_1, X_2) \)?