ACMS 40390: Sample Test II

February 27, 2019

By signing you confirm that you are following the honor code for this test.

Name: ............................................................

To receive credit you must show your work.

<table>
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<th>Maximum Points</th>
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Some Useful Results

The following result may be of use. You may assume it in any argument you need to give.

**Runge-Kutta Method of Order Four** For the ordinary differential equation \( y' = f(t, y) \) on \([a, b]\) with initial condition \( y(a) = \alpha \) and stepsize \( h \) we have

\[
egin{align*}
w_0 & = \alpha \\
k_1 & = hf(t_i, w_i) \\
k_2 & = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right) \\
k_3 & = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right) \\
k_4 & = hf(t_i + h, w_i + k_3) \\
w_{i+1} & = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{align*}
\]

Letting \( f(x) \in C^5[a, b] \) with \( a < b \), we use the book’s notation \( S(f, a, b) \) for the Simpson rule approximation to \( \int_a^b f(x)dx \). We use

\[
\text{Err}(f, a, b) := \frac{\left| S(f, a, b) - S\left(f, a, \frac{a+b}{2}\right) - S\left(f, \frac{a+b}{2}, b\right) \right|}{15}
\]

as an estimate for the error

\[
\left| \int_a^b f(x)dx - \left( S\left(f, a, \frac{a+b}{2}\right) + S\left(f, \frac{a+b}{2}, b\right) \right) \right|.
\]
Problems

In the following you must show your work.

Problem 1 (7 points total) We want to approximate

$$\int_{0}^{1} (1 + 1.06\pi x)\sqrt{\sin(\pi x)} \, dx$$

with at most an error of 0.005. We use the Adaptive Simpson’s rule and find

$$\frac{|S(f, 0, 1) - S(f, 0, 0.5) - S(f, 0.5, 1)|}{15} = 0.00508.$$

Should we accept the result or should we subdivide further and if so how? Explain your answer.

Since 0.00508 > 0.005, we need to subdivide further. Use the subintervals [0, 0.25], [0.25, 0.5], [0.5, 0.75], [0.75, 1.0].
Problem 2 (12 points total) Let \( y' = t^2 y^2 \) on \([0, 1]\) with initial condition \( y(0) = 24 \).

1. What is an explicit local solution near 0?

2. Is it unique and if so why?

3. Is there a continuously differentiable solution on all of \([0, 1]\)? (Either show there is by explicitly constructing the solution or show there is no solution on all of \([0, 1]\).)

\[
y = \frac{1}{24 - t^2/3}
\]

The solution is unique since \( t^2 y^2 \) is continuously differentiable with respect to \( t \) and \( y \).

The solution does not exist at \( t = 0.5 \).
**Problem 3 (7 points total)** What integration method does the Runge-Kutta Method of Order Four applied to solving \( y' = f(t) \) on \([1, 2]\) with initial value \( y(1) = 0 \) and \( h = 1.0 \) reduce to. (To receive credit you must show this explicitly.)

\[
\begin{align*}
k_1 &= hf(1) \\
k_2 &= hf(1.5) \\
k_3 &= hf(1.5) \\
k_4 &= hf(2)
\end{align*}
\]

So we get

\[
\frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6} = \frac{(f(1) + 4f(1.5) + f(2))}{6},
\]

i.e., Simpson’s method.

**Problem 4 (7 points total)** Use the midpoint method to approximate the solution of \( y' = t \sin(y) \) on \([1, 1.5]\) with initial value \( y(1) = 2 \) and \( h = 0.5 \).

\[
f(t, y) = t \sin(y).
\]

\[
2.0 + hf(1 + h/2, 2.0 + f(1.0, 2)h/2) = 2.0 + 0.5(1.25 \sin (2.0 + \sin(2.0)0.25)).
\]
Problem 5 (7 points total) Use Taylor's method of order three to approximate the solution of $y' = ty$ on $[0, 0.5]$ with initial value $y(0) = -1$ and $h = 0.5$.

Note

$y' = ty$

$y'' = y + t^2 y$

$y''' = 2ty + ty + t^3 y = 3ty + t^3 y$.

So we get

$-1 + 0.5 \cdot 0 + 0.25/2 \cdot (-1 + 0 \cdot (-1)) + 0.125/6 \cdot 0 + 0 = -1 - 0.125 = -1.125$.

Problem 6 (7 points) Use the Euler method with $h = 0.5$ to solve $y'' - 3y' + 2y$ on $[1, 1.5]$ and the initial conditions $y(0) = 2, y'(0) = 5$.

Letting $y_1 = y$ and $y_2 = y'$ we have

$$
\begin{bmatrix}
    y_1 \\
    y_2
\end{bmatrix}' = \begin{bmatrix}
    y_2 \\
    3y_2 + 2y_1
\end{bmatrix}
$$

with initial condition

$$
\begin{bmatrix}
    y_1(0) \\
    y_2(0)
\end{bmatrix} = \begin{bmatrix}
    2 \\
    5
\end{bmatrix}.
$$

Therefore we get

$$
\begin{bmatrix}
    2 \\
    5
\end{bmatrix} + \begin{bmatrix}
    5 \\
    19
\end{bmatrix} \cdot 0.5 = \begin{bmatrix}
    4.5 \\
    14.5
\end{bmatrix}.$$


Problem 7 (7 points) Let

\[ v = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \]

Compute \( ||v||_1 \), \( ||v||_2 \), and \( ||v||_\infty \).

4, \( \sqrt{10} \), 3.
Problem 8 (7 points) Let

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \]

Compute \( \|A\|_\infty \).

Problem 9 (7 points) Let

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

Compute \( \|A\|_1 \).
Problem 10 (10 points) Let

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

Compute \(\|A\|_2\).

We need to take the largest of the square roots of absolute values of the eigenvalues of \(A^t \cdot A\).

This gives us

\[3 + \sqrt{5}\]
Problem 11 (10 points) Let

\[ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

Use the Gram-Schmidt process on \( v_1, v_2 \) to find a set of orthogonal vectors.

First let

\[ e_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}. \]

The we have

\[ e_2 = v_2 - (v_2^t e_1) e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 4/5 \\ -2/5 \end{bmatrix}. \]
Problem 12 (12 points) Let

\[ A = \begin{bmatrix}
  \frac{5\pi}{2} & \frac{1\pi}{2} \\
  \frac{1\pi}{2} & \frac{5\pi}{2}
\end{bmatrix} \]

Compute \( \cos(A) \) explicitly.

Note the eigenvalues of \( A \) are \( 3\pi \) and \( 2\pi \) with eigenvectors \([1, 1]^t\) and \([-1, 1]^t\) respectively. Therefore letting

\[ T = \begin{bmatrix}
  1 & -1 \\
  1 & 1
\end{bmatrix}, \]

we have

\[ T^{-1} = \begin{bmatrix}
  1/2 & 1/2 \\
  -1/2 & 1/2
\end{bmatrix} \quad \text{and} \quad A = T \begin{bmatrix}
  3\pi & 0 \\
  0 & 2\pi
\end{bmatrix} \cdot T^{-1}.
\]

Therefor

\[ \cos(A) = T \cdot \cos \left( \begin{bmatrix}
  3\pi & 0 \\
  0 & 2\pi
\end{bmatrix} \right) \cdot T^{-1} = T \cdot \begin{bmatrix}
  -1 & 0 \\
  0 & 1
\end{bmatrix} \cdot T^{-1} = \begin{bmatrix}
  0 & -1 \\
  -1 & 0
\end{bmatrix}. \]