ACMS 40390: Test 2

April 3, 2019

By signing you confirm that you are following the honor code for this test.

Name: ....

To receive credit you must show your work.

Problem Number	Maximum Points	Points attained
1	7	
2	12	
3	7	
4	7	
5	10	
6	7	
7	9	
8	7	
9	7	
10	10	
11	10	
12	7	
TOTAL	100	

Notation

Letting $f(x) \in C^5[a,b]$ with $a < b$, we use the book's notation $S(f,a,b)$ for the Simpson rule approximation to $\int_a^b f(x)\mathrm{dx}$. We use

$$\mathrm{Err(f,a,b)} := \frac{\left|\mathrm{S(f,a,b)} - \mathrm{S}\left(\mathrm{f,a,\frac{a+b}{2}}\right) - \mathrm{S}\left(\mathrm{f,\frac{a+b}{2},b}\right)\right|}{15}$$

as an estimate for the error

$$\left|\int_a^b f(x)\mathrm{dx} - \left(\mathrm{S}\left(\mathrm{f,a,\frac{a+b}{2}}\right) + \mathrm{S}\left(\mathrm{f,\frac{a+b}{2},b}\right)\right)\right|.$$

Problems To obtain credit in the following problems you must show your work.

Problem 1 (7 points) *We want to approximate*

$$\int_0^1 (1+\pi x)^2 \sqrt{\sin(\pi x)}\ \mathrm{d}x$$

with at most an error of 0.002*. We use the Adaptive Simpson's rule and find*

$$\frac{|S(f,0,1) - S(f,0,0.5) - S(f,0.5,1)|}{15} = 0.04976.$$

Therefore we subdivide and find

$$\frac{|S(f,0,0.5) - S(f,0,0.25) - S(f,0.25,0.5)|}{15} = 0.00097.$$

and

$$\frac{|S(f,0.5,1) - S(f,0.5,0.75) - S(f,0.75,1)|}{15} = 0.01417.$$

Should we accept the result or should we subdivide further and if so how? Circle one and justify that answer.

1. Accept the result.
2. Subdivide each of $[0, 0.25], [0.25, 0.5], [0.5, 0.75], [0.75, 1.0]$.
3. Subdivide only $[0, 0.25]$ and $[0.25, 0.5]$.
4. Subdivide only $[0.5, 0.75]$ and $[0.75, 1.0]$.
5. None of the above.

The answer is d) Since $0.00097 < 0.001$, we may use $S(f,0,0.25) + S(f,0.25,0.5)$ as the approximation for

$$\int_0^0 .5(1+\pi x)^2 \sqrt{\sin(\pi x)}\ \mathrm{d}x.$$

Since $0.01417 > 0.001$, we need to further subdivide $[0.5, 0.75]$ and $[0.75, 1.0]$.

Problem 2 (12 points total) *Let $y' = y^2$ on $[-1, 1]$ with initial condition $y(-1) = 1$.*

1. *What is an explicit local solution near -1?*
2. *Is there a continuously differentiable solution on all of $[-1, 1]$? (Either show there is by explicitly constructing a solution or show there is no solution on all of $[-1, 1]$.)*

Since

$$\frac{dy}{y^2} = dt$$

integrates to give

$$y = -\frac{1}{t+c},$$

we obtain (using $y(-1) = 1$) the local solution

$$y = -\frac{1}{t}$$

around $t = -1$.

Since this function doesn't exist at $t = 0$, we see the answer to the second question is no.

Problem 3 (7 points total) *What integration method does the modified Euler method applied to solving* $y' = f(t)$ *on* $[1, 2]$ *with initial value* $y(1) = 0$ *and* $h = 1.0$ *reduce to. (To receive credit you must show this explicitly.)*

The step of the modified Euler method is given by

$$w := 0 + (f(1) + f(1 + h))\,\frac{h}{2} = \frac{f(1) + f(2)}{2},$$

which is the trapezoid method.

Problem 4 (7 points total) *Use the midpoint method to approximate the solution of* $y' = t\sin(y)$ *on* $[0, 0.5]$ *with initial value* $y(0) = \pi$ *and* $h = 0.5$.

Letting $f(t, y) = t\sin(y)$, the step of the midpoint method is given

$$w_1 = w_0 + f(0.25, w_0 + f(0, w_0)0.25)0.5.$$

This gives

$$\pi + f(0.25, \pi)0.5 = \pi + 0.25\sin(\pi)0.5 = \pi.$$

Problem 5 (10 points total) *Use Taylor's method of order two to approximate the solution of $y' = t\sin(y)$ on $[1, 1.5]$ with initial value $y(1) = e$ and $h = 0.5$.*

Differentiating, we see $y(t)'' = \sin(y) + t\cos(y)t\sin(y)$. Thus the step of the Taylor method of order two is given by

$$w_1 = e + \sin(e)0.5 + \cos(e)\sin(e)\frac{0.5^2}{2}.$$

Problem 6 (7 points) *Use the Euler method with $h = 1$ to solve*

$$\begin{aligned} u_1' &= -u_2 \\ u_2' &= u_1 \end{aligned}$$

on $[0, 1]$ *with the initial condition*

$$\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0. \end{bmatrix}.$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem 7 (9 points) *Let*

$$v = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

be a vector in $\mathbb{R}^3$. *Compute*

1. $||v||_1 = 6$

2. $||v||_2 = \sqrt{14}$

3. $||v||_\infty = 3$.

Problem 8 (7 points) *Let*

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Compute $||A||_\infty$.

$$||A||_\infty = 7$$

Problem 9 (7 points) *Let*

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Compute $||A||_1$.

$$||A||_1 = 5$$

Problem 10 (10 points) *Let*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Compute $||A||_2$.

The easiest way to do this is to note the matrix is self-adjoint and therefore that the largest of the absolute values of the eigenvalues, i.e.,

$$\frac{3+\sqrt{5}}{2}$$

is the norm.

The other way is to compute

$$A^t A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix},$$

and then compute the square root of the largest of the absolute values of the eigenvalues of $A^t A$, i.e.,

$$\sqrt{\frac{7+3\sqrt{5}}{2}}.$$

Problem 11 (10 points) *Let*

$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Use the Gram-Schmidt process on v_1, v_2 to find a set of orthogonal vectors.

Let

$$e = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}.$$

Then we obtain

$$v_2 - (e^t \cdot v_2)e = \begin{bmatrix} \frac{16}{25} \\ -\frac{12}{25} \end{bmatrix}.$$

Problem 12 (7 points) *Let*

$$A = \begin{bmatrix} 5 & 2 & 2 & 2 \\ 1 & 4 & 1 & 0 \\ 1 & 1 & 4 & 2 \\ 1 & 0 & 0 & 5 \end{bmatrix}.$$

Using the Geršgorin Circle Theorem, show that A is nonsingular, i.e., that zero is not an eigenvalue of A.

The horizontal Geršgorin circles all contain 0, so we use the vertical Geršgorin circles

$$\begin{aligned} |\lambda - 5| &< 3 \\ |\lambda - 4| &< 3 \\ |\lambda - 4| &< 3 \\ |\lambda - 5| &< 4 \end{aligned}$$