Problems for Mathematics 617

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10 November 22, 2004: Due November 29, 2004

Problem 10.1 *Set up and solve on the square* $0 \leq x \leq 1;\ 0 \leq y \leq 1$

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 4$$

with the boundary conditions $u(x,y) = 0$ *on the boundary of the square. Use* $N = 1$*, i.e.,* $x_0 = 0, x_1 = 0,5, x_2 = 1$*,* $y_0 = 0, y_1 = 0,5, y_2 = 1$*. The setting up of the problem is the main point of the problem, i.e., computing the approximate solution* $u_{1,1}$ *requires almost no arithmetic after you have set it up.*

y" + y= with the true solution .

Problem 10.2 *You wish to solve* $y'' + y = x * (1 - x) - 2$ *on* $[0,1]$ *where* $y(0) = y(1) = 0$*. Use the Galerkin method to compute an approximate solution* $\widehat{y}(x)$ *using the hat functions and* $N = 3$*, i.e., the* x *nodes are at* $x_0 = 0$*,* $x_1 = 0,25$*,* $x_2 = 0.5$*,* $x_3 = 0.75$*,* $x_4 = 1.0$*. Compare the approximate solution* $\widehat{y}(x)$ *with the true solutions* $sin(x)$*, e.g., compute* $||\widehat{y}(x) - x(1-x)||_2$.
`Warning: the term` $\phi(x)''$ ***is not*** 0 `for` $\phi(x)$ `a hat function!`

9 November 15, 2004: Due November 22, 2004

Problem 9.1 This is worth 3 points *Solve the boundary value problem on* $[1,2]$

$$y'' = \frac{x^2(y')^2 - 9y^2 + 4x^6}{x^5};\ y(1) = 0\ \&\ y(2) = \ln(256)$$

by the shooting method (with Newton's method at $x = 2$*). Use the Runge-Kutta method of order 4 with stepsize* $h = 0.05$*, i.e.,* $x_i = 1 + i * 0.05$ *for*

$i = 0, \ldots, 20$. *For your initial guess for t in Newton's method, use $t_0 = 1.7$. Stop when $|t_i - t_{i-1}| < 10^{-6}$. Letting $\widehat{y} = (y_0, \ldots, y_{20})$ be your approximate solution of the boundary value problem with $t = t_i$, and noting that $x^3 \ln(x)$ is the exact solution, what is the estimate in the infinity norm of the absolute error, i.e.,*

$$\max_{0 \leq i \leq 20} \left\{ \left| y_i - x_i^3 \ln(x_i) \right| \right\}.$$

8 November 8, 2004: Due November 15, 2004

Problem 8.1 *Use Gerschgorin's Theorem (both the vertical and horizonal forms) to determine bounds for the eigenvalues of*

$$B := \begin{bmatrix} 1.0 & 0.25 & -0.5 & 1.0 \\ 0.25 & 1.0 & -0.3 & 0.3 \\ -0.5 & 1 & 3 & 1.1 \\ 1 & 0.3 & 1.1 & 4 \end{bmatrix}$$

Compare the bounds with the actual eigenvalues.

Problem 8.2 *With the matrix B from Problem 8.1, do 25 iterations of the QR-algorithm, e.g.,*

```
  (Q,R) := QRDecomposition(B);
  for j from 1 to 24 do
       C:= MatrixMatrixMultiply(R,Q);
   (Q,R):= QRDecomposition(C);
   od:
 MatrixMatrixMultiply(R,Q);
```

How do the these approximations compare with the eigenvalues computed by your favorite software package?

Problem 8.3 *With the matrix*

$$A := \begin{bmatrix} 1.0 & 0.25 & -0.5 & 1.0 \\ 0.25 & 1.0 & -0.3 & 0.3 \\ -0.5 & -0.3 & 3 & 1.1 \\ 1 & 0.3 & 1.1 & 4 \end{bmatrix}$$

do 25 iterations of the QR-algorithm. Why is the last RQ almost diagonal?

Problem 8.4 *Let E be the perturbation matrix*

$$E := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

such that $B = A + E$ for the matrices of the last two problems. The eigenvalue 2.6010781441 *of B is a distance of* 0.283111719 *from $\rho(A)$. How does this compare to the estimate we obtained in class?*

Problem 8.5 *If A is a unitary matrix and $A = QR$ is the QR-factorization, what can you say about R? Assume here that we are normalizing the QR-factorization so that the diagonal entries of R are positive.*

7 October 29, 2004: Due November 8, 2004

Problem 7.1 *Let A be an $m \times n$ complex matrix. Show that $||A||_1 = ||A^*||_\infty$.*

Problem 7.2 *Let A, B be $m \times n$ complex matrices. Show that $||A + B|| \leq ||A|| + ||B||$ for any matrix norm.*

Problem 7.3 *Let A be an $m \times n$ complex matrix and let B be and $n \times p$ complex matrix. Show that $||A \cdot B|| \leq ||A|| \cdot ||B||$ for any matrix norm.*

Problem 7.4 *Let A be the Hilbert matrix*

$$H_n := \begin{bmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & \cdots & 1/(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/(n+1) & \cdots & 1/(2n+1) \end{bmatrix}.$$

Using only the SVD computed by your favorite program (Maple, Matlab,...)

1. *compute* $\mathrm{cond_2(H_n)}$ *for $n = 10, 20, 30$; and*
2. *compute the absolute values of the eigenvalues of H_{10} and H_{20}.*

6 October 15, 2004: Due October 29, 2004

Problem 6.1 *Find the solution of the difference equation*

$$y_{n+3} = 3y_{n+2} + 4y_{n+1} - 12y_n + 6,$$

with the initial conditions $y_0 = 1$, $y_1 = 1$, $y_2 = 6$ *and show that*

$$\lim_{n\to\infty} \frac{y_n}{3^n} = 1.$$

In part 2) of the Problem 6.2 you will need to use the Adams-Bashforth four-step method as predictor and the Adams-Moulton three-step method as corrector: both are fourth order methods. Recall from the Maple Worksheet `multistep.mws`, that to solve the IVP on $[a, b]$

$$y' = f(x, y) \text{ with } y(a) = \alpha,$$

with $h = \dfrac{b-a}{n}$ for some integer $n > 0$, the *Adams-Bashforth four-step method* is given by

$$y_{i+1} = y_i + [55f(x_i, y_i) - 59f(x_{i-1}, y_{i-1}) + 37f(x_{i-2}, y_{i-2}) - 9f(x_{i-3}, y_{i-3})] \frac{h}{24};$$

and the *Adams-Moulton three-step method* is given by

$$y_{i+1} = y_i + [9f(x_{i+1}, y_{i+1}) + 19f(x_i, y_i) - 5f(x_{i-1}, y_{i-1}) + f(x_{i-2}, y_{i-2})] \frac{h}{24}.$$

In both you need to start the process using some other fourth order method.

Problem 6.2 *Consider the IVP (i.e., initial value problem)*

$$y'' - 2y' + y = te^t \text{ on } [0,1];\ y(0) = 1;\ y'(0) = 2.$$

1. *Using the equivalent formulation as a first order initial value problem use the Runge-Kutta method of order 4 with* $h = 0.05$ *to find an approximation to* $(y(1), y'(1))$.

2. *Using the equivalent formulation as a first order initial value problem use the Adams 4th order predictor-corrector method with* $h = 0.05$ *(using Runge-Kutta of order four to get started) to find an approximation to* $(y(1), y'(1))$

3. *Using the exact solution to the IVP at the start of this problem*

$$y(t) = e^t \left(\frac{t^3}{6} + t + 1 \right),$$

give the absolute error in the infinity norm of the approximations to $(y(1), y'(1)) = (5.8896, 9.9670)$ *of part 1) and 2) of this problem.*

5 October 8, 2004: Due October 15, 2004

Problem 5.1 *A person wishes to find a zero of* $f(x) = x^3 - 5$ *for* $1 \leq x \leq 2$. *That person decides to use the bisection method to accomplish this.*

1. *The first approximation to a solution by the bisection method is* 1.5 *with* $f(1.5) = \frac{-13}{8}$. *What is the second approximation to a solution by the bisection method?*

2. *You would like to find an approximation to a zero of* $f(x)$ *on* $[1, 2]$ *with an absolute error of no more than* 0.001. *Using the bisection method as outlined in the previous part of this problem, and the error estimate for the bisection method, which is the smallest integer* n *for which you know that on the* n*-th approximation you will be within* 0.001 *of the correct answer. Compute this approximation.*

Problem 5.2 *A person wishes to find a zero of* $f(x) = x^3 - 5$ *for* $1 \leq x \leq 2$. *That person decides to use Newton's method (also known as the Newton-Raphson method) for finding a solution of the equation* $f(x) = 0$ *on the interval,* $[1, 2]$.

1. *Write down the iteration formula that Newton's method gives for solving $f(x) = 0$. Then using 32 digit arithmetic with* 2.0 *as a starting guess, find the first six approximations to a solution of $f(x) = 0$ given by this formula.*

2. *How do these approximations compare with the tenth approximation of the bisection method computed in the previous problem.*

Problem 5.3 *A person wishes to find a zero of $f(x) = x^3 - 5$ for $1 \le x \le 2$. That person decides to use the secant method for finding a solution of the equation $f(x) = 0$ on the interval,* $[1, 2]$.

1. *Write down the iteration formula that the secant method gives for solving $f(x) = 0$. Then using 32 digit arithmetic with* $1.0, 2.0$ *as the starting guesses, find the first six approximations to a solution of $f(x) = 0$ given by this formula.*

2. *For i from 1 to 6, compare the i-th approximation with the i-th approximation by Newton's method.*

Problem 5.4 *Consider the initial value problem $y' = \frac{(y^2+y)}{t}$ on* $[1, 2]$ *with* $y(1) = -3.0$.

1. *Write down the Taylor's methods of order 1, 2, 3, and 4 for solving this problem.*

2. *Using each of these methods with stepsize* $h = 0.05$, *find the approximations to $y(2)$.*

3. *Using the solution $y(t) = \dfrac{3t}{2 - 3t}$, we see the true value of $y(2)$ is* -1.5. *What are the absolute errors of the approximations to $y(2)$ given in part 2) of this problem by the four methods?*

Problem 5.5 *Consider the IVP (i.e., initial value problem)*

$$y'' - 2y' + y = te^t \text{ on } [0, 1];\ y(0) = 1;\ y'(0) = 2.$$

1. *Write the first order initial value problem this is equivalent to; and*

2. *with this formulation use Taylor's method of order 2 with* $h = 0.05$ *to find an approximation to $(y(1), y'(1))$.*

3. *Using the exact solution to the above IVP*

$$y(t) = e^t \left(\frac{t^3}{6} + t + 1 \right),$$

give the absolute error in the infinity norm of your approximation to $(y(1), y'(1)) = (5.8896, 9.9670)$.

4 October 1, 2004: Due October 8, 2004

Problem 4.1 *On the interval* $[-1, 1]$ *find the Gaussian one point, two point and three point rules. Do this by hand.*

Problem 4.2 *On* $[a, b]$, *for* h *be a positive real number such that* $(b-a)/N$ *is a positive integer, let*

$$T(f, h) = \left(f(a) + 2 \sum_{i=1}^{N-1} f(a + ih) + f(b) \right) \frac{h}{2}$$

denote the composite trapezoid rule.

1. *Assume you know* $T(f, b-a)$ *and* $T(f, (b-a)/2)$. *What is the rule you obtain by using Richardson extrapolation.*
2. *Assume you know* $T(f, b-a)$ *and* $T(f, (b-a)/3)$. *What is the rule you obtain by using Richardson extrapolation.*

You may assume that

$$T(f, h) = \int_a^b f(x)dx + a_1 h^2 + O(h^4)$$

where a_1 *is independent of* h.

Problem 4.3 *By hand compute the Bernoulli polynomials* $B_2(x), B_3(x), B_4(x)$ *and the corresponding Bernoulli numbers. You may assume that* $B_0(x) = 1$ *and* $B_1(x) = x - 1/2$.

3 September 15, 2004: Due September 22, 2004

Problem 3.1 *Given a function* $f(x) \in C^4[a,b]$*, write down the integration rule* $f(x) \to I(f)$ *obtained by using*

$$I(f) := \int_a^b p_3(x)\mathrm{dx}$$

to approximate

$$\int_a^b f(x)\mathrm{dx},$$

where $p_3(x)$ *is the polynomial of degree* ≤ 3 *such that* $p(a) = f(a)$, $p'(a) = f'(a)$, $p(b) = f(b)$, $p'(b) = f'(b)$.

Show that

$$\left\|\left|\int_a^b f(x)\mathrm{dx} - \mathrm{I(f)}\right|\right\| := \max_{[a,b]}\left|\int_a^b f(x)\mathrm{dx} - \mathrm{I(f)}\right| \le \frac{\max_{[a,b]}|f^{(4)}(x)|}{720}(b-a)^5.$$

Problem 3.2 *Compute (by hand) the spline*

$$B_0(x) := 2(t-x)_+^3[0,1,1,1,2].$$

Plot $B_0(x)$ *for* x *from* -1 *to* 3 *(you can use any program you want for this).*

Problem 3.3 *Compute the weights for Newton-Cotes integration on* $[a,b]$ *for* n *from 1 to 14. (I strongly recommend you do not do this by hand!) Which values of* n *have all their weights positive?*

2 September 8, 2004: Due September 15, 2004

Problem 2.1 *For* $x \in [1,2]$ *and given the knots* $w := \{0,1,2,3\}$*, compute by hand the B-spline of order 3, which has support on* $[0,1,2,3]$.

Problem 2.2 *Plot the B-spline of order 10 with knots at* $0, 0.5, 1, 2, 3, 4, 4.5, 6, 7, 8, 9$, *and its* 4*-th derivative.*

Problem 2.3 *Compute by hand the Newton form of the interpolating polynomial* p *of degree* ≤ 5 *such that* $p(0) = 0$, $p'(0) = 1$, $p(1) = 2$, $p'(1) = 0$, $p(2) = 1$, $p'(2) = 0$.

Problem 2.4 *Compute by hand the Newton form of the interpolating polynomial* p *of degree* ≤ 3 *such that* $p(a) = \alpha_0$, $p'(a) = \alpha_1$, $p(b) = \beta_0$, $p'(b) = \beta_1$.

1 September 1, 2004: Due September 8, 2004

Problem 1.1 *1. Using* 3 *digit arithmetic compute* $(1.2 + 0.004) + 0.004$. *What is the absolute and relative error?*

2. Using 3 *digit arithmetic with rounding compute* $1.2 + (0.004 + 0.004)$. *What is the absolute and relative error?*

Problem 1.2 *Prove Rolle's Theorem.*

Problem 1.3 *Write down a Newton form of the unique interpolating polynomial* $p(x)$ *of degree* ≤ 4 *such that* $p(1) = 1$, $p(2) = 1$, $p(3) = 2$, $p(4) = 3$, *and* $p(5) = 5$. *Compute this by hand with details given.*

Problem 1.4 *Plot on the same graph:*

1. *the function* $f(x) := \dfrac{1}{1 + 25x^2}$ *on* $[-1, 1]$*; and*

2. *the interpolation polynomial* $p_{10}(x)$ *of degree* ≤ 10 *with* $p(x_i) = f(x_i)$ *for* $x_i = -1 + 0.2i$ *for* $i = 0, \ldots, 10$.

Do this a second time with $p_{16}(x)$ *in place of* $p_{10}(x)$ $x_i = -1 + 0.125i$ *for* $i = 0, \ldots, 16$.

Problem 1.5 *Let* $x_0, \ldots, x_n$ *be* $n + 1$ *distinct real or complex points. Let*

$$L_i(x) := \Pi_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

be the i*th Lagrange polynomial of degree* $\leq n$ *for these points, i.e., the unique interpolation polynomial of degree* $\leq n$ *such that* $L_i(x_k) = 0$ *for* $i \neq k$ *and* $= 1$ *for* $i = k$*. Prove that* $\displaystyle\sum_{i=0}^{n} L_i(x) = 1$.