Problems for Mathematics 690

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December 3, 2009

9 December 3, 2009: Due December 10, 2009

Problem 9.1 *First redo an earlier homework problem: plot on the same graph:*

1. *the function* $f(x) := \dfrac{1}{1+25x^2}$ *on* $[-1, 1]$*; and*
2. *the interpolation polynomial* $p_{12}(x)$ *of degree* ≤ 12 *with* $p(x_i) = f(x_i)$ *for* $x_i = -1 + \frac{i}{6}$ *for* $i = 0, \ldots, 12$.

Now do this a second time with the points x_i *the roots of the Chebychev polynomial* $T_{13}(x)$*, i.e.,* $x_i = \cos\left(\dfrac{\pi}{2 \cdot 13} + i\dfrac{\pi}{13}\right)$ *for* $i = 0, \ldots, 12$.

8 November 5, 2009: Due November 12, 2009

Problem 8.1 *Consider the Initial Value Problem on* $[0, 1]$

$$y' = -1000y + 1000; \quad y(0) = 2$$

1. *Check that the solution is* $y(x) = e^{-1000x} + 1$*. For practical purposes this is 1 almost immediately!*
2. *Solve using the Euler method with* $N = 1000$*,* $h = 1/N$ *and grid-points* $x_j = j * h$ *for* $j = 0, \ldots, N$.
3. *Plot the above solution and compare it to the true solution.*
4. *Plot it with* $N = 500$ *and* $h = 1/N$*. Any thoughts on the deviation from the true solution.*

7 October 29, 2009: Due November 5, 2009

Problem 7.1 *Let H_{10} be the Hilbert matrix as in Problem 6.1.*

1. *Compute the eigenvalues of H_{10} by the built-in command of Maple, Matlab, or Mathematica.*

2. *Use the Power Method with 5 iterations to estimate the largest eigenvalue of H_{10} and compare the answer to the estimate in item 1).* `If you do not have at least 4 digits correct, you are doing something wrong!`

3. *Use the Inverse Power Method with 5 iterations to estimate the smallest eigenvalue of H_{10} and compare the answer to the estimate in item 1).* `If you do not have at least 4 digits correct, you are doing something wrong!`

4. *Do 5 steps of the QR algorithm on H_{10}.*

5. *How does the estimates from item 4) compare to the computations in item 1)?*

6 September 24, 2009: Due October 27, 2009

Problem 6.1 *Let H_N be the $N \times N$ matrix with entries $H_{i,j} = \dfrac{1}{i+j-1}$, e.g.,*

$$H_5 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}.$$

H_N is called the Hilbert matrix and is a naturally arising matrix, which is numerically difficult.

1. *For $N = 10$; compute the SVD of A, i.e., compute 10×10 unitary matrices U, V such that V^*AU is a diagonal matrix Σ_{10} with nonnegative diagonal entries $\sigma_1, \ldots, \sigma_{10}$ satisfying $\sigma_1 \geq \ldots \geq \sigma_{10} \geq 0$.*

2. *Verify that $A - V\Sigma_{10}U^*$ is close to zero, i.e., compute the Matrix 1-norm of the difference.*

3. *For N from 1 to 40 compute the condition number of H_N using 10 Digits.*

4. *For N from 1 to 40 compute the condition number of H_N using 60 Digits.*

5. *You should see a difference for the computations with 10 and 60 Digits. Why?*

`I suggest using Maple. Matlab does not have allow changing Digits from 10 to 60. The LinearAlgebra package has commands to compute the SVD and the Condition Number. You may use the default condition number from that package.`

5 October 8, 2009: Due October 6, 2009

Problem 5.1 *A person wishes to find a zero of* $f(x) = x^3 + 3x - 5$ *for* $1 \le x \le 2$. *That person decides to use Newton's method for finding a solution of the equation* $f(x) = 0$ *on the interval,* $[1, 2]$.

1. *Write down the iteration formula that Newton's method gives for solving* $f(x) = 0$. *Then using 32 digit arithmetic with* 2.0 *as a starting guess, find the first six approximations to a solution of* $f(x) = 0$ *given by this formula.*

2. *How do these approximations compare with the tenth approximation of the bisection method computed in the previous problem.*

4 September 17, 2009: Due September 24, 2009

Problem 4.1 *Fix the interval* $[0, 1]$ *and the inner product*

$$(f, g) = \int_0^1 f(x)g(x)\mathrm{d}x$$

on the vector space P_{10} *of polynomials with real coefficients of degree at most ten.*

1. *What is the dimension of* P_{10}*?*

2. *Compute and graph the polynomial* $p_{10}(x)$ *of degree ten that is orthogonal to all polynomials of degree at most nine.*

3. *Compute the weights for the Gaussian integration methods with weights at the roots of* $p_{10}(x)$.

Problem 4.2 *On an interval* $[a, b]$*, let* $f(x)$ *and* $\mathrm{Simp}(f, a, b, n)$ *be as in Problem 3.2.*

1. *Compute the sixth order integration rule arising from Romberg integration*
$$\frac{16\mathrm{Simp}(f, a, b, 2) - \mathrm{Simp}(f, a, b, 1)}{15}.$$

2. *The above is an integration method having nodes* $a + jh$ *with* $h = \frac{b-a}{4}$ *and* $j = 0, 1, 2, 3, 4$*. Compute the Newton-Cotes rule with these nodes and verify the two rules are different.*

Problem 4.3 *Do problems 3.3.1 and 3.3.7 in the lecture notes.*

3 September 10, 2009: Due September 17, 2009

`For the following two problems, the worksheet integrationComparison.doc on the class website might be useful.`

Problem 3.1 *Fix an interval* $[a, b]$ *and a function* $f \in C^2[a, b]$*. The composite trapezoid rule for the points* $x_i = a + i(b - a)/n$ *with* $i = 0, \ldots, n$ *is*

$$\mathrm{trap}(f, a, b, n) = \left(f(a) + 2\sum_{j=1}^{n-1} f\left(a + \frac{j(b-a)}{n}\right) + f(b)\right)\left(\frac{b-a}{2n}\right)$$

The Runge method of error control asserts that for f *as above with* $f^{(2)}(x)$ *not changing too fast*

$$\frac{|\mathrm{trap(f, a, b, n)} - \mathrm{trap(f, a, b, 2n)}|}{3}$$

is close to the error

$$\left|\int_a^b f(x)\mathrm{dx} - \mathrm{trap(f, a, b, 2n)}\right|.$$

In particular, this should be true for all sufficiently large n.

Compare $\dfrac{|\mathrm{trap(f, a, b, n)} - \mathrm{trap(f, a, b, 2n)}|}{3}$ *and* $\left|\int_a^b f(x)\mathrm{dx} - \mathrm{trap(f, a, b, 2n)}\right|$ *for* $f(x) = \sin(x)$*,* $a = 0$*,* $b = \pi$*, and* n *from 1 to 10.*

Problem 3.2 *For the composite Simpson rule*

$$\text{Simp}(f,a,b,n) = \left[f(a) + f(b) + 2\sum_{j=1}^{n-1} f(x_j) + 4\sum_{j=1}^{n} f(y_j) \right] \left(\frac{b-a}{6n} \right),$$

with $x_j = a + \frac{j(b-a)}{n}$ *and* $y_j = a + \frac{(2j-1)(b-a)}{2n}$, *make the analogous comparison of*

$$\frac{|\text{Simp(f, a, b, n)} - \text{Simp(f, a, b, 2n)}|}{15}$$

and

$$\left| \int_a^b f(x)\text{dx} - \text{Simp(f, a, b, 2n)} \right|$$

using $f(x) = \sin(x)$, $a = 0$, $b = \pi$, *and* n *from 1 to 10.*

2 September 3, 2009: Due September 10, 2009

Problem 2.1 *Given a function* $f(x) \in C^4[a,b]$, *write down the integration rule* $f(x) \to I(f)$ *obtained by using*

$$I(f) := \int_a^b p_3(x)\text{dx}$$

to approximate

$$\int_a^b f(x)\text{dx},$$

where $p_3(x)$ *is the polynomial of degree* ≤ 3 *such that* $p(a) = f(a)$, $p'(a) = f'(a)$, $p(b) = f(b)$, $p'(b) = f'(b)$.

Show that

$$\left\| \int_a^b f(x)\text{dx} - \text{I(f)} \right\| := \left| \int_a^b f(x)\text{dx} - \text{I(f)} \right| \leq \frac{\max_{[a,b]} |f^{(4)}(x)|}{720}(b-a)^5.$$

Problem 2.2 *Given a function* $f(x) \in C^2[-1,1]$, *write down the integration rule* $f(x) \to I(f)$ *obtained by using*

$$I(f) := \int_{-1}^1 p_1(x)\text{dx}$$

to approximate

$$\int_{-1}^1 f(x)\text{dx},$$

where $p_1(x)$ is the polynomial of degree at most one such that $p_1\left(-\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}\right)$ *and* $p_1\left(\frac{1}{\sqrt{3}}\right) = f\left(\frac{1}{\sqrt{3}}\right)$.

Show that if $g(x)$ is a polynomial of degree at most three, then

$$\int_{-1}^{1} g(x)\mathrm{dx} = \mathrm{g}\left(-\frac{1}{\sqrt{3}}\right) + \mathrm{g}\left(\frac{1}{\sqrt{3}}\right).$$

1 August 27, 2009: Due September 3, 2009

Problem 1.1

1. *Using* 3 *digit arithmetic compute* $(1.3 + 0.004) + 0.004$. *What is the absolute and relative error?*

2. *Using* 3 *digit arithmetic with rounding compute* $1.3 + (0.004 + 0.004)$. *What is the absolute and relative error?*

Problem 1.2 *Remember "rounding to even."*

1. *Using* 3 *digit arithmetic compute* $(1.3 + 0.005) + 0.005$.

2. *Using* 3 *digit arithmetic with rounding compute* $1.3 + (0.005 + 0.005)$.

3. *Using* 3 *digit arithmetic with rounding compute* $1.31 + (0.005 + 0.005)$.

4. *Using* 3 *digit arithmetic with rounding compute* $(1.31 + 0.005) + 0.005$.

Problem 1.3 *Write down a Newton form of the unique interpolating polynomial $p(x)$ of degree ≤ 3 such that $p(1) = 2$, $p(2) = 5$, $p(3) = 10$, $p(4) = 17$. Compute this by hand giving details.*

Problem 1.4 *Plot on the same graph:*

1. *the function* $f(x) := \dfrac{1}{1 + 25x^2}$ *on* $[-1, 1]$*; and*

2. *the interpolation polynomial $p_5(x)$ of degree ≤ 5 with $p(x_i) = f(x_i)$ for $x_i = -1 + 0.4i$ for $i = 0, \ldots, 5$.*

Do this a second time with $p_{12}(x)$ in place of $p_5(x)$ $x_i = -1 + \frac{i}{6}$ for $i = 0, \ldots, 12$.

Problem 1.5 *Let* $x_0, \ldots, x_n$ *be* $n+1$ *distinct real or complex points. Let*

$$L_i(x) := \Pi_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

be the i*th Lagrange polynomial of degree* $\leq n$ *for these points, i.e., the unique interpolation polynomial of degree* $\leq n$ *such that* $L_i(x_k) = 0$ *for* $k \neq i$ *and* $= 1$ *for* $k = i$*. Prove that* $\sum_{i=0}^{n} L_i(x) = 1$.