Problems with Answers for Mathematics 690

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4 September 17, 2009: Due September 24, 2009

Problem 4.1 Fix the interval \([0, 1]\) and the inner product 
\[(f, g) = \int_0^1 f(x)g(x)dx\]
on the vector space \(P_{10}\) of polynomials with real coefficients of degree at most ten.

1. What is the dimension of \(P_{10}\)?
11 since there is a one-to-one and onto correspondence between the vectors \((a_0, \ldots, a_{10})\) and the polynomials \(s_0 + a_1x + \ldots + a_{10}x^{10}\).

2. Compute and graph the polynomial \(p_{10}(x)\) of degree ten that is orthogonal to all polynomials of degree at most nine.
See Hmwk4.doc on the class website.

3. Compute the weights for the Gaussian integration methods with weights at the roots of \(p_{10}(x)\).
See Hmwk4.doc on the class website.

Problem 4.2 On an interval \([a, b]\), let \(f(x)\) and \(\text{Simp}(f, a, b, n)\) be as in Problem 3.2.

1. Compute the sixth order integration rule arising from Romberg integration
\[\frac{16\text{Simp}(f, a, b, 2) - \text{Simp}(f, a, b, 1)}{15}\]
See Hmwk4.doc on the class website.
2. The above is an integration method having nodes \( a + jh \) with \( h = \frac{b-a}{4} \) and \( j = 0, 1, 2, 3, 4 \). Compute the Newton-Cotes rule with these nodes and verify the two rules are different.

See Hmwk4.doc on the class website.

**Problem 4.3** Do problems 3.3.1 and 3.3.7 in the lecture notes. Come and see me if you had difficulty with this problem.

3 September 10, 2009: Due September 17, 2009

See rungeErrorControl.doc on the class website.

**Problem 3.1** Fix an interval \([a, b]\) and a function \( f \in C^2[a, b] \). The composite trapezoid rule for the points \( x_i = a + i(b - a)/n \) with \( i = 0, \ldots, n \) is

\[
\text{trap}(f, a, b, n) = \left( f(a) + 2 \sum_{j=1}^{n-1} f \left( a + \frac{j(b-a)}{n} \right) + f(b) \right) \left( \frac{b-a}{2n} \right)
\]

The Runge method of error control asserts that for \( f \) as above with \( f^{(2)}(x) \) not changing too fast

\[
\frac{|\text{trap}(f, a, b, n) - \text{trap}(f, a, b, 2n)|}{3}
\]

is close to the error

\[
\left| \int_a^b f(x)dx - \text{trap}(f, a, b, 2n) \right|.
\]

In particular, this should be true for all sufficiently large \( n \).

Compare \( \frac{|\text{trap}(f, a, b, n) - \text{trap}(f, a, b, 2n)|}{3} \) and \( \left| \int_a^b f(x)dx - \text{trap}(f, a, b, 2n) \right| \)

for \( f(x) = \sin(x), a = 0, b = \pi, \) and \( n \) from 1 to 10.

**Problem 3.2** For the composite Simpson rule

\[
\text{Simp}(f, a, b, n) = \left[ f(a) + f(b) + 2 \sum_{j=1}^{n-1} f(x_j) + 4 \sum_{j=1}^{n} f(y_j) \right] \left( \frac{b-a}{6n} \right),
\]

with \( x_j = a + \frac{j(b-a)}{n} \) and \( y_j = a + \frac{(2j-1)(b-a)}{2n} \), make the analogous comparison of

\[
\frac{|\text{Simp}(f, a, b, n) - \text{Simp}(f, a, b, 2n)|}{15}
\]

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and
\[ \left| \int_a^b f(x)dx - \text{Simp}(f, a, b, 2n) \right| \]

using \( f(x) = \sin(x) \), \( a = 0 \), \( b = \pi \), and \( n \) from 1 to 10.

2 September 3, 2009: Due September 10, 2009

**Problem 2.1** Given a function \( f(x) \in C^4[a, b] \), write down the integration rule \( f(x) \rightarrow I(f) \) obtained by using

\[ I(f) := \int_a^b p_3(x)dx \]

to approximate

\[ \int_a^b f(x)dx, \]

where \( p_3(x) \) is the polynomial of degree \( \leq 3 \) such that \( p(a) = f(a) \), \( p'(a) = f'(a) \), \( p(b) = f(b) \), \( p'(b) = f'(b) \).

Show that

\[ \left| \int_a^b f(x)dx - I(f) \right| := \left| \int_a^b f(x)dx - I(f) \right| \leq \frac{\max_{[a,b]} |f^{(4)}(x)|}{720} (b - a)^5. \]

**Problem 2.2** Given a function \( f(x) \in C^2[-1, 1] \), write down the integration rule \( f(x) \rightarrow I(f) \) obtained by using

\[ I(f) := \int_{-1}^1 p_1(x)dx \]

to approximate

\[ \int_{-1}^1 f(x)dx, \]

where \( p_1(x) \) is the polynomial of degree at most one such that \( p \left( -\frac{1}{\sqrt{3}} \right) = f \left( -\frac{1}{\sqrt{3}} \right) \) and \( p \left( \frac{1}{\sqrt{3}} \right) = f \left( \frac{1}{\sqrt{3}} \right) \).

Show that if \( g(x) \) is a polynomial of degree at most three, then

\[ \int_{-1}^1 g(x)dx = g \left( -\frac{1}{\sqrt{3}} \right) + g \left( \frac{1}{\sqrt{3}} \right). \]
1 August 27, 2009: Due September 3, 2009

Problem 1.1

1. Using 3 digit arithmetic compute $(1.3 + 0.004) + 0.004$. What is the absolute and relative error?
   Answer: $(1.3 + 0.004) + 0.004 = 1.304 + 0.004 = 1.30 + 0.004 = 1.30$. The absolute and relative errors are 0.008 and $\frac{0.008}{1.30} \approx 0.006$ respectively.

2. Using 3 digit arithmetic with rounding compute $1.3 + (0.004 + 0.004)$.
   What is the absolute and relative error?
   Answer: $1.3 + (0.004 + 0.004) = 1.3 + 0.008 = 1.308 = 1.31$. The absolute and relative errors are 0.002 and $\frac{0.002}{1.308} \approx 0.0015$ respectively.

Problem 1.2 Remember “rounding to even.”

1. Using 3 digit arithmetic compute $(1.3 + 0.005) + 0.005$.
   Answer: 1.3.

2. Using 3 digit arithmetic with rounding compute $1.3 + (0.005 + 0.005)$.
   Answer: 1.3.

3. Using 3 digit arithmetic with rounding compute $(1.31 + 0.005) + 0.005$.
   Answer: 1.32.

4. Using 3 digit arithmetic with rounding compute $(1.31 + 0.005 + 0.005)$.
   Answer: 1.32.

Problem 1.3 Write down a Newton form of the unique interpolating polynomial $p(x)$ of degree $\leq 3$ such that $p(1) = 2$, $p(2) = 5$, $p(3) = 10$, $p(4) = 17$. Compute this by hand giving details.
   Answer (without the work): $2 + 3(x - 1) + (x - 1)(x - 2)$.

Problem 1.4 Plot on the same graph:

1. the function $f(x) := \frac{1}{1 + 25x^2}$ on $[-1,1]$; and

2. the interpolation polynomial $p_5(x)$ of degree $\leq 5$ with $p(x_i) = f(x_i)$ for $x_i = -1 + 0.4i$ for $i = 0, \ldots, 5$. 

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Do this a second time with \( p_{12}(x) \) in place of \( p_5(x) \) \( x_i = -1 + \frac{i}{6} \) for \( i = 0, \ldots, 12 \).

Answer: See the Maple Worksheet: RungeFunction.mws

**Problem 1.5** Let \( x_0, \ldots, x_n \) be \( n + 1 \) distinct real or complex points. Let

\[
L_i(x) := \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}
\]

be the \( i \)th Lagrange polynomial of degree \( \leq n \) for these points, i.e., the unique interpolation polynomial of degree \( \leq n \) such that \( L_i(x_k) = 0 \) for \( i \neq k \) and \( = 1 \) for \( i = k \). Prove that \( \sum_{i=0}^{n} L_i(x) = 1 \).

Answer: Note that \( \sum_{i=0}^{n} L_i(x) \) is degree at most \( n \) and equals 1 at \( x_0, \ldots, x_n \). Thus by uniqueness it is 1.

You could also approach this via partial fractions. For example, we know that there are constants \( c_0, \ldots, c_n \) such that

\[
\frac{1}{\prod_{i=0}^{n}(x - x_i)} = \sum_{i=0}^{n} \frac{c_i}{x - x_i}. 
\] (1)

Multiplying both sides by \( x - x_i \) and setting \( x = x_j \) we see that

\[
\frac{1}{\prod_{j \neq i}(x_j - x_i)} = c_i.
\]

Substitute this in Equation 1 and multiply both sides by

\[
\prod_{i=0}^{n}(x - x_i).
\]

This gives the identity we wanted to prove. Note that proving the partial fraction expansion comes down to the same key point as the first proof, i.e., the Euclidean algorithm.