Problems with Answers for Mathematics 690

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September 24, 2009

4 September 17, 2009: Due September 24, 2009

Problem 4.1 *Fix the interval* $[0,1]$ *and the inner product*

$$(f,g) = \int_0^1 f(x)g(x)\mathrm{d}x$$

on the vector space P_{10} *of polynomials with real coefficients of degree at most ten.*

1. *What is the dimension of* P_{10}*?*
   `11 since there is a one-to-one and onto correspondence between the vectors` $(a_0, \ldots, a_{10})$ `and the polynomials` $s_0 + a_1x + \ldots + a_nx^n$.

2. *Compute and graph the polynomial* $p_{10}(x)$ *of degree ten that is orthogonal to all polynomials of degree at most nine.*
   `See Hmwk4.doc on the class website.`

3. *Compute the weights for the Gaussian integration methods with weights at the roots of* $p_{10}(x)$.
   `See Hmwk4.doc on the class website.`

Problem 4.2 *On an interval* $[a,b]$*, let* $f(x)$ *and* $\mathrm{Simp}(f,a,b,n)$ *be as in Problem 3.2.*

1. *Compute the sixth order integration rule arising from Romberg integration*
   $$\frac{16\mathrm{Simp}(f,a,b,2) - \mathrm{Simp}(f,a,b,1)}{15}.$$
   `See Hmwk4.doc on the class website.`

2. *The above is an integration method having nodes $a + jh$ with $h = \frac{b-a}{4}$ and $j = 0, 1, 2, 3, 4$. Compute the Newton-Cotes rule with these nodes and verify the two rules are different.*
   `See Hmwk4.doc on the class website.`

Problem 4.3 *Do problems 3.3.1 and 3.3.7 in the lecture notes.* `Come and see me if you had difficulty with this problem.`

3 September 10, 2009: Due September 17, 2009

`See rungeErrorControl.doc on the class website.`

Problem 3.1 *Fix an interval $[a, b]$ and a function $f \in C^2[a, b]$. The composite trapezoid rule for the points $x_i = a + i(b - a)/n$ with $i = 0, \ldots, n$ is*

$$\mathrm{trap}(f, a, b, n) = \left(f\left(a\right) + 2\sum_{j=1}^{n-1} f\left(a + \frac{j\left(b-a\right)}{n}\right) + f\left(b\right)\right)\left(\frac{b-a}{2n}\right)$$

The Runge method of error control asserts that for f as above with $f^{(2)}(x)$ not changing too fast

$$\frac{|\mathrm{trap(f,a,b,n)} - \mathrm{trap(f,a,b,2n)}|}{3}$$

is close to the error

$$\left| \int_a^b f(x)\mathrm{dx} - \mathrm{trap(f,a,b,2n)} \right|.$$

In particular, this should be true for all sufficiently large n.

Compare $\dfrac{|\mathrm{trap(f,a,b,n)} - \mathrm{trap(f,a,b,2n)}|}{3}$ *and* $\left| \int_a^b f(x)\mathrm{dx} - \mathrm{trap(f,a,b,2n)} \right|$ *for $f(x) = \sin(x)$, $a = 0$, $b = \pi$, and n from 1 to 10.*

Problem 3.2 *For the composite Simpson rule*

$$\mathrm{Simp}(f, a, b, n) = \left[f(a) + f(b) + 2\sum_{j=1}^{n-1} f(x_j) + 4\sum_{j=1}^{n} f(y_j) \right]\left(\frac{b-a}{6n}\right),$$

with $x_j = a + \frac{j(b-a)}{n}$ and $y_j = a + \frac{(2j-1)(b-a)}{2n}$, make the analogous comparison of

$$\frac{|\mathrm{Simp(f,a,b,n)} - \mathrm{Simp(f,a,b,2n)}|}{15}$$

and

$$\left| \int_a^b f(x)\mathrm{dx} - \mathrm{Simp(f, a, b, 2n)} \right|$$

using $f(x) = \sin(x)$*,* $a = 0$*,* $b = \pi$*, and* n *from 1 to 10.*

2 September 3, 2009: Due September 10, 2009

Problem 2.1 *Given a function* $f(x) \in C^4[a, b]$*, write down the integration rule* $f(x) \to I(f)$ *obtained by using*

$$I(f) := \int_a^b p_3(x)\mathrm{dx}$$

to approximate

$$\int_a^b f(x)\mathrm{dx},$$

where $p_3(x)$ *is the polynomial of degree* ≤ 3 *such that* $p(a) = f(a)$*,* $p'(a) = f'(a)$*,* $p(b) = f(b)$*,* $p'(b) = f'(b)$*.*

Show that

$$\left\| \int_a^b f(x)\mathrm{dx} - \mathrm{I(f)} \right\| := \left| \int_a^b f(x)\mathrm{dx} - \mathrm{I(f)} \right| \leq \frac{\max_{[a,b]} |f^{(4)}(x)|}{720}(b-a)^5.$$

Problem 2.2 *Given a function* $f(x) \in C^2[-1, 1]$*, write down the integration rule* $f(x) \to I(f)$ *obtained by using*

$$I(f) := \int_{-1}^1 p_1(x)\mathrm{dx}$$

to approximate

$$\int_{-1}^1 f(x)\mathrm{dx},$$

where $p_1(x)$ *is the polynomial of degree at most one such that* $p\left(-\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}\right)$ *and* $p\left(\frac{1}{\sqrt{3}}\right) = f\left(\frac{1}{\sqrt{3}}\right)$*.*

Show that if $g(x)$ *is a polynomial of degree at most three, then*

$$\int_{-1}^1 g(x)\mathrm{dx} = \mathrm{g}\left(-\frac{1}{\sqrt{3}}\right) + \mathrm{g}\left(\frac{1}{\sqrt{3}}\right).$$

1 August 27, 2009: Due September 3, 2009

Problem 1.1

1. *Using* 3 *digit arithmetic compute* $(1.3 + 0.004) + 0.004$. *What is the absolute and relative error?*
   `Answer:` $(1.3+0.004)+0.004 = 1.304+0.004 = 1.30+0.004 = 1.30$. `The absolute and relative errors are` 0.008 `and` $\frac{0.008}{1.308} \approx 0.006$ `respectively.`

2. *Using* 3 *digit arithmetic with rounding compute* $1.3 + (0.004 + 0.004)$. *What is the absolute and relative error?*
   `Answer:` $1.3 + (0.004 + 0.004) = 1.3 + 0.008 = 1.308 = 1.31$. `The absolute and relative errors are` 0.002 `and` $\frac{0.002}{1.308} \approx 0.0015$ `respectively.`

Problem 1.2 *Remember "rounding to even."*

1. *Using* 3 *digit arithmetic compute* $(1.3 + 0.005) + 0.005$.
   `Answer:` 1.3.

2. *Using* 3 *digit arithmetic with rounding compute* $1.3 + (0.005 + 0.005)$.
   `Answer:` 1.3.

3. *Using* 3 *digit arithmetic with rounding compute* $(1.31+0.005)+0.005$.
   `Answer:` 1.32.

4. *Using* 3 *digit arithmetic with rounding compute* $1.31+(0.005+0.005)$. $(1.31 + 0.005) + 0.005$.
   `Answer:` 1.32.

Problem 1.3 *Write down a Newton form of the unique interpolating polynomial* $p(x)$ *of degree* ≤ 3 *such that* $p(1) = 2$, $p(2) = 5$, $p(3) = 10$, $p(4) = 17$. *Compute this by hand giving details.*
`Answer (without the work):` $2 + 3(x - 1) + (x - 1)(x - 2)$.

Problem 1.4 *Plot on the same graph:*

1. *the function* $f(x) := \dfrac{1}{1 + 25x^2}$ *on* $[-1, 1]$*; and*

2. *the interpolation polynomial* $p_5(x)$ *of degree* ≤ 5 *with* $p(x_i) = f(x_i)$ *for* $x_i = -1 + 0.4i$ *for* $i = 0, \ldots, 5$.

Do this a second time with $p_{12}(x)$ in place of $p_5(x)$ $x_i = -1 + \frac{i}{6}$ for $i = 0, \ldots, 12$.
Answer: See the Maple Worksheet: RungeFunction.mws

Problem 1.5 *Let $x_0, \ldots, x_n$ be $n+1$ distinct real or complex points. Let*

$$L_i(x) := \Pi_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

be the ith Lagrange polynomial of degree $\leq n$ for these points, i.e., the unique interpolation polynomial of degree $\leq n$ such that $L_i(x_k) = 0$ for $i \neq k$ and $= 1$ for $i = k$. Prove that $\sum_{i=0}^{n} L_i(x) = 1$.

Answer: Note that $\sum_{i=0}^{n} L_i(x)$ is degree at most n and equals 1 at $x_0, \ldots, x_n$. Thus by uniqueness it is 1.

You could also approach this via partial fractions. For example, we know that there are constants $c_0, \ldots, c_n$ such that

$$\frac{1}{\Pi_{i=0}^{n}(x - x_i)} = \sum_{i=0}^{n} \frac{c_i}{x - x_i}. \tag{1}$$

Multiplying both sides by $x - x_i$ and setting $x = x_j$ we see that

$$\frac{1}{\Pi_{j \neq i}(x_j - x_i)} = c_i.$$

Substitute this in Equation 1 and multiply both sides by

$$\Pi_{i=0}^{n}(x - x_i).$$

This gives the identity we wanted to prove. *Note that proving the partial fraction expansion comes down to the same key point as the first proof, i.e., the Euclidean algorithm.*