Case Study: Alt's Problem

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Joint work with

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Details

- **Reference on the area up to 2005:**
  - Andrew Sommese and Charles Wampler, *Numerical solution of systems of polynomials arising in engineering and science*, (2005), World Scientific Press.

- **Recent articles are available at**
  - www.nd.edu/~sommese

Case Study: Alt's Problem

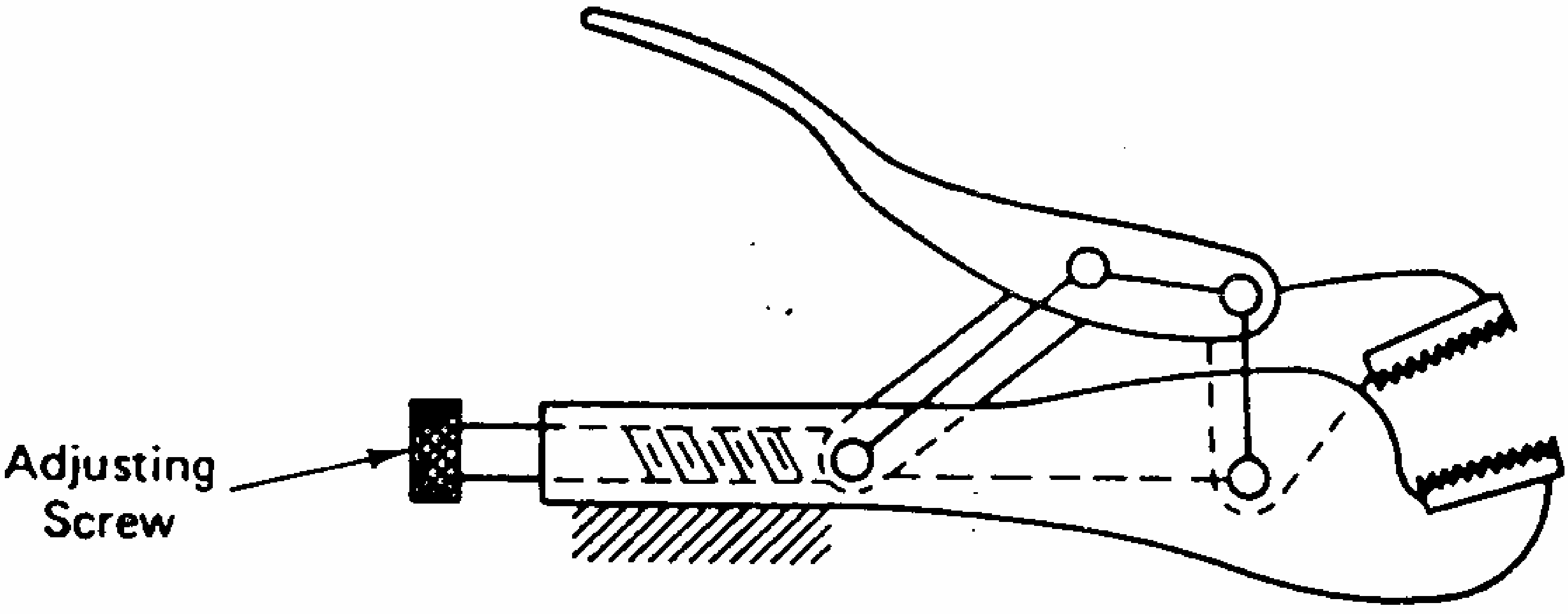
- We follow

C. Wampler, A. Morgan, and A.J. Sommese, Complete solution of the nine-point path synthesis problem for four-bar linkages, ASME Journal of Mechanical Design 114 (1992), 153–159.

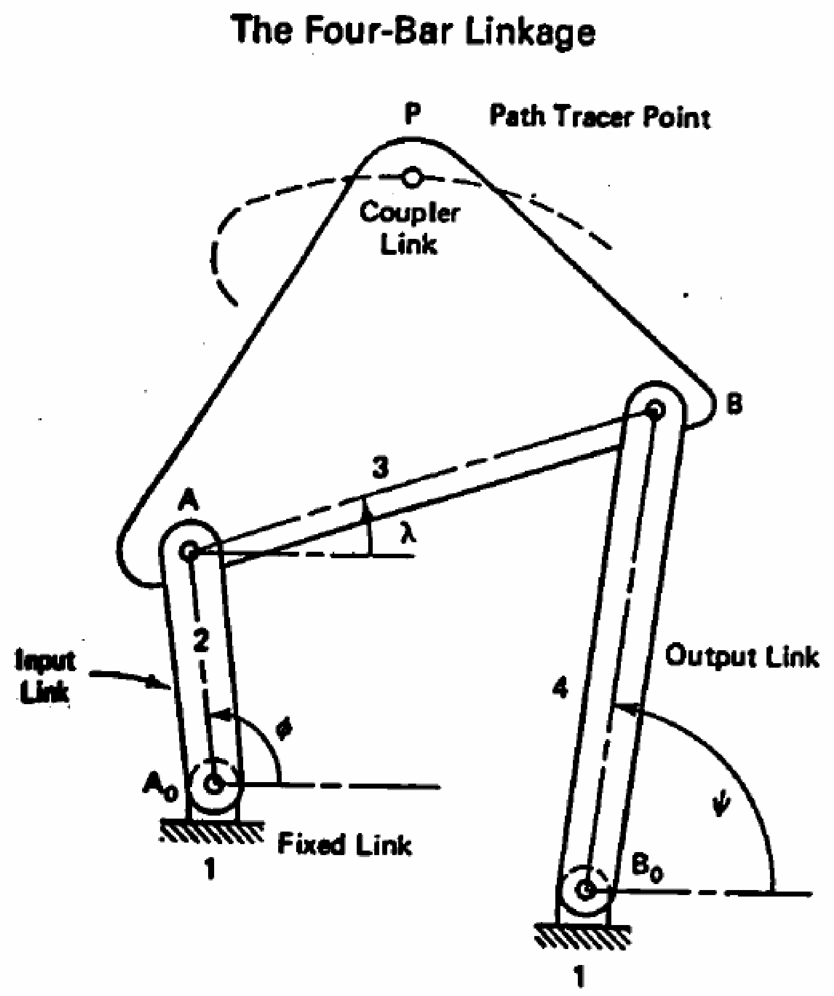
Four-bar planar linkages

- A four-bar planar linkage is a planar quadrilateral with a rotational joint at each vertex.
- They are useful for converting one type of motion to another.
- They occur everywhere.

A simple four-bar



More Abstractly



How Do Mechanical Engineers Find Mechanisms?

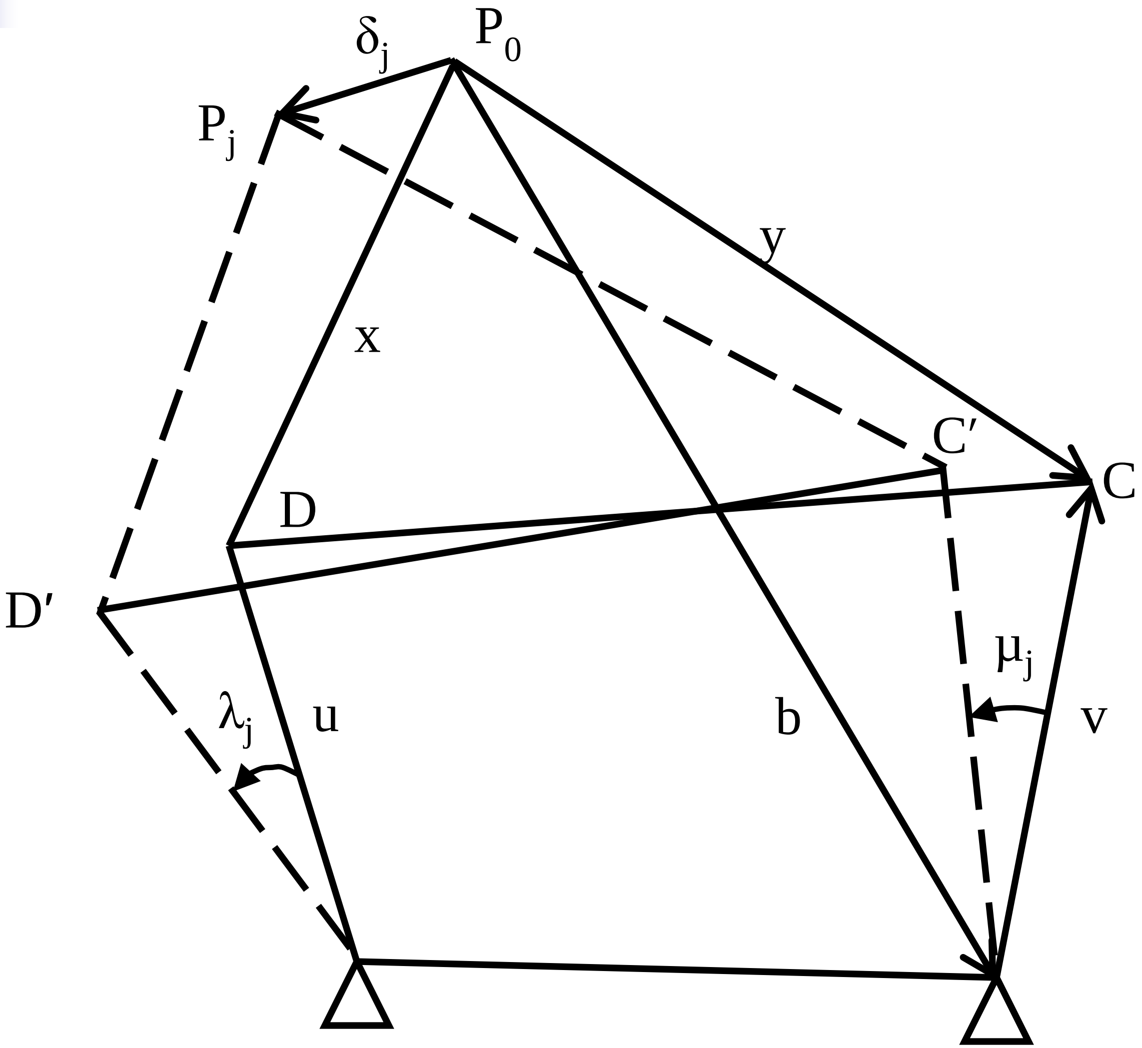
- Pick a few points in the plane (called precision points)
- Find a coupler curve going through those points
- If unsuitable, start over.

- Having more choices makes the process faster.
- By counting constants, there will be no coupler curves going through more than nine points.

Nine Point Path-Synthesis Problem

H. Alt, Zeitschrift für angewandte Mathematik und Mechanik, 1923:

- Given nine points in the plane, find the set of all four-bar linkages, whose coupler curves pass through all these points.



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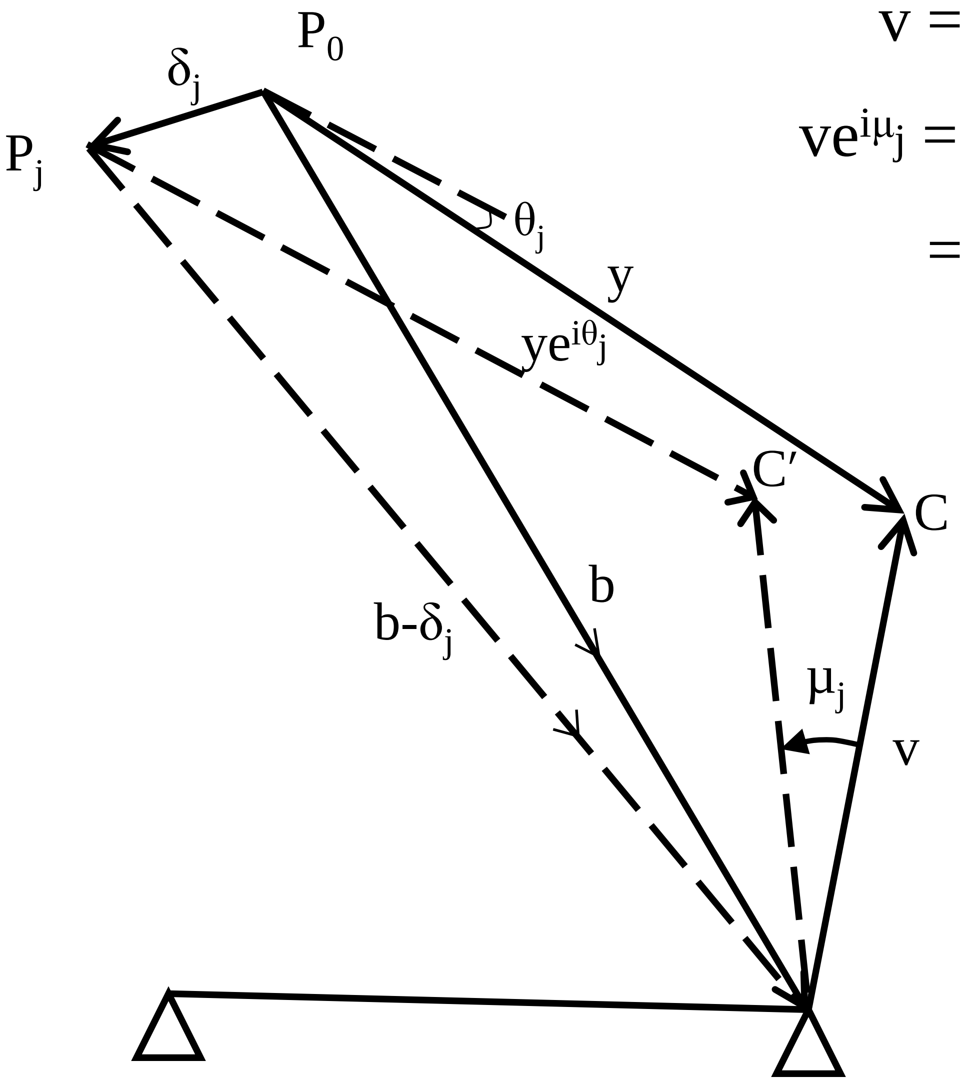
$$v = y - b$$

$$ve^{i\mu_j} = ye^{i\theta_j} - (b - \delta_j)$$

$$= ye^{i\theta_j} + \delta_j - b$$



We use complex numbers (as is standard in this area)

Summing over vectors we have 16 equations

$$(y-b)e^{i\mu_j} = ye^{i\theta_j} + \delta_j - b$$

$$(x-a)e^{i\lambda_j} = xe^{i\theta_j} + \delta_j - a$$

plus their 16 conjugates

$$(\bar{y}-\bar{b})e^{-i\mu_j} = \bar{y}e^{-i\theta_j} + \bar{\delta}_j - \bar{b}$$

$$(\bar{x}-\bar{a})e^{-i\lambda_j} = \bar{x}e^{-i\theta_j} + \bar{\delta}_j - \bar{a}$$

This gives 8 sets of 4 equations:

$$(x-a)e^{i\lambda_j} = xe^{i\theta_j} + \delta_j - a$$

$$(y-b)e^{i\mu_j} = ye^{i\theta_j} + \delta_j - b$$

$$(\bar{y}-\bar{b})e^{-i\mu_j} = \bar{y}e^{-i\theta_j} + \bar{\delta}_j - \bar{b}$$

$$(\bar{x}-\bar{a})e^{-i\lambda_j} = \bar{x}e^{-i\theta_j} + \bar{\delta}_j - \bar{a}$$

in the variables a, b, x, y, $\bar{a}$, $\bar{b}$, $\bar{x}$, $\bar{y}$, and

$\lambda_j, \mu_j, \theta_j$ for j from 1 to 8.

- Multiplying the first and fourth equation and the second and third equations eliminates the μ’s and the λ’s.
- Setting $\gamma_j = e^{i\theta_j} - 1$ and using the relation

$$\gamma_j \bar{\gamma}_j + \gamma_j + \bar{\gamma}_j = 0$$

- Now replace conjugate variables by new variables with hats.

Alt's System

$$\left[(\hat{a}-\bar{\delta}_j)x\right]\gamma_j+\left[(a-\delta_j)\hat{x}\right]\hat{\gamma}_j+\delta_j(\hat{a}-\hat{x})+\bar{\delta}_j(a-x)-\delta_j\bar{\delta}_j=0$$

$$\left[(\hat{b}-\bar{\delta}_j)y\right]\gamma_j+\left[(b-\delta_j)\hat{y}\right]\hat{\gamma}_j+\delta_j(\hat{b}-\hat{y})+\bar{\delta}_j(b-y)-\delta_j\bar{\delta}_j=0$$

$$\gamma_j+\hat{\gamma}_j+\gamma_j\hat{\gamma}_j=0$$

in the 24 variables $a, b, x, y, \hat{a}, \hat{b}, \hat{x}, \hat{y}$ and $\gamma_j, \hat{\gamma}_j$ with j from 1 to 8.

Note we have 24 equations of which 16 are degree 3 and 8 are degree 2. This would give a total possible number of solutions, $2^8 3^{16} = 11,019,960,576$—allowing 1 second a path it would take over 300 years to solve this system.

Freudenstein and Roth system

Using Cramers rule and substitution we have what is essentially the Freudenstein-Roth system consisting of 8 equations of degree 7. In 1991, this was impractical to solve:

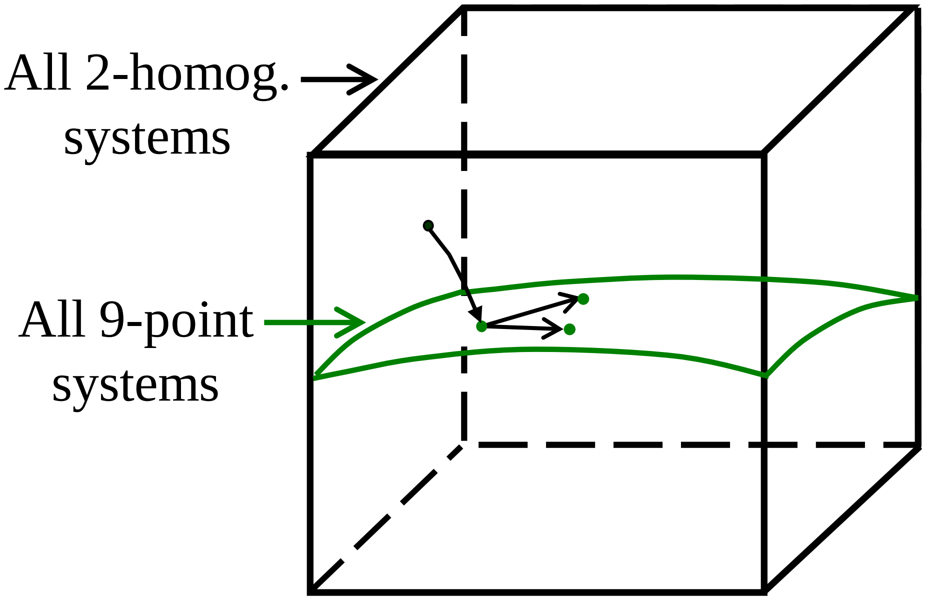
$7^8 = 5{,}764{,}801$solutions.

- Newton’s method doesn’t find many solutions: Freudenstein and Roth used a simple form of continuation combined with heuristics.

- Tsai and Lu using methods introduced by Li, Sauer, and Yorke found only a small fraction of the solutions: their method requires starting from scratch each time the problem is solved for different parameter values.

We followed a different route by introducing new variables $n, \hat{n}, m, \hat{m}$ so that $n = a\hat{x},\ \hat{n} = \hat{a}x,\ m = b\hat{y},\ \hat{m} = \hat{b}y$. We group the variables into 10 groups $\{\gamma_j, \hat{\gamma}_j\}$, $\{x, \hat{x}, a, \hat{a}, n, \hat{n}\}$, $\{y, \hat{y}, b, \hat{b}, m, \hat{m}\}$ for $j = 1, \ldots, 8$. Introducing homogeneous coordinates into each group, we use Cramer's rule to reduce to a system of 12 equations in 12 unknowns: 4 quadrics and 8 quartics. Though the Bézout number is 1,048,576, the 2-homogeneous Bézout number is 286,720, and there is an involution reducing the work to following 143,360 paths. There is also an order 3 symmetry…

Solve by Continuation



→ "numerical reduction" to test case (done 1 time)

→ synthesis program (many times)

Summary

- Analytical Reduction
  - Initial formulation .... $\approx 10^{10}$
  - Roth & Freudenstein 5,764,801
  - Our elimination 1,048,576
  - Multi-homogenization 286,720
  - Symmetry 143,360
- Numerical Reduction
  - Nondegenerate 4326
  - Roberts cognates.................. 1442
- Synthesis program tracks 1442 solution paths.