Problems for Mathematics 790

Instructor: Andrew Sommese

March 17, 2010

6 March 18, 2010: Due March 25, 2010

Problem 6.1 *Consider the heat equation*

$$u_t = u_{xx}$$

for $(x,t) \in [0,1] \times [0,\infty)$ *with* $u(0,t) = u(1,t) = 0$ *and* $u(x,0) = x(1-x)$. *For a positive integer* M, *and with* $h = 1.0/11$ *and* $k = 1.0/M$, *you want to find an approximate solution* $u_{i,j}$ *using finite differences on the* $10 \times M$ *grid* $(x_i, t_j) = (ih, jk)$ *for* $i = 1, \ldots, 10$ *and* $j = 1, \ldots, M$. *Setting*

1. $x_0 = 0$, $x_{11} = 1$, *and* $t_0 = 0$;
2. $u_{0,j} = u_{11,j} = 0$ *for all* j; *and*
3. $u_{i,0} = ih(1 - ih)$

you do this by computing

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

for j *from* 1 *to* M.

Compute the infinity norm of the vector $(u_{1,M}, \ldots, u_{10,M})$ *for* $M = 20$, *for* $M = 100$, *for* $M = 200$, *and for* $M = 250$.

5 February 18, 2010: Due February 25, 2010

Problem 5.1 *Assume you wish to find the eigenvalues and eigenvectors for an* $n \times n$ *complex matrix* A, *i.e., you wish to solve* $Ax - \lambda x = 0$ *where* x *is an* $n \times 1$ *vector. This is a polynomial system in the* $n+1$ *variables,* $x_1, \ldots, x_n$ *and* λ. *There are only* n *equations, but the eigenvectors are only well-defined* *up to a nonzero multiple. The simplest way of making this into a system with the same number of equations as variables is to add an extra random linear equation in the x variables, e.g.,*

$$\begin{bmatrix} a_{11}x_1 & + & \cdots & + & a_{1n}x_n - \lambda x_1 \\ & & \vdots & & \\ a_{n1}x_1 & + & \cdots & + & a_{nn}x_n - \lambda x_n \\ b_1x_1 & + & \cdots & + & b_nx_n - 1 \end{bmatrix} = 0.$$

We have here n quadratic equations and 1 linear equation in $n+1$ variables giving a total degree of 2^n. We expect n solutions most of the time, so we have to track $2^n - n$ extra paths using this approach. For a size 10,000 matrix, this means we must track $2^{10{,}000} \approx 10^{3{,}010}$ extra paths.

A different approach presented in articles by T.Y. Li, Michigan State University uses bihomogeneity. Since x is only well-defined up to nonzero multiple, we consider it as a point in $\mathbb{P}^{n-1}$. Homogenizing with respect to λ we get the system $Ax\lambda_0 - \lambda x = 0$ on $\mathbb{P}^{n-1} \times \mathbb{P}^1$ in the homogeneous variables $[x_1, \ldots, x_n]$ and $[\lambda_0, \lambda]$, i.e.,

$$\begin{bmatrix} a_{11}x_1\lambda_0 & + & \cdots & + & a_{1n}x_n\lambda_0 - \lambda x_1 \\ & & \vdots & & \\ a_{n1}x_1\lambda_0 & + & \cdots & + & a_{nn}x_n\lambda_0 - \lambda x_n \end{bmatrix} = 0.$$

So we have n equations on $\mathbb{P}^{n-1} \times \mathbb{P}^1$ each of bidegree $(1,1)$.

1. What is the number of paths we need to follow?
2. *If B is a diagonal $n \times n$ matrix with all eigenvalues different from one another and 0, what are the solutions of the system on $\mathbb{P}^{n-1} \times \mathbb{P}^1$ associated to $Bx - \lambda x = 0$.*

Extra Credit: Write down a homotopy based on the system in 2) as a start system to find the solutions of the system $Ax\lambda_0 - \lambda x = 0$ on $\mathbb{P}^{n-1} \times \mathbb{P}^1$.

4 February 4, 2010: Due February 18, 2010

Problem 4.1 *Write a continuation solver for constant stepsize h to solve a polynomial system*

$$f(z) = \begin{bmatrix} f_1(z_1, z_2) \\ f_2(z_1, z_2) \end{bmatrix} = 0$$

with $\deg f_i = d_i$ *for $i = 1, 2$. Use the homotopy*

$$H(t, z) = (1-t) \begin{bmatrix} f_1(z_1, z_2) \\ f_2(z_1, z_2) \end{bmatrix} + \gamma t \begin{bmatrix} z_1^{d_1} - 1 \\ z_2^{d_2} - 1 \end{bmatrix}$$

as t goes from 1 to 0 with γ a random complex number. Use Euler's method and one Newton correction at each step. Using your solver, solve the systems

$$f(z) = \begin{bmatrix} x^2 + y^2 - 1 \\ \frac{x^2}{16} + \frac{y^2}{9} - 1 \end{bmatrix} = 0$$

and

$$f(z) = \begin{bmatrix} xy \\ (x-1)(y-1) \end{bmatrix} = 0$$

for stepsizes $h = 0.2$, $h = 0.1$, and $h = 0.01$.

3 January 28, 2010: Due February 4, 2010

Problem 3.1 *Given the polynomial $p(x, y) = y(y - x)^2(y - 2x) + 6(y - x)x^2 - 8x^2 + 1$*

1. *Homogenize the polynomial to a homogeneous polynomial $q(x, y, z)$.*
2. *Compute the solutions at infinity of $p(x, y) = 0$, i.e., compute the solutions to $q(x, y, 0) = 0$.*
3. *From the solutions to part 2) compute the slopes of the lines asymptotic to the curve.*

2 January 21, 2010: Due January 28, 2010

Problem 2.1 *What is the discriminant of $p(x) = x^3 + ax^2 + bx + c$? (I.e., compute the polynomial $D(u, v, w)$, such that $D(a, b, c) = 0$ if and only if $x^3 + ax^2 + bx + c$ has a multiple zero.)*

Problem 2.2 *Use resultants to find the common zeros of $x^2 + y^2 - 10 = 0$ and $\dfrac{x^2}{16} + \dfrac{y^2}{9} - 1 = 0$.*

1 January 14, 2010: Due January 21, 2010

Problem 1.1 *Write a continuation solver for constant stepsize h to solve a polynomial $p(z) = 0$ of degree d. Use the homotopy*

$$H(t, z) = (1 - t)p(z) + \gamma t(z^d - 1)$$

as t *goes from* 1 *to* 0 *with* γ *a random complex number. Use Euler's method and one Newton correction at each step. Using your solver, solve* $z^5 - 15 * z + 11 = 0$ *for stepsizes* $h = 0.2$, $h = 0.1$, *and* $h = 0.01$.