On the second adjunction mapping and the very ampleness of the triadjoint bundle

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In Memoriam: Professor Wei-Liang Chow

ABSTRACT. Let $\widehat{L}$ be a very ample line bundle on a three dimensional projective manifold $\widehat{X}$, i.e., assume that $\widehat{L} \approx i^*\mathcal{O}_{\mathbb{P}^N}(1)$ for some embedding $i : \widehat{X} \hookrightarrow \mathbb{P}^N$. In this article, we continue a study, started in previous articles, of the rational map, $\psi : \widehat{X} \to \mathbb{P}^{h-1}$, associated to $|K_{\widehat{X}} + \widehat{L}|$, where $h := h^0(K_{\widehat{X}} + \widehat{L})$ under the assumption that the Kodaira dimension of $K_{\widehat{X}} + \widehat{L}$ is three. The degree properties of the map ψ when $\dim \psi(\widehat{X}) = 2$ are studied, e.g., it is shown that in this case if $h \geq 6$, e.g., if $\widehat{X}$ is of nonnegative Kodaira dimension, then the degree of the mapping $\psi_{\widehat{S}}$ for $\widehat{S} \in |\widehat{L}|$ is at most 13, and the degree of the finite part of the Remmert-Stein factorization of ψ is at most 4. If $\dim \psi(\widehat{X}) = 3$ then the degree of the mapping ψ is at most 53. Further it is shown that the morphism associated to $|3(K_{\widehat{X}} + \widehat{L})|$ factors as the natural morphism from $\widehat{X}$ to the second reduction of $(\widehat{X}, \widehat{L})$ followed by an embedding.

Introduction

For simplicity let $\widehat{X} \subset \mathbb{P}^N$ be a connected three dimensional algebraic submanifold of N-dimensional complex projective space. The $n \geq 4$ dimensional analogues of the results we discuss below are easy consequences of the three dimensional results. The very complete theory [**3**] of the adjunction mapping, i.e., the mapping associated to $|K_{\widehat{X}} + 2\widehat{L}|$, is a powerful tool to study the relations between the curve sections on $\widehat{X}$ and the projective geometry of $\widehat{X}$. The second adjunction mapping, i.e., the mapping associated to $|K_{\widehat{X}} + \widehat{L}|$, is a natural tool to study the relations between surface sections and the projective geometry of $\widehat{X}$. Our knowledge of this mapping is in a much more primitive state than our knowledge of the adjunction mapping. In a series of articles [**18, 4, 5**] we have been studying different aspects of

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this mapping. We know that the mapping exists [**18**] except for a very special class of degenerate varieties. Moreover [**4**] we have some strong lower bounds for the number of sections of $|K_{\widehat{X}}+\widehat{L}|$, e.g., if the Kodaira dimension of $\widehat{X}$ is nonnegative, then $h^0(K_{\widehat{X}}+\widehat{L})\geq 5$ with equality only if $(\widehat{X},\widehat{L})$ is a quintic hypersurface of $\mathbb{P}^4$. In [**5**] we studied the case when the mapping associated to $|K_{\widehat{X}}+\widehat{L}|$ has a one dimensional image. In analogy with surface results of Beauville [**1**] we showed that in this case the mapping is well behaved, e.g., if the Kodaira dimension of $\widehat{X}$ is nonnegative, and the mapping associated to $|K_{\widehat{X}}+\widehat{L}|$ has a one dimensional image, then the mapping is a morphism and, if $h:=h^0(K_X+L)\gg 0$ (e.g., $h\geq 21$), then the image of the morphism is a curve of genus ≤ 1 with $K3$ surfaces as fibers.

In this article we study the degree of the mapping associated to $|K_{\widehat{X}}+\widehat{L}|$ when its image is two or three dimensional, and we further give a complete description of the mapping associated to $|3(K_{\widehat{X}}+\widehat{L})|$.

In §1 we introduce background material we use in the form we need it and prove a number of preliminary technical lemmas.

In §2 we prove some general facts about the rational mapping $\psi:\widehat{X}\to\mathbb{P}^{h-1}$ associated to $|K_{\widehat{X}}+\widehat{L}|$.

In §3 we study the degree properties of the map ψ when $\dim\psi(\widehat{X})=2$ and the Kodaira dimension of $K_{\widehat{X}}+\widehat{L}$ is three. For example, we show that in this case if $h^0(K_{\widehat{X}}+\widehat{L})\geq 6$, e.g., if $\widehat{X}$ is of nonnegative Kodaira dimension, then the degree of the mapping $\psi_{\widehat{S}}$ for $\widehat{S}\in|\widehat{L}|$ is at most 13, and the degree of the finite part of the Remmert-Stein factorization of ψ is at most 4.

In §4 we study the degree properties of the map ψ when $\dim\psi(\widehat{X})=3$. For example, we show that the degree of the mapping ψ is at most 53. In §4 we also show that the morphism associated to $|3(K_{\widehat{X}}+\widehat{L})|$ factors as the natural morphism (described in §1) from $\widehat{X}$ to the second reduction of $(\widehat{X},\widehat{L})$ followed by an embedding.

We will study the mapping associated to $|2(K_{\widehat{X}}+\widehat{L})|$ in a sequel.

1. Background material

We refer to our book [**3**] for all the results on adjunction theory that we need in this paper and also for a complete list of references to the original sources.

We work over the complex numbers $\mathbb{C}$. Throughout the paper we deal with projective varieties V. We denote by $\mathcal{O}_V$ the structure sheaf of V and by K_V the canonical bundle. For any coherent sheaf $\mathcal{F}$ on V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V,\mathcal{F})$.

Let L be a line bundle on V. The line bundle L is said to be *numerically effective* (*nef*, for short) if $L\cdot C\geq 0$ for all effective curves C on V. L is said to be *big* if $\kappa(L)=\dim V$, where $\kappa(L)$ denotes the Kodaira dimension of L. If L is nef then this is equivalent to $c_1(L)^n>0$, where $c_1(L)$ is the first Chern class of L and $n=\dim V$.

1.1. **Notation.** The notation used in this paper are standard from algebraic geometry. Let us only fix the following.
$\approx$ (respectively $\sim$), linear (respectively numerical) equivalence of line bundles;
$\chi(L)=\sum_i(-1)^ih^i(L)$, the Euler characteristic of a line bundle L;
$|L|$, the complete linear system associated to a line bundle L on a variety V,

$\Gamma(L) = H^0(L)$, the space of the global sections of L;
$e(V)$, the topological Euler characteristic of V;
$c_n(V)$, the n-th Chern class of the tangent bundle of V for V smooth, which for V compact equals $e(V)$;
$\kappa(V) := \kappa(K_V)$, the Kodaira dimension, for V smooth;
$\lceil x \rceil$, the smallest integer bigger or equal to a rational number x.

We say that a sheaf on a variety is spanned if global sections span it at all points of the variety. Line bundles and divisors are used with little (or no) distinction. Hence we shall freely switch from the multiplicative to the additive notation and vice versa.

1.2. **Genus formula.** For a line bundle L on a variety V of dimension n the sectional genus, $g(L) = g(V, L)$, of (V, L) is defined by $2g(L) - 2 = (K_V + (n-1)L) \cdot L^{n-1}$. Note that if $|L|$ contains $n-1$ elements $H_1, \ldots, H_{n-1}$ meeting in a reduced irreducible curve C, then $g(L) = g(C) = 1 - \chi(\mathcal{O}_C)$, the arithmetic genus of C.

1.3. **Reductions.** (see e.g., [**3**, Chapters 7, 12]). Let $(\widehat{X}, \widehat{L})$ be a smooth projective variety of dimension $n \geq 2$ polarized with a very ample line bundle $\widehat{L}$. A smooth polarized variety (X, L) is called a (*first*) *reduction* of $(\widehat{X}, \widehat{L})$ if there is a morphism $\pi : \widehat{X} \to X$ expressing $\widehat{X}$ as the blowing up of X at a finite set of points, B, such that $L := (\pi_* \widehat{L})^{**}$ is ample and $\widehat{L} \approx \pi^* L - [\pi^{-1}(B)]$ or, equivalently, $K_{\widehat{X}} + (n-1)\widehat{L} \approx \pi^*(K_X + (n-1)L)$.

Note that there is a one-to-one correspondence between smooth divisors of $|L|$ which contain the set B and smooth divisors of $|\widehat{L}|$.

Except for an explicit list of well understood pairs $(\widehat{X}, \widehat{L})$ (see in particular [**3**, §§7.2, 7.3, 7.4]) we can assume:

a) $K_{\widehat{X}} + (n-1)\widehat{L}$ is spanned and big, and $K_X + (n-1)L$ is very ample. Note that this reduction, (X, L), is unique up to isomorphism. We will refer to it as *the first reduction* of $(\widehat{X}, \widehat{L})$;
b) $K_X + (n-2)L$ is nef and big, for $n \geq 3$.

Note that $h^0(K_{\widehat{X}} + (n-2)\widehat{L}) = h^0(K_X + (n-2)L)$. Indeed one has (see [**3**, (7.6.1)])

$$\Gamma(aK_{\widehat{X}} + b\widehat{L}) \cong \Gamma(aK_X + bL) \tag{1}$$

for integers a, b with $b \leq a(n-1)$. Hence in particular

$$H^0(K_{\widehat{X}} + \widehat{L}) \cong H^0(K_X + L).$$

It thus follows that the rational map $\widehat{\psi}$ associated to $|K_{\widehat{X}} + \widehat{L}|$ has the same degree and the same dimension and degree of the image as the rational map ψ associated to $|K_X + L|$.

Note also that $\kappa(K_{\widehat{X}} + (n-2)\widehat{L}) \geq 0$ implies $\kappa(K_{\widehat{X}} + (n-1)\widehat{L}) = n$, so that the first reduction (X, L) exists, as well as $K_X + (n-2)L$ is nef (see [**3**, (7.6.9)]).

Since by the above we can assume that $K_X + (n-2)L$ is nef and big, from the Kawamata-Shokurov base point free theorem (see [**12**], §3) we know that $|m(K_X + (n-2)L)|$, for $m \gg 0$, gives rise to a morphism $\varphi : X \to X'$ with connected fibers and normal image. Thus there is an ample line bundle $\mathcal{K}'$ on X' such that $K_X+(n-2)L \approx \varphi^*\mathcal{K}'$. The pair $(X', \mathcal{K}')$ is known as the *second reduction* of $(\widehat{X}, \widehat{L})$. The morphism φ is very well behaved (see e.g., [**3**, §§7.5, 7.6, 7.7 and Chapter 12] for complete results). Furthermore X' has terminal, 2-Gorenstein (i.e., $2K_{X'}$ is a line bundle) isolated singularities and $\mathcal{K}' \approx K_{X'} + (n-2)L'$, where $L' := (\varphi_* L)^{**}$ is a 2-Cartier divisor such that $2L \approx \varphi^*(2L') - \mathcal{D}$ for some effective Cartier divisor $\mathcal{D}$ on X which is φ-exceptional (see [**3**, (7.5.7)]). For definition and properties of terminal singularities we also refer to [**12**].

We will use in a crucial way the following theorem [**3**, (12.2.1)] which gives the structure of the second reduction in the threefolds case.

THEOREM 1.4. *Let $\widehat{X}$ be a connected smooth threefold with $\widehat{L}$ a very ample line bundle on $\widehat{X}$. Assume that the first reduction (X, L) and the second reduction $(X', \mathcal{K}')$, $\varphi : X \to X'$, as in (1.3) exist. Then φ is an isomorphism outside of an algebraic subset Z of X' such that* $\dim Z \leq 1$. *We have:*

a) *Any 1-dimensional component of Z is a smooth curve, C, such that $C \subset$* $\operatorname{reg}(X')$, *$\varphi^{-1}(C) = D \cup \{\cup D_i\}_{i\in I}$ where D and the D_i's are smooth irreducible divisors. The restriction φ_D of φ to D is a $\mathbb{P}^1$-bundle $\varphi_D : D \to C$ and $\mathcal{N}_{D/X|F} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ for a fiber F of φ_D. Furthermore if I is not empty, $\varphi(D_i)$ is a point for each $i \in I$. Let $x_i = \varphi(D_i)$, $i \in I$. Then $\varphi^{-1}(x_i) = \mathbb{F}_1$, and for each $i \in I$, D meets $\mathbb{F}_1$ transversally along the irreducible curve, E, of $\mathbb{F}_1$, with $(E^2)_{\mathbb{F}_1} = -1$.*

b) *If x is a 0-dimensional component of Z then $\varphi^{-1}(x)$ is a reduced divisor, D, and either*

   b1) *$D \cong \mathbb{P}^2$, $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{P}^2}(-2)$;*

   b2) *$D \cong \mathbb{P}^2$, $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1)$;*

   b3) *$D \cong \widetilde{\mathbb{F}}_2$, $\mathcal{N}_{D/X} \cong -G$ where $-2G \cong K_D$;*

   b4) *$D \cong \mathbb{F}_0$, $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{F}_0}(-1,-1)$; or*

   b5) *$D = D_1 \cup D_2$ where $D_1 \cong \mathbb{F}_2$, $D_2 \cong \mathbb{P}^2$, $\mathcal{N}_{\mathbb{F}_2/X} \approx -E - f$, $\mathcal{N}_{\mathbb{P}^2/X} \cong \mathcal{O}_{\mathbb{P}^2}(-2)$ and D_1, D_2 meet transversally in a smooth rational curve which is the irreducible curve, E, of $\mathbb{F}_2$ with $(E^2)_{D_1} = -2$ and is a line on $\mathbb{P}^2$.*

REMARK 1.5. Let $\widehat{X}$ be a smooth n-fold with $\widehat{L}$ a very ample line bundle. Assume that the first reduction (X, L) of $(\widehat{X}, \widehat{L})$ exists. Further assume that $K_X + (n-2)L$ is nef. Then $2(K_X + (n-2)L)$ is spanned by global sections.

Let $p : X \to Y$ be the morphism with connected fibers and normal image associated to $|m(K_X + (n-2)L|$, $m \gg 0$. Then $K_X + (n-2)L \approx p^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Y.

If $\dim Y = 3$ the result is proved in [**18**, Note added in proof].

Assume that $\dim Y = 2$. Then (X, L) is a quadric fibration under p and from [**6**, §8] we know that Y is smooth and $\mathcal{L} \approx K_Y + \mathcal{M}$ for some ample line bundle $\mathcal{M}$ on Y. Thus $2\mathcal{L} \approx K_Y + \mathcal{L} + \mathcal{M}$.

We claim that $(\mathcal{L} + \mathcal{M})^2 \geq 5$. Indeed, if not,

$$(\mathcal{L} + \mathcal{M})^2 = \mathcal{L}^2 + 2\mathcal{L}\cdot\mathcal{M} + \mathcal{M}^2 = 4,$$

whence $\mathcal{L}^2 = \mathcal{L}\cdot\mathcal{M} = \mathcal{M}^2 = 1$. Therefore $\mathcal{L} \sim \mathcal{M}$ by the Hodge index theorem. Then $K_Y \approx \mathcal{O}_Y$. This contradicts the parity condition from genus formula (1.2).

Thus by Reider's theorem either $2\mathcal{L} \approx K_Y + (\mathcal{L} + \mathcal{M})$ is spanned, and hence $2(K_X + (n-2)L)$ is spanned, or there exists an effective curve C on Y such that

$$(\mathcal{L} + \mathcal{M}) \cdot C \leq C \cdot C < \frac{(\mathcal{L} + \mathcal{M}) \cdot C}{2} < 1.$$

This contradicts the ampleness of $\mathcal{L}$ and $\mathcal{M}$.

Assume that $\dim Y = 1$. Then (X, L) is a Del Pezzo fibration under p and exactly the same argument as in [**18**, (0.5.2)] applies to say that $2\mathcal{L}$ is spanned.

1.6. **Pluridegrees.** Let $(\widehat{X}, \widehat{L})$, (X, L) be as in (1.3) with $n = 3$. Define the *pluridegrees*, for $j = 0, 1, 2, 3$, by

$$\widehat{d}_j := (K_{\widehat{X}} + \widehat{X})^j \cdot \widehat{L}^{3-j} \ \text{ and } \ d_j := (K_X + L)^j \cdot L^{3-j}.$$

If γ denotes the number of points blown up under $\pi : \widehat{X} \to X$, then because $K_{\widehat{X}} + \widehat{L} \cong K_X + L + \sum_i E_i$, the invariants $\widehat{d}_j$, d_j are related by

$$\widehat{d}_0 = d_0 - \gamma \, ; \ \widehat{d}_1 = d_1 + \gamma \, ; \ \widehat{d}_2 = d_2 - \gamma \, ; \ \widehat{d}_3 = d_3 + \gamma.$$

We put $\widehat{d} := \widehat{d}_0$, $d := d_0$. If $K_X + L$ is nef, by the generalized Hodge index theorem (see e.g., [**3**, (2.5.1), (13.1)], [**9**, (1.2)]) one has

$$d_1^2 \geq d d_2 \ \text{ and } \ d_2^2 \geq d_1 d_3 \tag{2}$$

and the parity Lemma (13.1.1) of [**3**] says that

$$d \equiv d_1 \bmod(2) \, ; \ d_2 \equiv d_3 \bmod(2). \tag{3}$$

Moreover if $K_X + L$ is nef and big the numbers d_j are positive.

We need the following versions of the double point formula (for the proofs and full details see [**4**, (0.5)] and also [**3**, (8.2), (13.1)]).

Proposition 1.7. *Let $(\widehat{X}, \widehat{L})$ be a smooth projective 3-fold, polarized with a very ample line bundle, $\widehat{L}$. Assume $\kappa(K_{\widehat{X}} + \widehat{L}) \geq 2$. Let (X, L) and $\pi : \widehat{X} \to X$ be the first reduction and the first reduction map respectively. Let $\widehat{d}_j$, d_j, $0 \leq j \leq 3$, be the pluridegrees of $(\widehat{X}, \widehat{L})$ and (X, L) respectively. Let γ be the number of points blown up by π. Let $\widehat{S}$ be a smooth element in $|\widehat{L}|$. Then*

$$44h^0(K_{\widehat{X}} + \widehat{L}) + 58\chi(\mathcal{O}_{\widehat{S}}) + 2h^0(K_{\widehat{X}}) + 4 \geq 12d_2 + 17d_1 + d_3 + (20 - \widehat{d})\widehat{d} + 5\gamma.$$

There is the following slightly stronger version of the usual double point formula (1.7).

Proposition 1.8. *Let $(\widehat{X}, \widehat{L})$ be a smooth projective 3-fold, polarized with a very ample line bundle, $\widehat{L}$. Let (X, L) and $\pi : \widehat{X} \to X$ be the first reduction and the first reduction map respectively. Let $\widehat{d}_j$, d_j, $0 \leq j \leq 3$, be the pluridegrees of $(\widehat{X}, \widehat{L})$ and (X, L) respectively. Let γ be the number of points blown up by π. Let $\widehat{S}$ be a smooth element in $|\widehat{L}|$. Let $q := h^1(\mathcal{O}_X)$. Then*

$$44h^0(K_{\widehat{X}} + \widehat{L}) + 60\chi(\mathcal{O}_S) + 2h^0(K_{\widehat{X}}) + 2q - 2 \geq 13d_2 + 17d_1 + d_3 + (20 - \widehat{d})\widehat{d} + 5\gamma.$$

Proof. Let S be a smooth surface in $|L|$ corresponding to $\widehat{S}$. Let $p_g := h^0(K_X)$, $p_g(S) = h^0(K_S)$. The proof is the same as in [**4**, (0.5.2)]. The only difference is to leave the q term and not using the Noether inequality $K_S^2 \geq 2p_g(S) - 4$ in relation (0.5.2.5) of [**4**]. Thus, instead of that relation we find,

$$e(\widehat{X}) \leq -4h^0(K_{\widehat{X}} + \widehat{L}) + 2p_g + 24\chi(\mathcal{O}_S) - 2d_2 + 2q - 2 + 2\gamma.$$

Substituting in (0.5.2.1) of [**4**] we get the result. □

The following inequalities follow from the log version of the usual Yau inequality given in [**20**, §5] (see [**4**, (0.6)] or [**3**, (13.1.7), (13.1.8)] for full details).

Proposition 1.9. (Tsuji's inequality) *Let $(\widehat{X}, \widehat{L})$ be a smooth projective 3-fold, polarized with a very ample line bundle, $\widehat{L}$. Assume that $\kappa(K_{\widehat{X}} + \widehat{L}) \geq 0$. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Let S be a smooth element of $|L|$. Then we have*

$$(K_X + L)^3 + \frac{8}{3}K_S \cdot L_S \leq 32(2h^0(K_X + L) - \chi(\mathcal{O}_S)),$$

or

$$h^0(K_X + L) \geq \frac{d_3}{64} + \frac{d_1}{24} + \frac{\chi(\mathcal{O}_S)}{2}.$$

1.10. **Castelnuovo's bound.** Let C be a reduced, irreducible projective curve. Assume that $\psi : C \to \mathbb{P}^N$ is a generically one-to-one morphism, and that $\psi(C)$ does not lie in any hyperplane. Let d denote the degree of $\psi(C)$ in $\mathbb{P}^N$. Let $g(C)$ be the arithmetic genus of C. Then Castelnuovo's bound (see e.g., [**8**, Theorem 3.7]) reads

$$g(C) \leq \text{Castel}(d, N) := \left[\frac{d-2}{N-1}\right]\left(d - N - \left(\left[\frac{d-2}{N-1}\right] - 1\right)\frac{N-1}{2}\right),$$

where $[x]$ means the greatest integer $\leq x$.

1.11. **Threefolds of log-general type.** Let $(\widehat{X}, \widehat{L})$, (X, L) be as in (1.3) with $n = 3$ and let $\widehat{d}_j$, d_j, $j = 0, 1, 2, 3$, be the pluridegrees as in (1.6). We say that $(\widehat{X}, \widehat{L})$ is of *log-general type* if $K_X + L$ is nef and big. Hence in particular the numbers d_j are positive in this case.

Assume $(\widehat{X}, \widehat{L})$ of log-general type. Let $\widehat{S}$ be a smooth element of $|\widehat{L}|$ and S the corresponding smooth surface in $|L|$. Since $K_X + L$ is nef and big the canonical bundle K_S of S is nef and big (see e.g., [**3**, (7.6.10)]), so that S is a minimal surface of general type. Hence we have

$$d_2 = K_S \cdot K_S < 9\chi(\mathcal{O}_S). \tag{4}$$

Indeed the Miyaoka inequality yields $d_2 \leq 9\chi(\mathcal{O}_S)$. Note that the equality cannot happen. Otherwise S is a ball quotient and hence a $K(\Pi, 1)$, which would contradict [**16**, (1.3)]. Throughout the paper we will often use (4) in the form

$$\frac{\chi(\mathcal{O}_S)}{2} \geq \frac{d_2 + 1}{18}. \tag{5}$$

Note that since $\chi(\mathcal{O}_S)$ is an integer we have the stronger integrality relation

$$\chi(\mathcal{O}_S) \geq \left\lceil \frac{d_2 + 1}{9} \right\rceil. \tag{6}$$

Note that if $(\widehat{X}, \widehat{L})$ is of log-general type then the restriction, φ_S, of the second reduction map φ to S coincides with the m-canonical map associated to $|mK_S|$, $m \geq 2$. This follows from the exact sequence

$$0 \to K_X + (m-1)(K_X + L) \to m(K_X + L) \to mK_S \to 0,$$

by noting that $K_X + L$ is nef and big. Therefore the image $S' := \varphi_S(S)$ is the canonical model of S.

Assume that $\kappa(X) \geq 0$. Then from [**17**, (1.5), (3.1)] (see also [**3**, (13.1.3)]), we know that

$$d_3 \geq d_2 \geq d_1 \geq d. \tag{7}$$

Note that if $\kappa(X) \geq 0$, then $(\widehat{X}, \widehat{L})$ is of log-general type. Indeed, if $h^0(tK_X) > 0$ for some positive integer t, then $t(K_X + L)$ gives a birational embedding, given on a Zariski open set by sections of $\Gamma(L)$, and thus $\kappa(K_X + L) = 3$.

The following result is not optimal, but it is more than enough for our purposes.

LEMMA 1.12. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Then either $|\widehat{L}|$ embeds $\widehat{X}$ in $\mathbb{P}^4$ or*

1. $d \geq \widehat{d} = \widehat{L}^3 \geq 8$*;*
2. $d_1 \geq 6$*,* $d_2 \geq 3$*, and* $d_3 \geq 1$*.*

PROOF. We can assume that $\Gamma(\widehat{L})$ embeds $\widehat{X}$ in $\mathbb{P}^N$ with $N \geq 5$. Since a smooth $\widehat{S} \in |L|$ is of general type we have by a result of Castelnuovo (see [**3**, Theorem (8.1.1)], [**14**, (0.6)]) that $\widehat{d} > 2(N-1-2)+2 \geq 6$. Thus $d \geq \widehat{d} \geq 7$. If we show that $\widehat{d} \neq 7$ we are done with showing the degree lower bound. To see this note that $N = 5$ since otherwise $\widehat{d} > 2(N-1-2)+2 \geq 8$. Thus by Castelnuovo's bound we conclude that $g(C) \leq 6$ for any smooth curve section of $\widehat{X} \subset \mathbb{P}^5$. Thus we conclude that $d_1 \leq \widehat{d}_1 \leq 3$. But then $d_2 \leq 1$ by the Hodge index relations (2). Since $K_X + L$ is nef and big we know that $d_3 \geq 1$ and thus by the Hodge index relations we conclude that $d_1 \leq 1$. This implies by the Hodge index relations the absurdity that $d = 1$.

The inequalities $d_1 \geq 6$ and $d_2 \geq 3$ follow from the Hodge index relations and parity conditions (3). $\square$

The following result of the authors and M. Schneider [**2**, Lemma (3.3)] will be useful.

LEMMA 1.13. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Then $6h^0(K_X + L) - d_2 + d_3 - 4\chi(\mathcal{O}_S) = 2h^0(2K_X + L)$. Furthermore $h^0(2K_X + L) = 0$ if either $2d_3 < d_2$, or $2d_2 < d_1$, or $2d_1 < d$.*

PROOF. The result [**2**, Lemma (3.3)] is stated for threefolds in $\mathbb{P}^5$, but the argument for the first assertion works with no change in general. The statements about $h^0(2K_X + L) = 0$ all follow by noting that $2K_X + L = 2(K_X + L) - L$ and that $K_X + L$ and L are nef and big. $\square$

The following will be useful.

LEMMA 1.14. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Assume that $h^0(\widehat{L}) \geq 6$ (which follows from $|K_{\widehat{X}} + \widehat{L}|$ not giving a birational mapping). Then*

1. *if $d_3 \geq 9$ it follows that $d \geq 9$, $d_1 \geq 9$, $d_2 \geq 9$;*
2. *if $d_3 \geq 18$ it follows that $d \geq 10$, $d_1 \geq 14$, $d_2 \geq 16$;*
3. *if $d_3 \geq 81$ it follows that $d \geq 11$, $d_1 \geq 23$, $d_2 \geq 44$;*
4. *if $d_2 \geq 9$ it follows that $d \geq 9$, $d_1 \geq 9$;*
5. *if $d_2 \geq 19$ it follows that $d \geq 10$, $d_1 \geq 14$;*
6. *if $d_2 \geq 41$ it follows that $d \geq 11$, $d_1 \geq 23$.*

PROOF. All of these are proved the same way. For example if $d_3 \geq 81$ then using the fact that $d \geq 8$, $d_1 \geq 6$, $d_2 \geq 3$ we see by using the Hodge index inequalities $dd_2 \leq d_1^1$, $d_1 d_3 \leq d_2^2$ repeatedly and taking account of parity (3) that $d_1 \geq 18$. Using Castelnuovo's inequality (1.10) we see that for degree at most 8 the genus g of a curve section of $\widehat{X}$ is bounded by 9. In this case we see by the genus formula that $d + d_1 \leq 16$. This implies $d \geq 9$. But in this case Castelnuovo's inequality implies that $g \leq 12$ and hence $d + d_1 \leq 22$. So $d \geq 10$. Using this, the Hodge index inequalities repeatedly and taking account of parity we conclude that $d_1 \geq 22$. Again using Castelnuovo's inequality we get the contradiction that $d_1 \leq 20$. Thus $d \geq 11$. Use $d \geq 11$ and the Hodge inequalities again to prove the inequalities $d_1 \geq 23$, $d_2 \geq 44$. □

We also need the following general fact.

LEMMA 1.15. *Let $\mathcal{F}$ be a rank r generically spanned vector bundle on an irreducible and reduced projective variety X such that $c_1(\det(\mathcal{F})) = 0$. Then $\mathcal{F}$ is trivial, i.e., $\mathcal{F} \cong \oplus^r \mathcal{O}_X$.*

PROOF. Choose r sections $s_j \in \Gamma(X, \mathcal{F})$ that span $\mathcal{F}$ at the generic point. Then $s := s_1 \wedge \cdots \wedge s_r$ is a non trivial section of $\det \mathcal{F}$ which is not zero at the generic point. Let $D = [s]$ be the divisor associated to s. Since $c_1(\det \mathcal{F}) = 0$ we see that D is empty, and hence S is nowhere zero. Thus the s_j's are nowhere zero and linearly independent at every point. Therefore they generate $\mathcal{F}$ and $\mathcal{F}$ is trivial. □

For the reader's convenience we also include the main result that we use from [**4**] and which allows us to assume $h := h^0(K_X + L) \geq 6$ in the nonnegative Kodaira dimension case.

THEOREM 1.16. *Let $\widehat{L}$ be a very ample line bundle on a projective manifold $\widehat{X}$ of dimension 3. If the Kodaira dimension of $K_{\widehat{X}} + \widehat{L}$ is 3 then $h^0(K_{\widehat{X}} + \widehat{L}) \geq 2$. If the Kodaira dimension of $\widehat{X}$ is nonnegative, then $h^0(K_{\widehat{X}} + \widehat{L}) \geq 5$ with equality only if $(\widehat{X}, \widehat{L})$ is a degree 5 hypersurface of $\mathbb{P}^4$.*

COROLLARY 1.17. *Let $\widehat{L}$ be a very ample line bundle on a projective manifold $\widehat{X}$ of dimension 3. If the Kodaira dimension of $K_{\widehat{X}} + \widehat{L}$ is 3 then $p_g(\widehat{S}) \geq 2$ for any smooth $\widehat{S} \in |\widehat{L}|$.*

PROOF. By Theorem (1.16) we know that $h^0(K_{\widehat{X}} + \widehat{L}) = h^0(K_X + L) \geq 2$. Then, since $\widehat{L}$ is very ample, we conclude from [**4**, (0.11)] that

$$p_g(S) = p_g(\widehat{S}) = h^0(K_{\widehat{S}}) = h^0(K_{\widehat{X}|\widehat{S}} + \widehat{L}_{\widehat{S}}) \geq 2.$$

□

2. Notation and general set up

Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized with a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type, which implies that the first reduction (X, L), $\pi : \widehat{X} \to X$, of $(\widehat{X}, \widehat{L})$ exists and furthermore $K_X + L$ is nef and big. Let $\widehat{S}$ be a general element of $|\widehat{L}|$ and let S be the corresponding smooth surface in $|L|$. Recall that the restriction $\pi_{\widehat{S}} : \widehat{S} \to S$ maps $\widehat{S}$ onto its minimal model, S, and S is of general type. Recall that $h^0(K_{\widehat{X}} + \widehat{L}) = h^0(K_X + L)$ by (1).

Let $\psi : X \to \mathbb{P}^n$ be the rational map associated to $|K_X + L|$. Let $\Sigma := \psi(X)$. Since from [**4**, (1.2)] we know that $h^0(K_X + L) \geq 2$, we have $\dim \Sigma \geq 1$. In this paper we deal with the case when $\dim \Sigma = 2, 3$ (see [**5**] for the case $\dim \Sigma = 1$). Let $h := h^0(K_X + L)$. Then $\Sigma \subset \mathbb{P}^{h-1}$, so that

$$\deg \Sigma \geq h - \dim \Sigma. \tag{8}$$

2.1. **General construction.** With the notation as above, let $\Gamma \subset X \times \Sigma$ be the graph of ψ. Let $\rho : \overline{X} \to \Gamma$ be a smooth resolution of Γ. Let $\sigma : \overline{X} \to X$ be the composition of ρ with the product projection $X \times \Sigma \to X$, and let $\overline{\psi} : \overline{X} \to \Sigma$ be the composition of ρ with the product projection $X \times \Sigma \to \Sigma$. Let $\overline{\psi} = s \circ \overline{r}$ be the Remmert-Stein factorization of $\overline{\psi}$, where $\overline{r}$ is a morphism with connected fibers and s is a finite morphism. Thus we have a commutative diagram

$$\begin{array}{ccc} \overline{X} - \sigma^{-1}(E) & \xrightarrow{\overline{r}} & C \\ & \overline{\psi} & \\ \sigma \downarrow & \searrow & \downarrow s \\ X - E & \xrightarrow{\psi} & \Sigma \end{array} \tag{9}$$

where E is the indeterminacy locus of ψ. Then there exists a rational map $r := \overline{r} \circ \sigma^{-1} : X \to C$ such that $\psi = s \circ r$. We say that $s \circ r$ is the *Remmert-Stein factorization of the rational map* ψ. Note also that the general fiber, F, of r is irreducible.

Thus we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \Sigma \\ & r & \\ \sigma \uparrow & \searrow & \uparrow s \\ \overline{X} & \xrightarrow{\overline{r}} & Y. \end{array}$$

We will denote $\mathcal{K} := K_X + L$. Then r is the meromorphic map given by the complete linear system $|\mathcal{K}|$. It is a general fact, see, e.g., [**11**, Chapter II, 7.17.3], that the smooth resolution $\rho : \overline{X} \to \Gamma$ can be chosen such that there exists a spanned line bundle $\overline{\mathcal{K}}$ on $\overline{X}$ such that

$$\overline{\mathcal{K}} \approx \sigma^* \mathcal{K} - \mathcal{E}, \tag{10}$$

for some effective divisor $\mathcal{E}$ on $\overline{X}$ with $\sigma(\mathcal{E})$ set theoretically equal to the base locus of $|\mathcal{K}|$, and $\Gamma(\mathcal{K}) \cong \Gamma(\overline{\mathcal{K}})$ under the map given by the inclusion $\sigma_* \overline{\mathcal{K}} \subset \mathcal{K}$. Note that $\overline{r}$ is the morphism given by $|\overline{\mathcal{K}}|$ and $\overline{\mathcal{K}} = \overline{r}^* H$ for some ample and spanned line bundle H on Y.

If $\dim \Sigma = 2$, let $\mathfrak{f}$ be a general fiber of ψ and let f be a general fiber of r. Then note that

$$\#s(L \cdot f) = L \cdot \mathfrak{f}, \tag{11}$$

where $\#s$ denotes the degree of s.

The following simple lemma is useful.

LEMMA 2.2. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Assume further that $|K_X + L|$ gives a rational map, ψ. If $h^0(K_X) > 0$ then ψ is a birational map.*

PROOF. Indeed if $h^0(K_X) > 0$ a section of $\Gamma(K_X)$ gives an embedding $\Gamma(L) \hookrightarrow \Gamma(K_X + L)$ and hence, since L is very ample off a finite set of points, $|K_X + L|$ defines a birational map, ψ, defined on a Zariski open set by sections of L. $\square$

We also need the following general fact.

LEMMA 2.3. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $\kappa(X) \geq 0$. Assume further that $|K_X + L|$ gives a rational map, ψ, having a 2-dimensional image. Let $\psi = s \circ r$ be the Remmert-Stein factorization of ψ as in* (2.1)*. Assume that the base locus $\mathcal{B}$ of $|K_X + L|$ is of codimension at least two. Then ψ is not a morphism and the general fiber f of r meets $\mathcal{B}$.*

PROOF. First note that $\mathcal{B}$ is not empty. To see this note that since $K_X + L$ is big, it would follow from $\mathcal{B}$ being empty that $K_X + L$ is the pullback of a line bundle from the lower dimensional image under ψ.

To see that ψ is not a morphism choose a small complex neighborhood U of $x \in \mathcal{B}$ such that $(K_X + L)_U$ is isomorphic to the trivial bundle and set $Z := \mathcal{B} \cap U$. Using this trivialization a basis of $H^0(K_X + L)$ restricts to a set of holomorphic sections $f_0, \ldots, f_N$ which only have Z as common zeros on U and such that the map $[f_0, \ldots, f_N]$ has a two dimensional image. For $\psi_U = [f_0, \ldots, f_N]$ to be a morphism at x it is necessary that there is a holomorphic function g on U and holomorphic functions $h_0, \ldots, h_N$ on U such that $f_i = gh_i$ for all i and $h_k(x) \neq 0$ for some k. If this happened then we would have all of the f_i zero on the nontrivial zero set of g. This contradicts the codimension at least two hypothesis since Z contains the zero set of g.

With the same notation as in (2.1), let $\mathcal{K} = K_X + L$, $\sigma^*\mathcal{K} \approx \overline{r}^*H + \mathcal{E}$, where $\sigma : \overline{X} \to X$, $\overline{r} : \overline{X} \to Y$ and H is an ample and spanned line bundle on Y. Let D_1, D_2 be general members of $|\overline{r}^*H|$. Since H is spanned we have that $D_1 \cap D_2 \sim (H \cdot H)\overline{f}$, where $\overline{f}$ is a general fiber of $\overline{r}$. From (8) and (1.16) we conclude that $H \cdot H \geq 3$. If the general fiber f of r does not meet $\mathcal{B}$, it thus follows that $\sigma(D_1) \cap \sigma(D_2)$ is disconnected on X, since otherwise $D_1 \cap D_2$ would consist of only one fiber.

Since the line bundle $\mathcal{L} := \mathcal{O}_X(\sigma(D_1))$ on X agrees with $K_X + L$ off the set $\mathcal{B}$ whose codimension is at least two, it follows that $K_X + L \approx \mathcal{L}$.

Look at the Koszul complex

$$0 \to -2\mathcal{L} \to -\mathcal{L} \oplus -\mathcal{L} \to \mathcal{O}_X \to \mathcal{O}_{\sigma(D_1) \cap \sigma(D_2)} \to 0.$$

Since $\mathcal{L}$ is nef and big $h^i(-2\mathcal{L}) = h^i(-\mathcal{L}) = 0$ for $i < 3$. On the other hand $h^0(\mathcal{O}_X) = 1$ and $h^0(\mathcal{O}_{\sigma(D_1) \cap \sigma(D_2)}) \geq 2$ by disconnectedness. This leads to a contradiction. Thus we conclude that the general fiber f of r meets $\mathcal{B}$. $\square$

3. The second adjunction mapping: the case of a 2-dimensional image

We follow the notation in §2. In this section we assume that the rational map $\psi : X \to \mathbb{P}^n$ associated to $|K_X + L|$ as an image $\Sigma := \psi(X)$ of dimension 2. Let $\mathfrak{f}$ be a general fiber of ψ.

We have the following general bounds.

LEMMA 3.1. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Assume further that $|K_X + L|$ gives a rational map, ψ, having a 2-dimensional image. Let $\mathfrak{f}$ be a general fiber of ψ and let $\mathcal{K} = K_X + L$. Let $h := h^0(K_X + L)$. Then*

1. $d_1 = \mathcal{K} \cdot L \cdot L > 0$;
2. $d_2 = L \cdot \mathcal{K} \cdot \mathcal{K} \geq L \cdot \mathfrak{f}(h-2)$;
3. $d_3 = \mathcal{K} \cdot \mathcal{K} \cdot \mathcal{K} \geq \mathcal{K} \cdot \mathfrak{f}(h-2) \geq h-2$.

PROOF. Notation as in §2. Since $\mathcal{K}$ is nef and big and L is ample one has $d_1 > 0$. Write $\mathcal{H} := \overline{\mathcal{K}}$. Then $\sigma^*\mathcal{K} \approx \mathcal{H} + \mathcal{E}$ by (10). Compute

$$\begin{aligned} d_2 &= \sigma^*\mathcal{K} \cdot \sigma^*\mathcal{K} \cdot \sigma^*L = \sigma^*\mathcal{K} \cdot (\mathcal{H}+\mathcal{E}) \cdot \sigma^*L \\ &= (\mathcal{H}+\mathcal{E}) \cdot \mathcal{H} \cdot \sigma^*L + \sigma^*\mathcal{K} \cdot \mathcal{E} \cdot \sigma^*L \\ &= \mathcal{H} \cdot \mathcal{H} \cdot \sigma^*L + \mathcal{E} \cdot \mathcal{H} \cdot \sigma^*L + \sigma^*\mathcal{K} \cdot \mathcal{E} \cdot \sigma^*L. \end{aligned}$$

Clearly $\sigma^*\mathcal{K} \cdot \mathcal{E} \cdot \sigma^*L \geq 0$. Also, $\mathcal{H} \cdot \mathcal{H} \cdot \sigma^*L = (L \cdot \mathfrak{f}) \deg \Sigma \geq L \cdot \mathfrak{f}(h-2)$ by (8). Noting that L and $\mathcal{H}$ are spanned and $\mathcal{E}$ is effective we have $\mathcal{E} \cdot \mathcal{H} \cdot \sigma^*L \geq 0$, and thus 2) is proved.

To show 3), compute

$$\begin{aligned} d_3 &= (\sigma^*\mathcal{K})^3 = (\sigma^*\mathcal{K})^2 \cdot (\mathcal{H}+\mathcal{E}) \\ &\geq (\sigma^*\mathcal{K})^2 \cdot \mathcal{H} = (\mathcal{H}+\mathcal{E}) \cdot \mathcal{H} \cdot \sigma^*\mathcal{K} \geq \mathcal{H} \cdot \mathcal{H} \cdot \sigma^*\mathcal{K}, \end{aligned}$$

where the two inequalities follow from the fact that $\mathcal{H}$, $\mathcal{E}$ are effective and $\sigma^*\mathcal{K}$ is nef. By using (8) and the fact that $\mathcal{K} \cdot \mathfrak{f} \geq \mathcal{K} \cdot f > 0$ since $\mathcal{K}$ is nef and big and f moves in a family covering X (see also [**3**, (2.1.2)]), we have $\mathcal{H} \cdot \mathcal{H} \cdot \sigma^*\mathcal{K} = (\mathcal{K} \cdot \mathfrak{f}) \deg \Sigma \geq \mathcal{K} \cdot \mathfrak{f}(h-2) \geq h-2$, so we are done. □

We can prove now the main result of this section.

THEOREM 3.2. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Assume further that $|K_X + L|$ gives a rational map, ψ, having a 2-dimensional image. Let $\mathfrak{f}$ be a general fiber of ψ and let $\mathcal{K} = K_X + L$. Let $h := h^0(K_X + L)$.*

1. *If $h \geq 6$, e.g., if $\kappa(X) \geq 0$, then $L \cdot \mathfrak{f} \leq 13$;*
2. *If $h = 5$, then $L \cdot \mathfrak{f} \leq 14$;*
3. *If $h = 4$, then $L \cdot \mathfrak{f} \leq 17$;*
4. *If $h = 3$ then $L \cdot \mathfrak{f} \leq 24$.*

PROOF. We can assume without loss of generality that $h^0(K_X) = 0$ since otherwise ψ has a 3-dimensional image (see (2.2)).

Assume that $L \cdot \mathfrak{f} \geq 14$. Using relation (5) and Lemma (3.1), 2) we have for any smooth $S \in |L|$ that $\chi(\mathcal{O}_S) \geq \dfrac{d_2+1}{9} \geq \dfrac{14(h-2)+1}{9}$ and thus by Lemma (1.13) that

$$d_3 + 6h \geq 4\frac{14(h-2)+1}{9} + 14(h-2),$$

i.e., that $d_3 \geq \dfrac{128}{9}h - 40$. Therefore from Tsuji's inequality (1.9) we obtain $d_1 \leq 51$. If $h \geq 15$ then $d_2 \geq 13 \cdot 14 = 182$. Using Castelnuovo's inequality and the Hodge inequality $dd_2 \leq d_1^2$ as in Lemma (1.14) we obtain $d \geq 14$ and $d_1 \geq 52$.

If $h = 14$ then proceeding the same way we obtain $d_2 \geq 168$ and $d_1 \geq 50$. Thus using Tsuji's inequality (1.9) and Lemma (1.13) we obtain $\chi(\mathcal{O}_S) \leq \dfrac{509}{27}$, and thus $\chi(\mathcal{O}_S) \leq 18$. Then Miyaoka's inequality (4) contradicts $d_2 \geq 168$.

If $h = 13$ then proceeding the same way we obtain $d_2 \geq 154$ and $d_1 \geq 45$. Thus using Tsuji's inequality (1.9) and Lemma (1.13) we obtain $\chi(\mathcal{O}_S) \leq \dfrac{53}{3}$, and thus $\chi(\mathcal{O}_S) \leq 17$. Again (4) contradicts $d_2 \geq 154$.

If $h = 12$ then proceeding the same way we obtain $d_2 \geq 140$ and $d_1 \geq 43$. Thus using Tsuji's inequality (1.9) and Lemma (1.13) we obtain $\chi(\mathcal{O}_S) \leq \dfrac{439}{27}$, and thus $\chi(\mathcal{O}_S) \leq 16$. Using the double point formula (1.7) we find that $d \geq 55$ and the Hodge inequality $d_1^2 \geq dd_2$, together with parity condition (3), gives $d_1 \geq 89$ contradicting $d_1 \leq 51$.

This same reasoning takes care of the cases $h = 7$ to 11.

If $h = 6$ then proceeding the same way we obtain $d_2 \geq 56$ and $d_1 \geq 25$. From Lemma (1.13) we get $d_3 \geq 4\chi(\mathcal{O}_S) + 20$. Using Tsuji's inequality with these bounds for d_1 and d_3 we find $\chi(\mathcal{O}_S) \leq 8$. Using the double point formula (1.7) and the Hodge inequalities gives $d \geq 39$ and $d_1 \geq 47$. Using $d_2 \geq 56$ and the Miyaoka inequality we obtain $\chi(\mathcal{O}_S) \geq 7$. Then Lemma (1.13) yields $d_3 \geq 48$. This gives a contradiction to Tsuji's inequality

$$6 \geq \frac{48}{64} + \frac{47}{24} + \frac{7}{2} > 6.$$

Assuming that $L \cdot \mathfrak{f} \geq 15$ the same reasoning leads to a contradiction in the case $h = 5$.

Now assume that $h = 4$ and $L \cdot \mathfrak{f} \geq 18$. From $d_2 \geq 18(h-2) \geq 36$ and relation (5) we conclude that $\chi(\mathcal{O}_S) \geq 5$. From Lemma (1.13) we conclude that $d_3 \geq 36 - 24 + 20 = 32$. From $d_2 \geq 36$, the inequality $d \geq 10$ from Lemma (1.14) and the Hodge inequality $dd_2 \leq d_1^2$, together with the parity condition, we conclude that $d_1 \geq 20$. From Tsuji's inequality we conclude that $\chi(\mathcal{O}_S) \leq 5$. Using the double point formula (1.7) with $h = 4$, $\chi(\mathcal{O}_S) \leq 5$ we conclude that $d \geq 28$ and $d_1 \geq 37$. Using this we conclude from Tsuji's inequality the contradiction that $\chi(\mathcal{O}_S) \leq 4$.

Assuming that $L \cdot \mathfrak{f} \geq 25$ the same reasoning used in the above paragraphs leads to a contradiction in the case $h = 3$. □

Let $\psi = s \circ r$ be the Remmert-Stein factorization of ψ as in §2. In the remaining part of this paragraph we will give a bound for the genus of a general (connected) fiber of r and for the degree of the finite morphism s.

First, we need the following general bound for the degree of a general fiber f of r.

Proposition 3.3. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Assume further that $|K_X + L|$ gives a rational map, ψ, having* *a 2-dimensional image. Let $\psi = s \circ r$ be the Remmert-Stein factorization of ψ and let f be a general fiber of the connected part r. Let $\mathcal{B}$ be the base locus of $|K_X + L|$. Then*

1. $L \cdot f \geq 3$;
2. $L \cdot f \geq 4$ *if* $\kappa(X) \geq 0$ *and* $\dim \mathcal{B} = 0$;
3. $L \cdot f \geq 4$ *if* $\kappa(X) = 3$.

PROOF. Let us prove that $L \cdot f \geq 3$. Assume $L \cdot f \leq 2$. Then $f \cong \mathbb{P}^1$. Since f moves in a $h^0(\mathcal{N}_{f/X})$-dimensional family, the normal bundle $\mathcal{N}_{f/X}$ of f in X is generically spanned and hence it is spanned since $f \cong \mathbb{P}^1$. Therefore $\det \mathcal{N}_{f/X} \cong \mathcal{O}_{\mathbb{P}^1}(a)$ for some integer $a \geq 0$ and the adjunction formula gives

$$K_{X|f} \approx K_f - \det(\mathcal{N}_{f/X}) \approx \mathcal{O}_{\mathbb{P}^1}(-2-a).$$

Thus $(K_X + L) \cdot f \leq -1$ if $L \cdot f = 1$ and $(K_X + L) \cdot f = 0$ if $L \cdot f = 2$. The first case contradicts the nefness assumption of $K_X + L$, the latter case contradicts the bigness assumption of $K_X + L$ (see e.g., [**3**, (2.1.2)]).

To show 3), assume $L \cdot f = 3$. Then Castelnuovo's bound gives $g(f) \leq 1$. Let $\widetilde{f}$ be a desingularization of f. Hence $g(\widetilde{f}) \leq g(f) \leq 1$, so that either $f \cong \mathbb{P}^1$ or f is an elliptic curve. In both the cases the additivity of the Kodaira dimension, $\kappa(X) \leq 2 + \kappa(f) \leq 2$, contradicts the assumption $\kappa(X) = 3$.

To show 2), assume $L \cdot f = 3$. Since $\kappa(X) \geq 0$, the same argument as above, by using the additivity of the Kodaira dimension, shows that the only possibility is $g(f) = 1$ and f is a smooth elliptic curve. Since f moves in a $h^0(\mathcal{N}_{f/X})$-dimensional family, the normal bundle $\mathcal{N}_{f/X}$ of f in X is generically spanned and therefore $\deg(\det \mathcal{N}_{f/X}) \geq 0$.

Assume $\deg(\det \mathcal{N}_{f/X}) > 0$. Then from $K_f \approx K_{X|f} + \det(\mathcal{N}_{f/X})$ we conclude that $K_X \cdot f < 0$. Since $\kappa(X) \geq 0$ this contradicts the fact that f moves.

So we can assume $\deg(\det \mathcal{N}_{f/X}) = 0$ and therefore Lemma (1.15) implies that $\mathcal{N}_{f/X}$ is trivial. Since $\dim \mathcal{B} = 0$, Lemma (2.3) applies to say that the general fiber f meets $\mathcal{B}$. Then there exists a pencil of fibers f through a point $x \in \mathcal{B}$. By deformation theory this implies that $\deg(\det \mathcal{N}_{f/X}) > 0$, contradicting the present assumption. □

PROPOSITION 3.4. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Assume further that $|K_X + L|$ gives a rational map, ψ, having a 2-dimensional image. Let $\psi = s \circ r$ be the Remmert-Stein factorization of ψ and let f be a general fiber of the connected part r. Let $h := h^0(K_X + L)$. Then*

1. $g(f) \leq 66$ *if* $h \geq 6$, *e.g., if* $\kappa(X) \geq 0$;
2. $g(f) \leq 78$ *if* $h = 5$;
3. $g(f) \leq 120$ *if* $h = 4$;
4. $g(f) \leq 253$ *if* $h = 3$;
5. $g(f) \leq 26$ *if* ψ *is a morphism and* $h \gg 0$ *(e.g.,* $h \geq 77$*).*

PROOF. By Theorem (3.2) and (11) we know that $L \cdot f \leq 13$ if $h \geq 6$, $L \cdot f \leq 14$ if $h = 5$, $L \cdot f \leq 17$ if $h = 4$ and $L \cdot f \leq 24$ if $h = 3$. Since the maximum genus $g(f)$ is taken when f is a plane curve, the genus formula gives the desired bounds in 1), 2), 3) and 4).

Assume now that ψ is a morphism, so that $K_{X|f} \cong K_f$ and hence, by using the lower bound $L \cdot f \geq 3$ from (3.3),

$$\mathcal{K} \cdot f = K_f + L \cdot f \geq 2g(f) + 1.$$

Thus Lemma (3.1) gives $d_2 \geq L \cdot f(h-2) \geq 3(h-2)$ and $d_3 \geq \mathcal{K} \cdot f(h-2) \geq (2g(f)+1)(h-2)$. Therefore Tsuji's inequality (1.9) and Miyaoka's inequality (5) yield

$$h \geq \frac{(2g(f)+1)(h-2)}{64} + \frac{d_1}{24} + \frac{h-2}{6} + \frac{1}{18}. \tag{12}$$

Assume $g(f) \geq 27$. Then (12) leads to

$$h \geq \frac{55}{64}(h-2) + \frac{h-2}{6} + \frac{1}{18},$$

which implies $h \leq 76$. □

We give now some bound for the degree of the finite morphism s of the Remmert-Stein factorization of the morphism $\psi = s \circ r$ as in §2.

PROPOSITION 3.5. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Assume further that $|K_X + L|$ gives a rational map, ψ, having a 2-dimensional image. Let $\mathcal{B}$ be the base locus of $|K_X + L|$. Let $\psi = s \circ r$ be the Remmert-Stein factorization of ψ and let f be a general fiber of the connected part r. Let $\#s$ be the degree of the finite morphism s. Let $h := h^0(K_X + L)$. Then*

1. $\#s \leq \dfrac{13}{L \cdot f}$ *(and hence $\#s \leq 4$) if $h \geq 6$, e.g., if $\kappa(X) \geq 0$;*
2. $\#s \leq 3$ *if $\kappa(X) \geq 0$ and $\dim \mathcal{B} = 0$;*
3. $\#s \leq 3$ *if $\kappa(X) = 3$.*

PROOF. Let $\mathfrak{f}$ be a general fiber of ψ. Recall that $\#s \leq \dfrac{13}{L \cdot f}$ by (11). Assume $h^0(K_X + L) \geq 6$. Thus Theorem (3.2) implies that $L \cdot \mathfrak{f} \leq 13$, so we conclude that $\#s \leq \dfrac{13}{L \cdot f}$. Hence in particular $\#s \leq 4$ since $L \cdot f \geq 3$ by (3.3), 1). This shows 1). By (3.3), 2) we get 2). If $\kappa(X) = 3$, (3.3), 3) yields $L \cdot f \geq 4$ and hence the argument above gives $\#s \leq \dfrac{14}{L \cdot f} \leq 3$. □

REMARK 3.6. Notation as in Proposition (3.5). Assume $\kappa(X) \geq 0$. Then if $\#s = 4$, the fiber f is a smooth elliptic cubic. Indeed, by looking over the proof of (3.5), wee see that $\#s = 4$ implies $L \cdot f = 3$. Then $g(f) = 1$ by Castelnuovo's bound. If f is singular, let $\widetilde{f}$ be a desingularization of f. Then $g(\widetilde{f}) = 0$, so that $\widetilde{f} \cong \mathbb{P}^1$ and hence we have the contradiction $\kappa(X) \leq \kappa(f) + \dim \psi(X) < 0$. Thus f is smooth and we are done.

Assume $\kappa(X) = 3$. Then if $\#s = 3$, the fiber f is a plane quartic. Indeed $\#s = 3$ implies $L \cdot f = 4$. If f lies in $\mathbb{P}^N$ with $N \geq 3$, Castelnuovo's bound yields $g(f) = 1$ and hence we have the contradiction $\kappa(X) \leq \kappa(f) + \dim \psi(X) \leq 2$. Thus f lies in $\mathbb{P}^2$.

We conclude this section by recalling an example from [**13**] which shows that the rational map ψ can have a 0-dimensional base locus.

EXAMPLE 3.7. From Fujita's lists of Del Pezzo 3-folds (see [**10**, (8.11)]) we know that there exists a polarized threefold (X, M), M ample line bundle on X, such that $K_X \approx -2M$ and $M^3 = 1$. In [**13**] it is shown that $L := 3M$ is very ample, $K_X + L \approx M$ is not spanned and has a single point as base locus. Furthermore $h^0(K_X + L) = 3$, so that $|K_X + L|$ gives a birational map $\psi : X \to \mathbb{P}^2$. The invariants are $d_3 = 1$, $d_2 = 3$, $d_1 = 9$, $d = 27$.

4. The second adjunction mapping: the case of a 3-dimensional image

Notation as in §2. In this section we assume that the rational map $\psi : X \to \mathbb{P}^n$ associated to $|K_X + L|$ as an image $\Sigma := \psi(X)$ of dimension 3. Our aim is to give a bound for the degree $t := \#\psi$ of ψ. From Lemma (2.2) we can assume

$$h^0(K_X) = 0. \tag{13}$$

Let $h := h^0(K_X + L)$. Then we have

$$d_3 \geq t(h - 3). \tag{14}$$

Indeed, $\Sigma \subset \mathbb{P}^{h-1}$ and therefore $\deg \Sigma \geq h - \dim \Sigma = h - 3$, so that

$$d_3 = (K_X + L)^3 \geq t \deg \psi(X) \geq t(h - 3).$$

Let S be a general smooth member of $|L|$. We claim that if $t \geq 5$ then

$$d_2 = K_S^2 \geq 3h - 7. \tag{15}$$

By Beauville's generalization of an inequality of Castelnuovo [**1**] we know that if the rational map given by $|K_S|$ is generically one-to-one then

$$d_2 \geq 3h^2(\mathcal{O}_S) - 7 \geq 3h - 7,$$

where the last inequality follows from (13). Thus we can assume that $|K_S|$ does not give a generically one-to-one map. Therefore, since L_S is very ample outside of a finite set of points, $h^0(K_S - L_S) = 0$. Thus since $h^0(K_X) = 0$, the map $H^0(K_X + L) \to H^0(K_{S|C})$ is injective for any smooth $C \in |L_S|$. Then $h \leq h^0(K_{S|C})$. We conclude from Clifford's theorem that

$$h^0(K_{S|C}) \leq \frac{\deg(K_{S|C})}{2} + 1 = \frac{d_1}{2} + 1$$

with equality implying that C is hyperelliptic. But since $K_X + 2L$ is very ample by the main result of [**19**] and its restriction to C is K_C we conclude that C is not hyperelliptic and thus that $d_1 \geq 2h - 1$. By this inequality, inequality (14), and the Hodge index theorem we conclude that

$$d_2^2 \geq d_3 d_1 \geq t(h - 3)(2h - 1).$$

Noting that $h \geq 4$ it is a straightforward check that for $t \geq 5$ we have $t(h-3)(2h-1) \geq (3h - 7)^2$, whence inequality (15).

Using this and Miyaoka's inequality (5) we conclude that for $t \geq 5$

$$\chi(\mathcal{O}_S) \geq \frac{h - 2}{3}. \tag{16}$$

Let us now give a bound for the degree of ψ.

PROPOSITION 4.1. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $|K_{\widehat{X}} + \widehat{L}|$ gives a rational map, $\widehat{\psi}$, having a 3-dimensional image. Let $h := h^0(K_{\widehat{X}} + \widehat{L})$ and let $t := \#\widehat{\psi}$ be the degree of $\widehat{\psi}$. Then $t \leq 53$.*

PROOF. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. Let ψ be the rational map associated to $|K_X + L|$. By recalling the discussion following (1), we can without loss of generality work with X, L, ψ.

Since the result is trivial for hypersurfaces in $\mathbb{P}^4$, we can assume without loss of generality that $h^0(\widehat{L}) \geq 6$. Thus we have that a smooth $\widehat{S} \in |\widehat{L}|$ is a general type surface embedded in $\mathbb{P}^m$ with $m \geq 4$ and its image on X is a smooth minimal model surface of general type S.

Now assume that $t \geq 54$. Thus $d_3 \geq 54(h-3)$ by (14). Using this and the inequality (16) in Tsuji's inequality (1.9) we conclude that

$$h + 4d_1 \leq 275. \tag{17}$$

Since ψ has a three dimensional image, it follows that $h \geq 4$. We use the inequality (17) and a bootstrap procedure to deal with the cases $h \geq 5$ and then turn with more a detailed argument to deal with the case $h = 4$.

If $h \geq 5$ then $d_3 \geq 108$. By Lemma (1.14) we conclude that $d \geq 11$. Using the Hodge inequalities and parity conditions repeatedly we conclude that $d_1 \geq 25$ and $d_2 \geq 52$. From Miyaoka's inequality (5) we conclude that $\chi(\mathcal{O}_S) \geq 6$. From Tsuji's inequality we obtain

$$h \geq \frac{54(h-3)}{64} + \frac{25}{24} + \frac{6}{2}.$$

Simplified this gives $h > 9$. If $h \geq 10$ then $d_3 \geq 378$ and by using the Hodge inequalities and the reasoning of Lemma (1.14) we conclude that $d \geq 13$, $d_1 \geq 41$, and $d_2 \geq 125$. From $d_2 \geq 125$ we conclude that $\chi(\mathcal{O}_S) \geq 14$. From Tsuji's inequality we obtain

$$h \geq \frac{54(h-3)}{64} + \frac{41}{24} + \frac{14}{2}.$$

Simplified this gives $h > 39$. Then (14) gives $d_3 \geq 1998$ and from the Hodge inequalities and parity conditions as in Lemma (1.14) we conclude that $d_1 \geq 80$. This contradicts the inequality (17).

We now do the case with $h = 4$. From the inequality (14) we conclude that $d_3 \geq 54$. By Lemma (1.14) we conclude that $d \geq 10$. Using the usual Hodge index relations and parity conditions repeatedly we conclude that

$$d \geq 10, \quad d_1 \geq 18, \quad d_2 \geq 32, \quad d_3 \geq 54.$$

By Miyaoka's inequality we have $\chi(\mathcal{O}_S) \geq 4$. Then Tsuji's inequality (1.9) gives

$$4 = h \geq \frac{54}{64} + \frac{18}{24} + \frac{\chi(\mathcal{O}_S)}{2},$$

which implies $\chi(\mathcal{O}_S) = 4$. Using the double point formula (1.7) with $h = \chi(\mathcal{O}_S) = 4$ with the Hodge inequalities and parity conditions repeatedly we conclude that

$$d \geq 45, \quad d_1 \geq 49, \quad d_2 \geq 52, \quad d_3 \geq 54.$$

In particular we have the contradiction $\chi(\mathcal{O}_S) \geq 6$ from Miyaoka's inequality. $\square$

REMARK 4.2. It can be shown by the same reasoning that $t \leq 32$ for $h \leq 17$. Unfortunately the lack of lower bound for d_1 involving h linearly prevents us from showing this for all h.

REMARK 4.3. It is worth emphasizing that if $h^0(K_X) > 0$ then $t = 1$ (see (2.2)). Further note that if $p_g(\Sigma) > 0$ where by $p_g(\Sigma)$ we mean $h^3(\mathcal{O}_{\overline{\Sigma}})$ for $\overline{\Sigma}$, a desingularization of $\Sigma := \dim \psi(X)$, then $h^0(K_X) > 0$ and $t = 1$.

THEOREM 4.4. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Assume further that $|K_{\widehat{X}} + \widehat{L}|$ gives a rational map, $\hat{\psi}$, having a 3-dimensional image. Let $h := h^0(K_{\widehat{X}} + \widehat{L})$ and let $t := \#\hat{\psi}$ be the degree of $\hat{\psi}$. Then $t \leq 31$ if $q(X) = 0$.*

PROOF. Let (X, L) be the first reduction of $(\widehat{X}, \widehat{L})$. As in the last proposition it suffices to work with X, L, and the rational map ψ associated to $|K_X + L|$.

We assume that $t \geq 32$ and deduce a contradiction. Since $q(X) = 0$ we have $q(S) = 0$ and using the exact sequence

$$0 \to K_X \to K_X + L \to K_S \to 0$$

we get

$$\chi(\mathcal{O}_S) \geq h + 1. \tag{18}$$

Then Tsuji's inequality (1.9), (14) and (18) yield

$$h \geq \frac{32(h-3)}{64} + \frac{h+1}{2} + \frac{d_1}{24},$$

or

$$d_1 \leq 24.$$

We claim that $d_1 \leq 24$ implies that $h \leq 6$. Indeed if $h \geq 7$ we have $d_3 \geq 128$ by (14). Using Lemma (1.14) we see that $d \geq 11$, $d_1 \geq 23$, $d_2 \geq 44$. Using the Hodge inequalities (2) and parity conditions (3) repeatedly we have the contradiction $d_1 \geq 27$.

If $h = 6$ then $d_3 \geq 96$ and Lemma (1.14) implies that $d \geq 11$, $d_1 \geq 23$, and $d_2 \geq 44$. One more application of the Hodge inequalities yields $d_2 \geq 48$. From Tsuji's inequality using $d_1 \geq 23$ and $d_3 \geq 96$ we conclude that $\chi(\mathcal{O}_S) \leq 7$. From the double point formula (1.7) with $\chi(\mathcal{O}_S) \leq 7$ and $h = 6$ combined with the Hodge inequalities we conclude that $d \geq 50$ and the contradiction $d_1 \geq 64$.

If $h = 5$ then $d_3 \geq 64$ and Lemma (1.14) implies that $d \geq 10$, $d_1 \geq 14$, $d_2 \geq 16$. Using the Hodge inequalities and parity conditions repeatedly yields that $d \geq 10$, $d_1 \geq 20$, and $d_2 \geq 36$. From Tsuji's inequality using $d_1 \geq 20$ and $d_3 \geq 64$ we conclude that $\chi(\mathcal{O}_S) \leq 6$. From the double point formula (1.7) with $\chi(\mathcal{O}_S) \leq 6$ and $h = 5$ combined with the Hodge inequalities we conclude that $d \geq 45$ and the contradiction $d_1 \geq 52$.

Finally if $h = 4$ and $d_3 \geq 32$ then Lemma (1.14) implies that $d \geq 10$, $d_1 \geq 14$, $d_2 \geq 16$. This and the Hodge inequalities, taking account of parity conditions, yields that $d \geq 10$, $d_1 \geq 16$, and $d_2 \geq 23$. From Tsuji's inequality using $d_1 \geq 16$ and $d_3 \geq 32$ we conclude that $\chi(\mathcal{O}_S) \leq 5$. From the double point formula (1.7) with $\chi(\mathcal{O}_S) \leq 5$ and $h = 4$ combined with the Hodge inequalities we conclude that $d \geq 37$ and the contradiction $d_1 \geq 37$. □

5. On embeddings of the second reduction

Notation as in §2. Consider the second reduction (X', L'), $\varphi : X \to X'$, $L' := (\varphi_* L)^{**}$ of (X, L). Recall that X' has terminal 2-Gorenstein singularities, and $2L'$ and $\mathcal{K}' := K_{X'} + L'$ are line bundles. In this section we will show that $3(K_{X'} + L')$ gives an embedding on X' (see Corollary (5.4) below). We will derive this fact from the following more general result.

THEOREM 5.1. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L), $\pi : \widehat{X} \to X$, (X', L'), $\varphi : X \to X'$ be the first and the second reduction as in (1.3). Let $\mathcal{L}$ be a line bundle on X'. Let $\widehat{S}$ be a smooth surface in $|\widehat{L}|$ and let $S := \pi(\widehat{S})$, $S' = \varphi(S)$. Let $\varphi_S : S \to S'$ and $\mathcal{L}_{S'}$ be the restrictions of φ and $\mathcal{L}$ to S and S' respectively. Assume that*

i) *the image of $\Gamma(\varphi^*\mathcal{L}) \to \Gamma(\varphi_S^*\mathcal{L}_{S'})$ is surjective and spans $\varphi_S^*\mathcal{L}_{S'}$;*
ii) *the morphism, σ, defined by $|\varphi_S^*\mathcal{L}_{S'}|$ factors as $\sigma = s \circ \varphi_S$, where s is an embedding.*

Then $\mathcal{L}$ is very ample.

PROOF. First note that $\mathcal{L}$, $\varphi^*\mathcal{L}$ are spanned by global sections. To see this, take a point $x \in X$ and let $\widehat{x} \in \widehat{X}$ such that $x = \pi(\widehat{x})$. By Bertini's type theorem (see e.g., [**3**, (1.7.9)]) there exists a smooth $\widehat{A} \in |\widehat{L}|$ passing through $\widehat{x}$. Then the image $A := \pi(\widehat{A})$ is a smooth divisor in $|L|$ passing through x. Let $A' := \varphi(A)$. Since by the assumption i) the image of $\Gamma(\varphi^*\mathcal{L}) \to \Gamma(\varphi_A^*\mathcal{L}_{A'})$ spans $\varphi_A^*\mathcal{L}_{A'}$, we see that $\varphi^*\mathcal{L}$ is spanned at x. Since this is true for any point $x \in X$, we conclude that $\varphi^*\mathcal{L}$ is spanned, and hence $\mathcal{L}$ is also spanned.

Thus we have a commutative diagram

$$\begin{array}{ccc} X & & \\ \varphi\downarrow & \overset{\Phi}{\searrow} & \\ X' & \xrightarrow{\Phi'} & \mathbb{P}_{\mathbb{C}}, \end{array} \tag{19}$$

where Φ, Φ' are the morphisms given by $\varphi^*\mathcal{L}$ and $\mathcal{L}$ respectively. To show that $\mathcal{L}$ is very ample is equivalent to show that Φ' is an embedding.

By condition ii) we have a commutative diagram

$$\begin{array}{ccc} S & & \\ \varphi_S\downarrow & \overset{\sigma}{\searrow} & \\ S' & \xrightarrow{s} & \mathbb{P}_{\mathbb{C}}, \end{array} \tag{20}$$

where s is an embedding. Since $\Gamma(\varphi^*\mathcal{L}) \cong \Gamma(\mathcal{L})$ and $\Gamma(\varphi_S^*\mathcal{L}_{S'}) \cong \Gamma(\mathcal{L}_{S'})$, the assumption ii) gives a surjection $\Gamma(\mathcal{L}) \to \Gamma(\mathcal{L}_{S'})$, and hence $s = \Phi'_{S'}$ is the restriction of Φ' to S'.

Take two distinct points $x_1', x_2' \in X'$ and let $\widehat{x}_1, \widehat{x}_2 \in \widehat{X}$ such that $\pi(\widehat{x}_i) := x_i$ and $\varphi(x_i) = x_i'$, $i = 1, 2$. By Bertini's type theorem (see e.g., [**3**, (1.7.9)]) we know that there exists a smooth divisor $\widehat{A} \in |\widehat{L}|$ passing through $\widehat{x}_1$, $\widehat{x}_2$. Then the image $A := \pi(\widehat{A})$ is a smooth divisor in $|L|$ passing through x_1, x_2. Furthermore $A' := \varphi(A)$ is a surface passing through x_1', x_2'. Now, since s is an embedding, we have $s(x_1') \neq s(x_2')$. But $s(x_i') = \sigma(x_i)$, $i = 1, 2$, by the commutativity of diagram (20). Therefore $\sigma(x_1) \neq \sigma(x_2)$. Since $\sigma(x_i) = \Phi(x_i) = \Phi'(x_i')$, $i = 1, 2$, by the commutativity of diagram (19), we thus conclude that $\Phi'(x_1) \neq \Phi'(x_2)$, and hence Φ' separates x_1, x_2. This shows that Φ' is one-to-one.

We need the structure of the second reduction map described in Theorem (1.4). Let Z be the algebraic subset of X', of dimension ≤ 1, such that φ is an isomorphism outside of Z. Let $Z_0 \subset Z$ be the subset of Z consisting of all 0-dimensional components of Z and let $Z_1 \subset Z$ be the set of points x' belonging to the 1-dimensional components of Z and such that $\varphi^{-1}(x')$ is a divisor (see (1.4), a)). Then from (1.4) we know that

- $Z_0 \cup Z_1$ is a finite set of points, X' is smooth outside of $Z_0 \cup Z_1$ and $\varphi^{-1}(x')$ is a reduced (possibly reducible) divisor for $x' \in Z_0 \cup Z_1$.

First consider the case when $x' \in X' \setminus Z$. Let v' be a tangent direction at x', i.e., v' is a non-zero element of $\mathcal{T}_{X',x'}$, where $\mathcal{T}_{X'}$ denotes the tangent bundle of X'. Choose points $\widehat{x} \in \widehat{X}$, $x \in X$ such that $\pi(\widehat{x}) = x$, $\varphi(x) = x'$. Let $\widehat{v}$, v be nonzero elements of $\mathcal{T}_{\widehat{X},\widehat{x}}$, $\mathcal{T}_{X,x}$ respectively such that $d\pi_{\widehat{x}}(\widehat{v}) = v$, $d\varphi_x(v) = v'$, where $d\pi_{\widehat{x}}$, $d\varphi_x$ denote the differential maps. Then $d(\varphi \circ \pi)_{\widehat{x}}(\widehat{v}) = v'$. By [**3**, (1.7.9)] there exists a smooth surface $\widehat{A} \in |\widehat{L}|$ with $\widehat{x} \in \widehat{A}$ and $\widehat{v} \in \mathcal{T}_{\widehat{A},\widehat{x}}$. Therefore the image $A := \pi(\widehat{A})$ is a smooth divisor in $|L|$ passing through x and with $v \in \mathcal{T}_{A,x}$. Furthermore $A' := \varphi(A)$ is a surface passing through x', x' is a smooth point of A' and $v' \in \mathcal{T}_{A',x'}$.

We want to show that $d\Phi'_{x'}(v') \neq 0$, where $d\Phi'_{x'}$ is the differential map. Since s is an embedding, we have $ds_{x'}(v') \neq 0$. By the commutativity of diagram (20), one has $ds_{x'}(v') = d\sigma_x(v) = d\Phi_x(v)$ and $d\Phi_x(v) = d\Phi'_{x'}(v')$ by the commutativity of diagram (19). Thus $d\Phi'_{x'}(v') \neq 0$ and we are done.

Assume now that $x' \in Z \setminus (Z_0 \cup Z_1)$ and that v' is a tangent direction at x'. Then by using the structure of the second reduction map given in (1.4), i.e., that the restriction $\varphi_{X \setminus \varphi^{-1}(Z \setminus (Z_0 \cup Z_1))}$ is the blowup of the smooth curve $Z \setminus (Z_0 \cup Z_1)$, we can find points $\widehat{x} \in \widehat{X}$, $x \in X$ and nonzero elements $\widehat{v}$, v of $\mathcal{T}_{\widehat{X},\widehat{x}}$, $\mathcal{T}_{X,x}$ respectively such that $d\pi_{\widehat{x}}(\widehat{v}) = v$, $d\varphi_x(v) = v'$with all the same properties above. Then the same argument as above applies to give the result.

It remains to consider the case when $x' \in Z_0 \cup Z_1$. By $\bullet$) we know that in both cases $D := \varphi^{-1}(x')$ is a reduced divisor.

Look at diagram (19). Note that Φ' being an embedding at x' is equivalent to

$$\Phi'^{*}\mathfrak{m}_{\Phi'(x')} = \mathfrak{m}_{x'}. \tag{21}$$

CLAIM 5.2. Let $D = \varphi^{-1}(x')$ and let $\mathcal{J}_D$ be the ideal sheaf of D in X. Then $(\Phi^{*}\mathfrak{m}_{\Phi(x)})_x = \mathcal{J}_{D,x}$ for every $x \in D$ implies relation (21).

PROOF. Note that $\varphi^{*}\mathfrak{m}_{x'} = \mathcal{J}_D$. Therefore, by the commutativity of diagram (19) and by the assumption, we have, for each $x \in D$,

$$((\Phi' \circ \varphi)^{*}\mathfrak{m}_{\Phi(x)})_x = (\Phi^{*}\mathfrak{m}_{\Phi(x)})_x = \mathcal{J}_{D,x} = (\varphi^{*}\mathfrak{m}_{x'})_x. \tag{22}$$

Now,

$$(\Phi' \circ \varphi)^{*}\mathfrak{m}_{\Phi(x)} = \varphi^{*}(\Phi'^{*}\mathfrak{m}_{\Phi(x)}) = \varphi^{*}\Phi'^{*}\mathfrak{m}_{\Phi'(x')}. \tag{23}$$

By combining (22) and (23) we get

$$(\varphi^{*}\Phi'^{*}\mathfrak{m}_{\Phi'(x')})_x = (\varphi^{*}\mathfrak{m}_{x'})_x, \ x \in D.$$

Therefore $\varphi^{*}\Phi'^{*}\mathfrak{m}_{\Phi'(x')} = \varphi^{*}\mathfrak{m}_{x'}$ and hence since X' is normal and φ has connected fibers $\Phi'^{*}\mathfrak{m}_{\Phi'(x')} = \mathfrak{m}_{x'}$. This proves the claim. □

Thus we are reduced to show that, for each $x \in X$,

$$(\Phi^{*}\mathfrak{m}_{\Phi(x)})_x = \mathcal{J}_{D,x}. \tag{24}$$

Recall that Φ is the morphism defined by $\varphi^{*}\mathcal{L}$. Write $\mathcal{M} := \varphi^{*}\mathcal{L}$. Since sections of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)$ vanishing at $\Phi(x)$ pull back to sections of $\mathcal{M} \otimes \mathcal{J}_{D,x}$, it is enough to show that $\Gamma(\mathcal{M} \otimes \mathcal{J}_D)$ spans $\mathcal{M} \otimes \mathcal{J}_D$ in a neighborhood U of D (i.e., that $\Gamma(\mathcal{M} \otimes \mathcal{J}_D)$ spans $\mathcal{M} \otimes \mathcal{J}_{D,x}$ at x for each $x \in D$).

CLAIM 5.3. Notation as above. Let $\mathcal{N} := \mathcal{N}_{D/X}$ be the normal bundle of D in X. If $\Gamma(\mathcal{M} \otimes \mathcal{J}_D)$ spans $\mathcal{M} \otimes \mathcal{N}^*$ then $\Gamma(\mathcal{M} \otimes \mathcal{J}_D)$ spans $\mathcal{M} \otimes \mathcal{J}_D$ in a neighborhood U of D.

PROOF. This is an immediate consequence of Nakayama's lemma. □

Thus by Claim (5.3) we are reduced to show that $\Gamma(\mathcal{M} \otimes \mathcal{J}_D)$ spans $\mathcal{M} \otimes \mathcal{N}^*_{D/X}$ or, equivalently, that

- $\Gamma(\mathcal{M} \otimes \mathcal{J}_D)$ spans $\mathcal{M} \otimes \mathcal{N}^*_{D/X}$ at x for each $x \in D$.

Let $\ell := D \cap S$ be the transversal intersection of D with a general $S \in |L|$ that contains x. We know that ℓ is either a smooth rational curve with $L \cdot \ell = 1$ or a reducible conic, i.e., the union of two distinct lines meeting in x. Let $\mathcal{M}_S$ be the restriction of $\mathcal{M} = \varphi^*\mathcal{L}$ to S, let $\mathcal{J}_\ell$ be the ideal sheaf of ℓ in S, and let $\mathcal{N}_\ell$ be the normal bundle of ℓ in S, $\mathcal{N} = \mathcal{N}_{D/X}$. We claim that the result follows from the following three facts.

a) $\Gamma(\mathcal{M}_S \otimes \mathcal{J}_\ell)$ spans $\mathcal{M}_S \otimes \mathcal{N}^*_\ell$ at x, for each $x \in D$;
b) the restriction map $\Gamma(\mathcal{M} \otimes \mathcal{J}_D) \to \Gamma(\mathcal{M}_S \otimes \mathcal{J}_\ell)$ is onto;
c) $(\mathcal{M} \otimes \mathcal{N}^*)_\ell \cong \mathcal{M}_S \otimes \mathcal{N}^*_\ell$.

Indeed, we have the diagram

$$\begin{array}{ccc} \Gamma(\mathcal{M}_S \otimes \mathcal{J}_\ell) & \xrightarrow{\alpha} & (\mathcal{M}_S \otimes \mathcal{N}^*_\ell)/\mathfrak{m}_x \cong \mathbb{C} \\ \beta \uparrow & & \uparrow \gamma \\ \Gamma(\mathcal{M} \otimes \mathcal{J}_D) & \xrightarrow{\rho} & (\mathcal{M} \otimes \mathcal{N}^*)/\mathfrak{m}_x \cong \mathbb{C} \end{array}$$

where $\mathfrak{m}_x$ is the ideal sheaf of x in S, the morphisms α, β are onto by conditions a), b) respectively. Furthermore by c) there exists a map γ which makes the diagram commute. Hence in particular γ is an isomorphism and therefore ρ is onto. This gives •).

Thus it remains to prove conditions a), b), c).

Proof of a). By the assumption ii), we know that $|\mathcal{M}_S| = |\varphi_S^*\mathcal{L}_{S'}|$ defines a morphism, σ, which factors as $\sigma = s \circ \varphi_S$, where s is an embedding. In particular s is an embedding at the points $x' \in S'$ such that $D := \varphi^{-1}(x')$, $\ell = D \cap S = \varphi_S^{-1}(x')$ (see diagram (20)). This means that the map

$$\Gamma(\mathcal{L}_{S'} \otimes \mathfrak{m}_{x'}) \to \Gamma(\mathcal{L}_{S'} \otimes (\mathfrak{m}_{x'}/\mathfrak{m}^2_{x'})) \to 0$$

is onto, where $\mathfrak{m}_{x'}$ is the ideal sheaf of x' in S'. Since $\mathcal{M}_S \otimes \mathcal{J}_\ell \cong \varphi_S^*(\mathcal{L}_{S'} \otimes \mathfrak{m}_{x'})$, this implies that $\Gamma(\mathcal{M}_S \otimes \mathcal{J}_\ell)$ spans $\mathcal{M} \otimes \mathcal{N}^*_\ell$ at x for each $x \in D$.

Proof of b). Let $\mathcal{M}_\ell$ be the restriction of $\mathcal{M}$ to ℓ. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \Gamma(\mathcal{M} \otimes \mathcal{J}_D) & \xrightarrow{\alpha} & \Gamma(\mathcal{M}) & \xrightarrow{\alpha'} & \Gamma(\mathcal{M}_D) \cong \Gamma(\mathcal{O}_D) \cong \mathbb{C} & \to & 0 \\ & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \\ 0 & \to & \Gamma(\mathcal{M}_S \otimes \mathcal{J}_\ell) & \xrightarrow{\rho} & \Gamma(\mathcal{M}_S) & \xrightarrow{\rho'} & \Gamma(\mathcal{M}_\ell) \cong \Gamma(\mathcal{O}_\ell) \cong \mathbb{C} & \to & 0 \end{array}$$

where α', ρ' are onto since $\mathcal{M}_{S'}$, $\mathcal{M}_\ell$ are spanned because $\mathcal{M} = \varphi^*\mathcal{L}$ is spanned. Therefore δ is an isomorphism. Take $a \in \Gamma(\mathcal{M}_S \otimes \mathcal{J}_\ell)$. Then since γ is surjective by assumption $\rho(a) = \gamma(b)$ for some $b \in \Gamma(\mathcal{M})$. But $\alpha'(b) = 0$ since $\alpha'(b) = \rho'(\gamma(b)) = \rho'\rho(a) = 0$. Thus $b \in \Gamma(\mathcal{M} \otimes \mathcal{J}_D)$ and therefore $a = \beta(b)$, i.e., β is onto.

Proof of c). Since D, S are both Cartier divisors on X, and ℓ is the transversal intersection $\ell = D \cap S$, we have

$$(\mathcal{N}^*_{D/X})_{|\ell} \cong (\mathcal{N}^*_{D/X})_{|S} \cong \mathcal{O}_D(-D)_S \cong \mathcal{O}_\ell(-\ell) \cong \mathcal{N}^*_\ell.$$

□

Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L), $\pi : \widehat{X} \to X$, be the first reduction of $(\widehat{X}, \widehat{L})$. For a general smooth surface $\widehat{S} \in |\widehat{L}|$, let $S = \pi(\widehat{S})$. Then

$$h^1(2K_S) = 0. \tag{25}$$

To see this note that since $K_X + L$ is nef and big, the restriction K_S is nef and big on S. Therefore $h^1(2K_S) = 0$ by the Kawamata-Viewheg vanishing theorem.

Corollary 5.4. *Let $(\widehat{X}, \widehat{L})$ be a smooth threefold polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L), $\pi : \widehat{X} \to X$, (X', L'), $\varphi : X \to X'$ be the first and the second reduction as in* (1.3). *Then $3(K_{X'} + L')$ is very ample.*

Proof. Let $\mathcal{K}' := K_{X'} + L'$. Then $\varphi^*\mathcal{K}' \approx \mathcal{K}$. Since the canonical model $S' := \varphi(S)$ of $\widehat{S}$ has rational Gorenstein singularities we have $\mathcal{K}'_{S'} \approx K_{S'}$. From [**18**, (2.2.2)] we know that $2(K_X + L)$ is spanned. Then $3(K_X + L) \approx \varphi^*(3\mathcal{K}')$ is spanned. Since $h^1(K_X + 2\mathcal{K}) = 0$, from the exact sequence

$$0 \to K_X + 2\mathcal{K} \to 3\mathcal{K} \to 3K_S \to 0$$

we infer that

$$\Gamma(\varphi^*(3\mathcal{K}')) \cong \Gamma(3\mathcal{K}) \to \Gamma(\varphi_S^*(3\mathcal{K}'_{S'})) \cong \Gamma(3K_S)$$

is onto.

By using Lemma (1.12), Corollary (1.17) and (25) we know from the canonical embedding theorem of surfaces (see [**7**, (1.2)]) that $3\mathcal{K}'_{S'} \approx 3K_{S'}$ gives an embedding, s, for any surface $S' = \varphi(S)$, S smooth member of $|L|$. This means that the morphism, σ, defined by $|\varphi_S^*(3\mathcal{K}'_{S'})|$ factors as $\sigma = s \circ \varphi_S$. Then Theorem (5.1) applies to say that $3\mathcal{K}'$ is very ample. □

Remark 5.5. Notation as in (5.4). Let $\widehat{S}$ be a smooth surface in $|\widehat{L}|$ and let $S := \pi(\widehat{S})$, $S' := \varphi(S)$. The same arguments as above show that if $|tK_{S'}|$ gives an embedding for each smooth $S \in |L|$ and if the restriction map $\rho_t : \Gamma(t(K_X + L)) \to \Gamma(tK_S)$ is onto for $t \geq 1$, then $t(K_{X'} + L')$ is very ample.

Note that for $t \geq 2$ the surjectivity of the map ρ_t follows from the exact sequence

$$0 \to tK_X + (t-1)L \to t(K_X + L) \to tK_S \to 0.$$

Indeed, $h^1(tK_X + (t-1)L) = h^1(K_X + (t-1)(K_X + L)) = 0$ by Kawamata-Viewheg vanishing theorem since $K_X + L$ is nef and big.

We conclude this section by pointing out an extension to higher dimension of the above results.

A straightforward modification of the proof of Theorem (5.1), which is in fact easier since the structure of the second reduction is simpler in dimension $n \geq 4$ (see [**3**, §7.5]), proves the following.

THEOREM 5.6. *Let $(\widehat{X}, \widehat{L})$ be a smooth n-fold, $n \geq 4$, polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L), $\pi : \widehat{X} \to X$, (X', L'), $\varphi : X \to X'$ be the first and the second reduction as in (1.3). Let $\mathcal{L}$ be a line bundle on X'. Let $\widehat{S}$ be a smooth surface obtained as transversal intersection of $n-2$ general members of $|\widehat{L}|$ and let $S := \pi(\widehat{S})$, $S' = \varphi(S)$. Let $\varphi_S : S \to S'$ and $\mathcal{L}_{S'}$ be the restrictions of φ and $\mathcal{L}$ to S and S' respectively. Assume that*

i) *the image of $\Gamma(\varphi^*\mathcal{L}) \to \Gamma(\varphi_S^*\mathcal{L}_{S'})$ is surjective and spans $\varphi_S^*\mathcal{L}_{S'}$;*

ii) *the morphism, σ, defined by $|\varphi_S^*\mathcal{L}_{S'}|$ factors as $\sigma = s \circ \varphi_S$, where s is an embedding.*

Then $\mathcal{L}$ is very ample.

The theorem above gives rise to the following n-dimensional version of Corollary (5.4).

COROLLARY 5.7. *Let $(\widehat{X}, \widehat{L})$ be a smooth n-fold, $n \geq 4$, polarized by a very ample line bundle $\widehat{L}$. Assume that $(\widehat{X}, \widehat{L})$ is of log-general type. Let (X, L), $\pi : \widehat{X} \to X$, (X', L'), $\varphi : X \to X'$ be the first and the second reduction as in (1.3). Then $3(K_{X'} + (n-2)L')$ is very ample.*

PROOF. Denote by $\widehat{X}_{n-1}$ a general smooth element of $|\widehat{L}|$ and let $\widehat{L}_{n-1} := \widehat{L}_{\widehat{X}_{n-1}}$. Similarly $\widehat{X}_{n-2}$ denotes a general smooth element of $|\widehat{L}_{n-1}|$ and in general $\widehat{X}_i$ denotes a smooth general element of $|\widehat{L}_{i+1}|$, where $\widehat{L}_{i+1} := \widehat{L}_{\widehat{X}_{i+1}}$, $i = 2, \dots, n-1$, and $\widehat{L}_n = \widehat{L}$, $\widehat{X}_n = \widehat{X}$. Then $\widehat{X}_2$ coincides with the surface $\widehat{S}$ obtained as transversal intersection of $n-2$ general members of $|\widehat{L}|$. Let π be the first reduction map and let $X_i = \pi(\widehat{X}_i)$, $i = 2, \dots, n-2$. Then X_2 coincides with the surface $S := \pi(\widehat{S})$. From the exact sequences

$$0 \to K_{X_{i+1}} + (m-1)(K_{X_{i+1}} + \mu L_{i+1}) \to m(K_{X_{i+1}} + (i-1)L_{i+1}) \to mK_{X_i} \to 0,$$

by noting that $\mu := \frac{m(i-1)-1}{m-1} \geq i-1$, so that $K_{X_{i+1}} + \frac{m(i-1)-1}{m-1} L_{i+1}$ is nef and big since $K_{X_{i+1}} + (i-1)L_{i+1} \approx (K_X + (n-2)L)_{X_{i+1}}$ is nef and big, we conclude that the restriction map $\Gamma(m(K_X + (n-2)L)) \to \Gamma(mK_S)$ is surjective and therefore the restriction, φ_S, of φ to S coincides with the m-canonical map associated to $|mK_S|$, $m \geq 2$.

Let $\mathcal{K}' := K_{X'} + (n-2)L'$. Then $\varphi^*\mathcal{K}' \approx \mathcal{K}$. Since the canonical model $S' := \varphi(S)$ of $\widehat{S}$ has rational Gorenstein singularities we have $\mathcal{K}'_{S'} \approx K_{S'}$. Now the same argument as in the proof of (5.4), by using Theorem (5.6), gives the result. $\square$

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