AMPLE AND SPANNED VECTOR BUNDLES OF MINIMAL CURVE GENUS

ANTONIO LANTERI, HIDETOSHI MAEDA AND ANDREW J. SOMMESE

ABSTRACT. Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $n-1$ on a compact complex manifold X of dimension $n \geq 3$ and let $g(X, \mathcal{E})$ be the curve genus, defined by $2g(X, \mathcal{E}) - 2 = (K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E})$. As a consequence of the Lefschetz-Sommese theorem, the inequality $g(X, \mathcal{E}) \geq h^{1,0}(X)$ holds. In this paper we characterize pairs $(X, \mathcal{E})$ as above satisfying the condition that $g(X, \mathcal{E}) = h^{1,0}(X)$.

Introduction

To study ample vector bundles $\mathcal{E}$ on a complex projective manifold X, Fujita [F2] introduced two notions of sectional genus: the c_1-sectional genus and the $\mathcal{O}(1)$-sectional genus, defined as the sectional genera of the polarized manifolds $(X, \det \mathcal{E})$ and $(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ respectively, where $H(\mathcal{E})$ stands for the tautological bundle of $\mathcal{E}$. In order to study some ample and spanned vector bundles of rank 2 on threefolds, Ballico [Ba] introduced another notion of sectional genus for an ample vector bundle $\mathcal{E}$ of rank $n-1$ on X of dimension n by setting

$$2g(X, \mathcal{E}) - 2 = (K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}).$$

In general $g(X, \mathcal{E})$ is much smaller than the c_1-sectional genus (see [Ba, p. 135]). If $\mathcal{E}$ has a section vanishing on a smooth curve $Z \subset X$, which certainly happens for the general section of $\mathcal{E}$, if $\mathcal{E}$ is ample and spanned, due to the Bertini theorem [Mu, Theorem 1.10], then $g(X, \mathcal{E})$ coincides with the genus of the smooth curve Z. This is the reason why in this paper we call the invariant $g(X, \mathcal{E})$ defined above, the curve genus. Note that in the very special case when $\mathcal{E} = L^{\oplus(n-1)}$, with L a line bundle on X, $g(X, \mathcal{E}) = g(X, L)$ is the usual sectional genus of (X, L). In this paper we consider the following set-up.

($*$) $\mathcal{E}$ is an ample and spanned vector bundle of rank $n-1$ on a complex projective manifold X of dimension $n \geq 3$. $Z := (s)_0$ will denote the zero locus of a general section $s \in \Gamma(\mathcal{E})$.

Thus Z is a smooth curve, and the restriction homomorphism $H^1(X, \mathbb{Z}) \to H^1(Z, \mathbb{Z})$ is injective and its cokernel is torsion free, by the Lefschetz-Sommese theorem [LM2, Theorem 1.3] (see also [S1, Proposition 1.16]). Thus $g(X, \mathcal{E}) = g(Z) \geq h^{1,0}(X)$, with the equality $g(Z) = h^{1,0}(X)$ implying $H^1(X, \mathbb{Z}) \cong H^1(Z, \mathbb{Z})$. The aim of this paper is to prove the following

1991 *Mathematics Subject Classification.* Primary 14J60; Secondary 14C20, 14F05, 14J40.

Theorem. *Let $(X, \mathcal{E})$ be as in $(*)$. Then $g(X, \mathcal{E}) = h^{1,0}(X)$ if and only if $(X, \mathcal{E})$ is one of the following pairs:*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)})$.
(2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$.
(3) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)})$.
(4) $X = \mathbb{P}_B(\mathcal{F})$ *for some vector bundle* $\mathcal{F}$ *of rank* n *on a smooth curve* B *and* $\mathcal{E} = H \otimes \pi^*\mathcal{G}$ *for some vector bundle* $\mathcal{G}$ *of rank* $n-1$ *on* B*, where* H *stands for the tautological bundle of* $\mathcal{F}$ *and* $\pi : X \to B$ *is the bundle projection.*

The same assertion of the Theorem holds also for $\dim X = 2$. In fact, in the special case when $\mathcal{E} = \oplus_{j=1}^{n-1} L_j$ with the L_j being ample and spanned line bundles on X for $n \geq 2$, the above result was proved several years ago [LP, Theorem 2.1].

We prove our Theorem in Section 1, after investigating some basic properties on ample and spanned vector bundles in Section 0. In Section 2 we present some corollaries and discuss some conjectures in the above set-up, which are suggested by well-known facts related to adjunction theory for surfaces.

The present paper was prepared when the second author was visiting the University of Milan in the winter of 1994-95. He would like to thank the Italian C.N.R. and the J.S.P.S. for financial support and for making this collaboration possible. The first author was partially supported by the M.U.R.S.T. of the Italian Government in the framework of the 40% research project "Geometria algebrica". The third author would like to thank the University of Notre Dame and the National Science Foundation (DMS 93–02121) for their support.

0. Preliminaries

In this paper we will work over the complex number field $\mathbb{C}$. We use the standard notation from algebraic geometry. The pull-back $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding $i : Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. A vector bundle $\mathcal{E}$ is said to be *spanned* if it is generated by its global sections. The canonical bundle of a smooth variety X is denoted by K_X. Following a current abuse, we will freely switch between the multiplicative and the additive notation for the tensor products of line bundles.

Before proceeding to prove the Theorem, we present two lemmas. More generally, let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \leq n-1$ on a complex projective manifold X of dimension n. Let $s \in H^0(\mathcal{E})$ be a global section of $\mathcal{E}$. Then s defines a homomorphism $\mathcal{O}_X \to \mathcal{E}$ by sending 1 to s. Let $\eta : \mathcal{E}^\vee \to \mathcal{O}_X$ be its dual homomorphism. The closed subscheme $(s)_0$ of X defined by the ideal $\mathcal{I} = \operatorname{Im}(\eta) \subset \mathcal{O}_X$ is called the *zero locus* of s. A global section s of $\mathcal{E}$ is called *nondegenerate* if the codimension of $(s)_0$ in X is equal to r.

We set $|\mathcal{E}| = \mathbb{P}(H^0(\mathcal{E})^\vee) = (H^0(\mathcal{E}) - 0)/\mathbb{C}^*$ and denote by $[s]$ the class of $s \in H^0(\mathcal{E})$ in $|\mathcal{E}|$. Since $\mathcal{E}$ is spanned, the evaluation homomorphism $\mathrm{ev} : H^0(\mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}$ is surjective. Its kernel $\mathcal{K}$ is the bundle

$$\mathcal{K} = \{(s, x) \in H^0(\mathcal{E}) \times X | s(x) = 0\}.$$

Let $\mathcal{F}$ be the dual of $\mathcal{K}$. Then $A := \mathbb{P}_X(\mathcal{F})$ is a closed submanifold of $|\mathcal{E}| \times X = \mathbb{P}_X(H^0(\mathcal{E})^\vee \otimes \mathcal{O}_X)$. Let f be the restriction of the first projection $|\mathcal{E}| \times X \to |\mathcal{E}|$ to A, and let $h^0(\mathcal{E}) = N+1$. Then f is a morphism of the $(N+n-r)$-dimensional projective manifold A to the N-dimensional projective space $|\mathcal{E}|$. Moreover, for any point $[s] \in |\mathcal{E}|$, the fibre $f^{-1}([s])$ is isomorphic to the zero locus of s. Since $\mathcal{E}$ is spanned, [Mu, Theorem 1.10] tells us that a general global section of $\mathcal{E}$ is nondegenerate and that its zero locus is smooth. Thus, if we let F be a general fibre of f, then F is smooth of dimension $n-r \geq 1$. This implies that f is surjective. We regard F as a closed subscheme of X. Then, since $\mathcal{E}$ is ample, we have $H^0(F,\mathbb{Z}) \cong H^0(X,\mathbb{Z}) = \mathbb{Z}$ by [LM2, Theorem 1.3, (1.3.1)] (note that its proof is valid without assuming the connectedness of F). Thus F is connected, hence irreducible. By the Stein factorization theorem, we can factor f into $h \circ f'$, where $f' : A \to Y$ is a proper surjective morphism with connected fibres onto a normal projective variety Y and $h : Y \to |\mathcal{E}|$ is a finite surjective morphism. But then h must be an isomorphism because a general fibre of f is irreducible. Thus f has connected fibres and we have $f_*\mathcal{O}_A = \mathcal{O}_{|\mathcal{E}|}$ by construction of f'. Now we use the projection formula to obtain

$$(0.1) \qquad f_*f^*\mathcal{O}_{|\mathcal{E}|}(1) \cong \mathcal{O}_{|\mathcal{E}|}(1).$$

Let $p : |\mathcal{E}| \times X \to |\mathcal{E}|$ be the first projection and $q : |\mathcal{E}| \times X \to X$ the second projection. Let G be any fibre of q. Then under the identification $p_{|G} : G \simeq |\mathcal{E}|$, we have $(p^*\mathcal{O}_{|\mathcal{E}|}(1))_G \cong \mathcal{O}_{|\mathcal{E}|}(1)$, so $p^*\mathcal{O}_{|\mathcal{E}|}(1)$ is the tautological line bundle on $|\mathcal{E}| \times X$ (see for example [F1, (4.6)]). Thus $H := f^*\mathcal{O}_{|\mathcal{E}|}(1) = (p^*\mathcal{O}_{|\mathcal{E}|}(1))_A$ is the tautological line bundle on A. By virtue of (0.1) we have

$$(0.2) \qquad H^0(H) \cong H^0(\mathcal{O}_{|\mathcal{E}|}(1)).$$

This implies that f is the morphism associated to the complete linear system $|H|$.

We need the following

(0.3) Lemma. *Let L be a general $(N-d)$-plane in $|\mathcal{E}|$ with $1 \leq d \leq N-1$. Then $f^{-1}(L)$ is smooth and irreducible.*

Proof. By (0.2) we can regard $f^{-1}(L)$ as the intersection of d general elements of the complete linear system $|H|$. Thus, since H is spanned, $f^{-1}(L)$ is smooth by Bertini's theorem. Moreover, it follows from [FuL, Theorem 1.1] that $f^{-1}(L)$ is irreducible. □

We need also the following Lemma, which will be used later.

(0.4) Lemma. *X is the closure of the union of a family of smooth zero loci of dimension $n-r$.*

Proof. Take an open dense subset U of $|\mathcal{E}|$ such that $f^{-1}([s])$ is a smooth variety of dimension $n-r$ for any $[s] \in U$. Since the restriction g of the second projection

$g : |\mathcal{E}| \times X \to X$ to A is just the bundle projection $A \to X$, g is surjective. Thus $g_{|f^{-1}(U)} : f^{-1}(U) \to X$ is dominant, and consequently

$$X = \overline{g(f^{-1}(U))} = \overline{g(\bigcup_{[s]\in U} f^{-1}([s]))} = \overline{\bigcup_{[s]\in U} g(f^{-1}([s]))} = \overline{\bigcup_{[s]\in U} (s)_0}. \quad \square$$

1. The proof

Set $g := g(X, \mathcal{E})$ for short. First of all, if $(X, \mathcal{E})$ is as in $(1) - (3)$, then a direct computation shows that $g = 0 = h^{1,0}(X)$. In case (4) we have $h^{1,0}(X) = g(B)$. Let s be a general section of $\mathcal{E}$. Then $s_F \in \Gamma(\mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)})$ for any fibre $F(\cong \mathbb{P}^{n-1})$ of π, and so $(s)_0 \cap F = (s_F)_0$ is a point. This implies that $(s)_0$ is a section of π; hence $(s)_0 \cong B$. Thus $g = g(B) = h^{1,0}(X)$. To prove the converse, first note that if $g = 0$, then [LM2, Theorem A] applies, and so $(X, \mathcal{E})$ is as in cases (1), (2), (3) or as in case (4) with $B = \mathbb{P}^1$. So let $g > 0$ and let Z be the zero locus of a general global section s of $\mathcal{E}$. Consider the Albanese morphism of X, $\alpha : X \to \mathrm{ALB}(X)$. Due to the assumption that $g = h^{1,0}(X)$, the inclusion of Z in X induces an isomorphism $\mathrm{ALB}(Z) \cong \mathrm{ALB}(X)$. So $\alpha_{|Z}$ is the Albanese morphism of Z, whose image B can be of course identified with Z itself. If we can prove that

$$\alpha(X) = \alpha(Z), \tag{1.0}$$

then α fibres X over B. Now let us combine this with a result of Wiśniewski [W] in order to prove our Theorem. Indeed, let F denote any smooth fibre of α. Then F is a smooth complex projective manifold of dimension $n-1$ and $\mathcal{E}_F$ is an ample and spanned vector bundle of rank $n-1$ on F with $c_{n-1}(\mathcal{E}_F) = 1$, since the restriction of our s to F, $s_F \in \Gamma(\mathcal{E}_F)$, vanishes at $Z \cap F$ which consists of a single point. Thus $(F, \mathcal{E}_F) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)})$ by virtue of [W, Theorem 3.4].

Claim. Every fibre of α is irreducible and reduced.

To see this, let F_0 be a fibre of α, and write $F_0 = \sum_{i=1}^{t} m_i A_i$, where the A_i are the irreducible components of F_0 and $m_i > 0$. Then $\dim A_i = n - 1$ for every A_i because α is flat. Furthermore, $\mathcal{E}_{A_i}$ is an ample and spanned vector bundle of rank $n - 1$ on A_i. It thus follows from [Fu, Example 12.1.7, (b)] that $c_{n-1}(\mathcal{E}_{A_i}) > 0$. Let us denote by Z the cycle associated with Z by abuse of notation. Then, since Z has codimension $n - 1$, we have $c_{n-1}(\mathcal{E}) = Z$, hence by the projection formula ([Fu, Theorem 3.2, (c)])

$$ZA_i = c_{n-1}(\mathcal{E})A_i = c_{n-1}(\mathcal{E}_{A_i}).$$

Thus

$$1 = ZF = ZF_0 = Z(\sum_{i=1}^{t} m_i A_i) = \sum_{i=1}^{t} m_i c_{n-1}(\mathcal{E}_{A_i}),$$

so $t = 1$ and $m_1 = 1$, which proves the Claim.

Now, by using the same argument as in [F3, (3.3)] we can see that there are no singular fibres of α. Consequently X is a $\mathbb{P}^{n-1}$-bundle over B and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for any fibre F of α. This gives case (4) of the Theorem.

In the following we will prove (1.0). First of all, note that (1.0) is obvious when $g = 1$, so we can assume that $g \geq 2$ in the sequel. Moreover, since $\alpha(X)$ is irreducible, to prove (1.0) it is enough to show that $\alpha(X)$ is 1-dimensional. To do this we need the following construction.

Choose two general sections $s_0, s_1 \in \Gamma(\mathcal{E})$ and let $[s_0]$, $[s_1] \in |\mathcal{E}| = (H^0(\mathcal{E}) - 0)/\mathbb{C}^*$ be the corresponding points. Let S denote the surface in $|\mathcal{E}| \times X$ consisting of points $([c], y)$, where $c = \lambda s_0 + \mu s_1 ((\lambda, \mu) \neq (0,0))$ and $y \in (c)_0$. Then S is a smooth surface by (0.3). Note that we have a natural morphism $\varphi : S \to \mathbb{P}^1$ induced by the first projection of $|\mathcal{E}| \times X$. Let C be a general fibre of the pencil $\varphi : S \to \mathbb{P}^1$. Let ρ be the natural homomorphism $H^0(\Omega_S^1) \to H^0(\Omega_C^1) = H^0(K_C)$ induced by the inclusion $C \subset S$.

(1.1) Lemma. *The homomorphism ρ is an isomorphism.*

Proof. By construction, S is a pencil of zero loci parametrized by $\mathbb{P}^1 \subset |\mathcal{E}|$ and the map $q_S : S \to X$ induced by the second projection of $|\mathcal{E}| \times X$ sends C to a zero locus. Hence, owing to the assumption, we have $g(C) = h^{1,0}(X)$, which implies that $H^1(X, \mathbb{Z}) \cong H^1(C, \mathbb{Z})$. Thus the homomorphism $H^0(\Omega_X^1) \to H^0(K_C)$ is an isomorphism. Since it factors through the natural homomorphisms $H^0(\Omega_X^1) \to H^0(q_S^*\Omega_X^1) \to H^0(\Omega_S^1)$ and ρ itself, i.e.,

$$H^0(\Omega_X^1) \to H^0(\Omega_S^1) \to H^0(K_C),$$

we conclude that ρ is surjective. Assume that ρ is not an isomorphism, i.e.,

(1.1.1) $$g(C) < h^{1,0}(S).$$

Then the image of C under the Albanese morphism of S, $a : S \to \mathrm{ALB}(S)$ generates a proper abelian subvariety $J(C)$ by (1.1.1). Consider the map

$$\beta : S \to A := \mathrm{ALB}(S)/J(C)$$

induced by a. Then the abelian variety A has positive dimension and β is nonconstant, since $a(S)$ generates $\mathrm{ALB}(S)$. Of course $\beta(C)$ is a point, by the definition of β. Thus, by the rigidity lemma [BS, Lemma 4.1.13, p. 89] there exists a nonconstant morphism $\mathbb{P}^1 \to A$, which is absurd. □

We need also the following

(1.2) Lemma. *Let L be a line bundle on a smooth projective surface Y. Assume that $h^0(L) \geq 2$ and that there is a section of L having a smooth zero locus C, which is connected and such that the natural homomorphism $H^0(\Omega_Y^1) \to H^0(K_C)$ induced by the inclusion $C \subset Y$ is an isomorphism. Then $h^{2,0}(Y) = 0$.*

Proof. Consider the exact sequence

(1.2.1) $$0 \to -L \to \mathcal{O}_Y \to \mathcal{O}_C \to 0.$$

According to our hypotheses, the homomorphisms

$$H^0(\mathcal{O}_Y) \to H^0(\mathcal{O}_C) \quad \text{and} \quad H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_C)$$

both are isomorphisms. Thus the exact cohomology sequence arising from (1.2.1) gives $h^1(-L) = 0$, which means that $h^1(K_Y \otimes L) = 0$, by Serre duality. Then, by using the exact cohomology sequence associated to

$$0 \to K_Y \to K_Y \otimes L \to K_C \to 0, \tag{1.2.2}$$

and the isomorphism $H^0(K_C) \to H^1(K_Y)$, the dual of the latter considered above, we see that $H^0(K_Y) \cong H^0(K_Y \otimes L)$. Therefore, $H^0(K_Y \otimes L) \to H^0(K_C)$ is the zero homomorphism for any smooth curve $C \in |L|$. But this is absurd if $h^0(K_Y) \neq 0$. To see this, take a smooth curve $D \in |L|$ and a 2-form $\omega \in H^0(K_Y)$ such that $\omega_D \neq 0$. Moreover, since the singular elements of $|L|$ constitute a Zariski closed subset of $|L|$, we can choose a smooth curve $D' \in |L|$ such that $D \neq D'$. Now if $s \in H^0(L)$ is a nontrivial section vanishing on D', we conclude that $\omega \otimes s$ cannot be identically zero when restricted to D. This gives a contradiction. □

Thanks to Lemma (1.1), if we let $Y = S$ and $L = \varphi^*\mathcal{O}_{\mathbb{P}^1}(1)$, Lemma (1.2) applies to our situation. Moreover, for a general fibre C of φ, we have $h^{1,0}(S) = g(C) \geq 2$ by our hypothesis. Thus, since $h^{2,0}(S) = 0$, we deduce that the Albanese morphism of S has a 1-dimensional image: the image of the Albanese morphism of C.

Now by (0.4) we have $X = \overline{\cup Z}$, where $\{Z = (s)_0\}$ is a family of smooth curves. Thus

$$\alpha(X) = \alpha(\overline{\cup Z}) \subseteq \overline{\alpha(\cup Z)} = \overline{\cup \alpha(Z)},$$

i.e., $\alpha(X) = \overline{\cup\alpha(Z)}$. This implies that $\alpha(X)$ is 1-dimensional. Indeed, since our surface S was constructed from a general line in $|\mathcal{E}|$, we have shown that most zero loci of sections of $\mathcal{E}$ have the same 1-dimensional image in $\mathrm{ALB}(X)$. This concludes the proof.

2. Corollaries and further comments

(2.1) Corollary. *Let $(X, \mathcal{E})$ be as in $(*)$. Then the following conditions are equivalent :*

(2.1.1) $g(X, \mathcal{E}) = h^{1,0}(X)$.
(2.1.2) $h^0(m(K_X + \det \mathcal{E})) = 0$ *for all* $m \geq 1$.
(2.1.3) $K_X + \det \mathcal{E}$ *is not nef.*

Proof. If (2.1.1) holds, then $(X, \mathcal{E})$ is as in (1) – (4) of our Theorem. In cases (1) – (3) we have $K_X + \det \mathcal{E} = \mathcal{O}_{\mathbb{P}}(-2)$, $\mathcal{O}_{\mathbb{P}}(-1)$ and $\mathcal{O}_{\mathbb{Q}}(-1)$ respectively, while in case (4) $(K_X + \det \mathcal{E})_F = \mathcal{O}_{\mathbb{P}}(-1)$ for any fibre F of π. So in all cases (2.1.2) holds. Now assume that (2.1.2) holds. Then the base point free theorem [KMM, Theorem 3-1-1] implies (2.1.3). Finally, if (2.1.3) holds, then $(X, \mathcal{E})$ is as in (1) – (4) of our Theorem, as was shown by Ye and Zhang [YZ, Theorem 3]. Thus (2.1.1) follows from our Theorem. □

(2.2) *Remark.* Note that the equivalence among (2.1.2), (2.1.3) and the condition that $(X, \mathcal{E})$ is as in (1) – (4) of our Theorem does not require the spannedness of $\mathcal{E}$. In case $n = 2$, i.e., $\mathcal{E} = \det \mathcal{E}$ is an ample and spanned line bundle, there is another equivalent condition, namely $h^0(K_X + \det \mathcal{E}) = 0$. What about the higher dimensions ? Let $(X, \mathcal{E})$ and Z be as in $(*)$ and consider the Koszul complex of Z

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \wedge^2 \mathcal{E} \to \ \dots \ \to \det \mathcal{E} \to \det \mathcal{E}_Z \to 0,$$

which is exact by our hypothesis. Twisting it by K_X and recalling that $\mathcal{E}_Z$ is the normal bundle of Z in X ([Fu, Example 6.3.4] or [Mu, (1.5)]), by adjunction we get the exact sequence

$$0 \to K_X \to K_X \otimes \mathcal{E} \xrightarrow{r_{n-1}} K_X \otimes \wedge^2 \mathcal{E} \xrightarrow{r_{n-2}} \ \dots \ \xrightarrow{r_2} K_X + \det \mathcal{E} \xrightarrow{r_1} K_Z \to 0.$$

Letting $\mathcal{F}_i = \mathcal{K}er(r_i)$, $i = 1, \dots, n-2$, we have the following short exact sequences:

$$0 \to \mathcal{F}_1 \to K_X + \det \mathcal{E} \xrightarrow{r_1} K_Z \to 0,$$

$$0 \to \mathcal{F}_i \to K_X \otimes \wedge^{n-i} \mathcal{E} \xrightarrow{r_i} \mathcal{F}_{i-1} \to 0,$$

$$0 \to K_X \to K_X \otimes \mathcal{E} \xrightarrow{r_{n-1}} \mathcal{F}_{n-2} \to 0.$$

Now consider the exact cohomology sequences they induce. By the Le Potier vanishing theorem [SS, Theorem 5.71] we have

(2.2.1) $$H^q(K_X \otimes \wedge^p \mathcal{E}) = 0 \quad \text{for} \quad q + p > n - 1 \quad \text{and } 1 \leq p \leq n - 1.$$

Thus, for $(q, p) = (1, n - 1)$, (2.2.1) gives $h^1(K_X + \det \mathcal{E}) = 0$, and so we get from the first cohomology sequence

(2.2.2) $$g(X, \mathcal{E}) = g(Z) = h^0(K_Z) \geq h^1(\mathcal{F}_1);$$

moreover, if the latter is a strict inequality, then the same exact sequence shows that $h^0(K_X + \det \mathcal{E}) > 0$. Similarly, by using (2.2.1) for $(q, p) = (2, n - 2), \dots, (n - 2, 2)$, we obtain

$$h^1(\mathcal{F}_1) \geq h^2(\mathcal{F}_2) \geq \ \dots \ \geq h^{n-2}(\mathcal{F}_{n-2})$$

and finally, the last exact cohomology sequence combined with (2.2.1) for $(q, p) = (n - 1, 1)$ gives

$$h^{n-2}(\mathcal{F}_{n-2}) \geq h^{n-1}(K_X).$$

Putting together all these inequalities we get

(2.2.3) $$g(X, \mathcal{E}) \geq h^{1,0}(X)$$

(which was already known from the Lefschetz-Sommese theorem). If the strict inequality in (2.2.3) would give rise to the strict inequality in (2.2.2), then we could conclude that $h^0(K_X + \det \mathcal{E}) > 0$.

Unfortunately we cannot prove this, but we state the following

(2.3) *Conjecture.* Let $(X, \mathcal{E})$ be as in $(*)$. If $h^0(K_X + \det \mathcal{E}) = 0$, then $g(X, \mathcal{E}) = h^{1,0}(X)$.

(2.4) Proposition. *Conjecture* (2.3) *is true for* $n = 3$.

Proof. Set $L = \det \mathcal{E}$. Then L is an ample and spanned line bundle on our threefold X. Due to the ampleness of $\mathcal{E}$, for any smooth rational curve $C \subset X$ we have

(2.4.1) $$LC = c_1(\mathcal{E}_C) \geq \text{rank}\mathcal{E}_C = 2.$$

In particular, this implies that if (X, L) admits a reduction (X', L') in the sense of adjunction theory, then

(2.4.2) $$(X, L) = (X', L').$$

Now, let $h^0(K_X + L) = 0$. Then, by [S3, Main Theorem] combined with (2.4.2), we get for (X, L) the following possibilities:

(a) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(1))$, $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}}(1))$, or a scroll over a smooth curve;

(b) a Del Pezzo 3-fold (i.e., $-K_X = 2L$), a quadric fibration over a smooth curve, or a scroll over a smooth surface;

(c) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(3))$, $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}}(2))$, or there exists a morphism $\phi : X \to B$ onto a smooth curve B such that $2K_X + 3L = \phi^* H$ for an ample line bundle $H \in \text{Pic}(B)$.

Note that for all pairs (X, L) as in (a) and (b), except for $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2))$ in (b), there exists a smooth rational curve $l \subset X$ such that $Ll = 1$. This is obvious unless (X, L) is a Del Pezzo 3-fold. In this case, however, X contains a smooth surface $S \in |L|$, which is non-minimal apart from the above exception. So, by taking an exceptional curve in S for l, we get

$$1 = (-K_S)l = (-K_X - L)_S l = (2L - L)l = Ll.$$

Thus by (2.4.1) in cases (a) and (b) only the pair $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2))$ is allowable. In this case, we immediately see that $\mathcal{E}$ is uniform, and so we conclude that $(X, \mathcal{E}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2})$, which corresponds to case (1) of the Theorem. Similarly, the first two pairs in case (c) lead to $(X, \mathcal{E}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1))$ and $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus 2})$ respectively, i.e., cases (2), (3) of the Theorem. Finally, in the last case of (c), let F be a general fibre of ϕ. Then, since its canonical bundle is the restriction of K_X, we have $2K_F + 3L_F = 0$. In particular, F is a Del Pezzo surface. Noting that $-K_F$ is divisible by 3, we see that $(F, L_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2))$. Since $2K_X + 3L = \phi^* H$ is nef, the line bundle $A := K_X + 2L$ is ample. Let G be any fibre of ϕ. Then $A^2 G = A^2 F = 1$, and so G is irreducible and reduced. Moreover, $\Delta(G, A_G) \leq \Delta(F, A_F) = 0$ by the upper-semicontinuity theorem. Thus $(G, A_G) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1))$ and we conclude that (X, A) is a scroll over B, hence $L_G = \mathcal{O}_{\mathbb{P}}(2)$. This gives case (4) of the Theorem. □

(2.5) Corollary. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $n - 1$ on a projective manifold X of dimension n. Assume that $\mathcal{E}$ is a direct sum of line bundles plus at most one vector bundle of rank 2. If $h^0(K_X + \det \mathcal{E}) = 0$, then $g(X, \mathcal{E}) = h^{1,0}(X)$.*

Proof. Let $\mathcal{E} = \bigoplus_{i=1}^{n-3} L_i \oplus \mathcal{F}$, where L_i is a line bundle and $\mathcal{F}$ is a vector bundle of rank 2. In view of (2.4) we can assume that $n \geq 4$. Consider the exact sequence

$$0 \to K_X \to K_X \otimes L_{n-3} \to K_A \to 0,$$

where $A \in |L_{n-3}|$ is a smooth divisor. Tensoring with $\bigotimes_{i=1}^{n-4} L_i \otimes \det \mathcal{F}$, we see that the assertion is true on X if the analogous assertion for $\bigoplus_{i=1}^{\dim A-3} L_{iA} \oplus \mathcal{F}_A$ is true on A. Thus, by induction, we reduce our situation to the case $n = 3$, for which the assertion is true by (2.4). Of course the same proof works for $\mathcal{E}$, a direct sum of line bundles, since by induction we can reduce our situation to the case $n = 2$. □

(2.6) *Remark.* If $(X, \mathcal{E})$ is as in $(*)$, then either $g(X, \mathcal{E}) = h^{1,0}(X)$ or $m(K_X + \det \mathcal{E})$ is spanned for some positive integer m. This follows from (2.1) combined with the base point free theorem.

Standing on this fact, we also pose the following

(2.7) *Conjecture.* Let $(X, \mathcal{E})$ be as in $(*)$. If $\mathcal{E}$ is further very ample, then either $g(X, \mathcal{E}) = h^{1,0}(X)$ or $K_X + \det \mathcal{E}$ itself is spanned.

A vector bundle $\mathcal{E}$ on X is called *very ample* if the tautological line bundle $H(\mathcal{E})$ on $\mathbb{P}_X(\mathcal{E})$ is very ample. Let us quote the following results related to Conjecture (2.7).

(2.7.1) The conjecture is true for $\dim X = 2$ [S2, Theorem 1.5].

(2.7.2) Let $\mathcal{E}$ be a very ample vector bundle on a complex projective manifold X of dimension n. If $\operatorname{rank}\mathcal{E} \geq n+1$, or if $\operatorname{rank}\mathcal{E} = n$ and $c_1(\mathcal{E})^n > (\operatorname{rank}\mathcal{E})^n$, then $K_X + \det \mathcal{E}$ is spanned [BSS, Corollary 2.4].

(2.7.3) Moreover, if $\dim X = 2$ and $\operatorname{rank}\mathcal{E} \geq 2$, then $K_X + \det \mathcal{E}$ is spanned unless $(X, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2})$ in the case when $\mathcal{E}$ is simply supposed to be ample and spanned [LM1, Theorem A].

References

[Ba] E. Ballico, *On vector bundles on 3-folds with sectional genus 1*, Trans. Amer. Math. Soc. **324** (1991), 135–147.

[BSS] M. C. Beltrametti, M. Schneider and A. J. Sommese, *Chern inequalities and spannedness of adjoint bundles*, Proceedings of the Hirzebruch 65 Conference, Bar - Ilan, 1993 (to appear).

[BS] M. C. Beltrametti and A. J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, Exp. Math., vol. 16, de Gruyter, Berlin - New York, 1995.

[F1] T. Fujita, *On the hyperplane section principle of Lefschetz*, J. Math. Soc. Japan **32** (1980), 153–169.

[F2] T. Fujita, *Ample vector bundles of small c_1-sectional genera*, J. Math. Kyoto Univ. **29** (1989), 1–16.

[F3] T. Fujita, *On adjoint bundles of ample vector bundles*, Complex Algebraic Varieties, Proc. Bayreuth 1990 (K. Hulek, T. Peternell, M. Schneider and F.-O. Schreyer, eds.), Lecture Notes in Math., vol. 1507, Springer - Verlag, Berlin - Heidelberg, 1992, pp. 105-112.

[Fu] W. Fulton, *Intersection Theory*, Ergeb. Math. Grenzgeb. (3), vol. 2, Springer - Verlag, Berlin - Heidelberg - New York - Tokyo, 1984.

[FuL] W. Fulton and R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, Algebraic Geometry (A. Libgober and P. Wagreich, eds.), Lecture Notes in Math., vol. 862, Springer-Verlag, Berlin-Heidelberg, 1981, pp. 26–92.

[KMM] Y Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Algebraic Geometry, Sendai, 1985 (T. Oda, ed.), Adv. Stud. Pure Math., vol. 10, Kinokuniya, Tokyo, 1987, pp. 283–360.

[LM1] A. Lanteri and H. Maeda, *Adjoint bundles of ample and spanned vector bundles on algebraic surfaces*, J. Reine Angew. Math. **433** (1992), 181–199.

[LM2] A. Lanteri and H. Maeda, *Ample vector bundles with sections vanishing on projective spaces or quadrics*, Internat. J. Math. (to appear).

[LP] A. Lanteri and M. Palleschi, *Some characterizations of projective bundles and projective spaces*, Geom. Dedicata **14** (1983), 203–208.

[Mu] S. Mukai, *Polarized K3 surfaces of genus 18 and 20*, Complex Projective Geometry (G. Ellingsrud, C. Peskine et al., eds.), London Math. Soc. Lecture Note Ser., vol. 179, Cambridge Univ. Press, Cambridge, 1992, pp. 264–276.

[SS] B. Shiffman and A. J. Sommese, *Vanishing Theorems on Complex Manifolds*, Progr. Math., vol. 56, Birkhäuser, Boston - Basel - Stuttgart, 1985.

[S1] A. J. Sommese, *Submanifolds of Abelian varieties*, Math. Ann. **233** (1978), 229-256.

[S2] A. J. Sommese, *Hyperplane sections of projective surfaces I – The adjunction mapping*, Duke Math. J. **46** (1979), 377-401.

[S3] A. J. Sommese, *On the nonemptiness of the adjoint linear system of a hyperplane section of a threefold*, J. Reine Angew. Math. **402** (1989), 211–220.

[W] J. A. Wiśniewski, *Length of extremal rays and generalized adjunction*, Math. Z. **200** (1989), 409–427.

[YZ] Y.-G. Ye and Q. Zhang, *On ample vector bundles whose adjunction bundles are not numerically effective*, Duke Math. J. **60** (1990), 671–687.

Antonio Lanteri
Dipartimento di Matematica "F. Enriques"
Università
Via C. Saldini, 50
I-20133 Milano
Italy
e-mail:lanteri@vmimat.mat.unimi.it

Hidetoshi Maeda
Graduate School of Mathematics
Kyushu University
4-2-1 Ropponmatsu, Chuo-ku
Fukuoka 810
Japan
e-mail:hmaeda@math.kyushu-u.ac.jp

Andrew J. Sommese
Department of Mathematics
University of Notre Dame
Notre Dame, INDIANA 46556
U.S.A.
e-mail:sommese.1@nd.edu