Notes on very ample vector bundles on 3-folds

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ABSTRACT. Let $\mathcal{E}$ be a very ample vector bundle of rank two on a smooth complex projective threefold X. An inequality about the third Segre class of $\mathcal{E}$ is provided when $K_X + \det \mathcal{E}$ is nef but not big, and when a suitable positive multiple of $K_X + \det \mathcal{E}$ defines a morphism $X \to B$ with connected fibers onto a smooth projective curve B, where K_X is the canonical bundle of X. As an application, the case where the genus of B is positive and $\mathcal{E}$ has a global section whose zero locus is a smooth hyperelliptic curve of genus ≥ 2 is investigated, and our previous result is improved for threefolds.

Introduction

In what follows, varieties are always assumed to be defined over the field $\mathbb{C}$ of complex numbers.

Let $\mathcal{E}$ be a very ample vector bundle of rank $n-1$ on a smooth projective variety X of dimension $n \geq 3$. Then there exists a global section $s \in \Gamma(\mathcal{E})$ whose zero locus $C = (s)_0$ is a smooth curve on X.

If $K_X + \det \mathcal{E}$ is nef and big, then there exist a birational morphism $\pi : X \to X'$ expressing X as the blow-up of a smooth projective variety X' along a finite set B of points (possibly empty) and an ample vector bundle $\mathcal{E}'$ of rank $n-1$ on X' such that $\mathcal{E} = \pi^* \mathcal{E}' \otimes \mathcal{O}_X(-\pi^{-1}(B))$ and that $K_{X'} + \det \mathcal{E}'$ is ample. The polarized pair $(X', \mathcal{E}')$ is called the *first reduction* of $(X, \mathcal{E})$. In this case, under the assumption that C is a hyperelliptic curve of genus $g \geq 2$, we have proved in [MS]

$$n \geq \tau \geq \frac{(n-1)(g+t+1)+2}{2(g-1)} - 1,$$

where τ is the nefvalue of the polarized pair $(X', K_{X'} + \det \mathcal{E}')$ and t is the number of exceptional divisors with respect to the first reduction morphism $\pi : X \to X'$.

As a continuation of the above research, we investigate the case where $K_X + \det \mathcal{E}$ is nef but not big in case $n = 3$. According to [BS1, Theorem 3.1], there are five possibilities in this case. In this article we restrict ourselves to especially the following three cases among them, where a suitable positive multiple of $K_X + \det \mathcal{E}$

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defines a morphism $X \to B$ with connected fibers onto a smooth projective curve B.

(a) X is a $\mathbb{P}^2$-bundle over a smooth curve B, and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)$ for any fiber F of the projection $\pi : X \to B$;
(b) X is a $\mathbb{P}^2$-bundle over a smooth curve B, and $\mathcal{E}_F \cong T_{\mathbb{P}}$ for any fiber F of the projection $\pi : X \to B$, where $T_{\mathbb{P}}$ is the tangent bundle of $\mathbb{P}^2$;
(c) there exists a surjective morphism $\pi : X \to B$ onto a smooth curve B such that a general fiber F of π is a smooth quadric surface $\mathbb{Q}^2$ in $\mathbb{P}^3$ with $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus 2}$.

We note that $K_X + \det \mathcal{E} = \pi^* H$ for some ample line bundle H on B in every case. The purpose of this article is to give an inequality in each of the above cases concerning the third Segre class of $\mathcal{E}$ by using the double point formula. The precise statement of our result is as follows:

Theorem 1. *Let $\mathcal{E}$ be a very ample vector bundle of rank two on a smooth projective 3-fold X, and assume that $(X, \mathcal{E})$ is one of* (a), (b) *or* (c). *Let D denote the third Segre class $s_3(\mathcal{E})$ of $\mathcal{E}$, let q be the genus of B, and let $\rho = \deg(H - K_B)$. Then*

(1) *$D(D - 20) \geq 200(q - 1) + 37\rho$ in case* (a)*;*
(2) *$D(D - 20) \geq 168(q - 1) + 21\rho$ in case* (b)*;*
(3) *$D(D - 20) \geq 168(q - 1) + 20\rho + k$ in case* (c)*, where k is the number of singular fibers of π.*

By Lemma 4, the singular fibers in case (c) are biholomorphic to irreducible and reduced quadric surfaces in $\mathbb{P}^3$.

This article is organized as follows. In Section 0 we collect preliminary material. Section 1 is devoted to the proof of Theorem 1. As an application of Theorem 1, in Section 2 we come back to the case where C is a hyperelliptic curve of genus ≥ 2, and show that case (a) does not occur and that case (b) is very restricted when the genus of B is positive. This allows us to improve [MS, Theorem 6] in case of 3-folds. It also complements the result [LPS, Theorem 2.4] as noted in [LPS, Remark 2.5].

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0. Background material

We use the standard notation from algebraic geometry. The tensor products of line bundles are denoted additively. The numerical equivalence is denoted by $\equiv$. The pullback $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding $i : Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. For a vector bundle $\mathcal{E}$ on X, the tautological line bundle on the projective space bundle $\mathbb{P}_X(\mathcal{E})$ associated to $\mathcal{E}$ is denoted by $H(\mathcal{E})$. A vector bundle $\mathcal{E}$ on a projective variety X is said to be *very ample*, if the tautological line bundle $H(\mathcal{E})$ on $\mathbb{P}_X(\mathcal{E})$ is very ample. We denote by K_X the canonical bundle of a smooth variety X. Let $\mathcal{E}$ be a vector bundle of rank $n - 1$ on a smooth projective variety X of dimension $n \geq 3$ such that there exists a global section $s \in \Gamma(\mathcal{E})$ whose zero locus $C = (s)_0$ is a smooth curve on X. Then we should note that $K_C = (K_X + \det \mathcal{E})_C$.

Let X be a smooth projective variety of dimension $n \geq 3$, and let $\mathcal{E}$ be a very ample vector bundle of rank $n-1$ on X. Assume that $K_X + \det \mathcal{E}$ is nef. Then, by the base point free theorem, a suitable positive multiple of $K_X + \det \mathcal{E}$ is spanned and defines a morphism $\pi : X \to B$ with connected fibers onto a normal projective variety B. Assume furthermore that $\dim B = 1$. Then, from [BS1, Theorem 3.1] for $n = 3$ and [ABW, Theorem B] for $n \geq 4$, $(X, \mathcal{E})$ is one of the following:

(a) X is a $\mathbb{P}^{n-1}$-bundle over a smooth curve B, and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for any fiber F of the projection $\pi : X \to B$;

(b) X is a $\mathbb{P}^{n-1}$-bundle over a smooth curve B, and $\mathcal{E}_F \cong T_{\mathbb{P}}$ for any fiber F of the projection $\pi : X \to B$, where $T_{\mathbb{P}}$ is the tangent bundle of $\mathbb{P}^{n-1}$;

(c) there exists a surjective morphism $\pi : X \to B$ onto a smooth curve B such that a general fiber F of π is a smooth hyperquadric $\mathbb{Q}^{n-1}$ in $\mathbb{P}^n$ with $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)}$.

It should be emphasized that $K_X + \det \mathcal{E} = \pi^* H$ for some ample line bundle H on B in every case.

In all that follows, suppose that $(X, \mathcal{E})$ is one of (a), (b) or (c). Since $\mathcal{E}$ is very ample, there exists a global section $s \in \Gamma(\mathcal{E})$ whose zero locus $C = (s)_0$ is a smooth curve on X. Let g denote the genus of C. We note that the restriction $\pi_C : C \to B$ of π to C is surjective. Hence the Riemann-Hurwitz formula tells us that

$$2g - 2 = d(2q - 2) + r,$$

where q is the genus of B, d is the degree of π_C, and r is the degree of the ramification divisor of π_C. Since $d = c_{n-1}(\mathcal{E}_F)$, we have $d = 2$ in cases (a), (c) and $d = n$ in case (b). On the other hand,

$$K_C = (K_X + \det \mathcal{E})_C = (\pi^* H)_C = \pi_C^* H,$$

so that

$$2g - 2 = d\delta,$$

where $\delta = \deg H$. Hence

$$d\delta = d(2q - 2) + r.$$

This implies that r is an integer multiple of d. Let $r = d\rho$. Then

$$\delta = 2q - 2 + \rho, \tag{0.1}$$

and we conclude that

$$\rho = \deg(H - K_B).$$

It should be kept in mind that

$$2g - 2 = d(2q - 2 + \rho). \tag{0.2}$$

The following lemma tells us that $\rho > 0$.

Lemma 2. *The morphism $\pi_C : C \to B$ can never be unramified.*

Proof. Let X_C denote the fiber product $X \times_B C$ of X and C over B. Then X_C is connected. Let $p_1 : X_C \to X$ be the first projection, and let $p_2 : X_C \to C$ be the second projection. Note that $p_1^*\mathcal{E}$ is ample and that $p_1^{-1}(C)$ is the zero locus of the pullback of the section s defining C. Thus $H^0(p_1^{-1}(C), \mathbb{Z}) \cong H^0(X_C, \mathbb{Z}) = \mathbb{Z}$ by the Lefschetz-Sommese theorem [LM2, Theorem 1.1] (note that its proof is valid without assuming the connectedness of $p_1^{-1}(C)$), so that $p_1^{-1}(C)$ is connected. On the other hand, $p_1^{-1}(C)$ decomposes into a curve C' and a curve C'', where C' is the image of the section of p_2 defined by sending $x \in C$ to $(x, x) \in X_C$. If π_C is unramified, then $C' \cap C'' = \emptyset$. This contradicts the connectedness of $p_1^{-1}(C)$. □

Moreover, if C is a hyperelliptic curve of genus $g \geq 2$ and if $q > 0$, then we have the following

Lemma 3. *Assume that C is a hyperelliptic curve of genus $g \geq 2$. If the genus q of B is positive, then ρ is either 1 or 2.*

Proof. By Lemma 2 we know that $\rho > 0$. Letting $H = K_B + \mathcal{H}$ for some line bundle $\mathcal{H}$ on B, we have $\rho = \deg \mathcal{H}$ and $K_C = \pi_C^*(K_B + \mathcal{H})$. Since C is hyperelliptic, $|K_B + \mathcal{H}|$ cannot take any smooth curve of positive genus as its image. Therefore $\deg \mathcal{H} \leq 2$. □

1. Proof of Theorem 1

From now on, throughout these notes, we assume that $n = 3$, and use the same notation as in Section 0. Let $M = \mathbb{P}_X(\mathcal{E})$ be the projective space bundle associated to $\mathcal{E}$, let $p : M \to X$ be the bundle projection, and let $H(\mathcal{E})$ be the tautological line bundle on M. Then $H(\mathcal{E})$ is very ample. As mentioned, we have $K_X + \det \mathcal{E} = \pi^* H$. Thus $K_M + 2H(\mathcal{E}) = p^*(K_X + \det \mathcal{E}) = p^*\pi^* H$.

Let Z be a general element of $|H(\mathcal{E})|$, let S be a general element of $|H(\mathcal{E})_Z|$, and let $\varphi : S \to B$ be the restriction of $\pi \circ p$ to S. Then $K_S = \varphi^* H$. Let G be a general fiber of $\pi \circ p$, and let $f = S \cap G$. Since S can be regarded as the zero locus of a section $t \in \Gamma(M, H(\mathcal{E})^{\oplus 2})$, f is the zero locus of the section $t_G \in \Gamma(G, H(\mathcal{E})_G^{\oplus 2})$. Hence $f \neq \emptyset$, and φ is surjective. In particular, this implies that f is a 1- equidimensional smooth fiber of φ. On the other hand, since $H^0(f, \mathbb{Z}) \cong H^0(G, \mathbb{Z}) = \mathbb{Z}$ by [LM2, Theorem 1.1], f is a smooth curve in S and we conclude that φ has connected fibers.

Now let us apply the double point formula [BS2, Theorem 13.1.5] to $(Z, H(\mathcal{E})_Z)$, which tells us that

$$e(Z) - 48\chi(\mathcal{O}_Z) + 84\chi(\mathcal{O}_S) - 11K_S^2 - 17K_S H(\mathcal{E})_S - (K_Z + H(\mathcal{E})_Z)^3 + D(D - 20) \geq 0,$$

where $D = H(\mathcal{E})^3_Z = H(\mathcal{E})^4$, and $e(Z)$ is the topological Euler characteristic of Z. Then

$$D = s_3(\mathcal{E}) = c_1(\mathcal{E})^3 - 2c_1(\mathcal{E})c_2(\mathcal{E}),$$

where $s_3(\mathcal{E})$ is the third Segre class of $\mathcal{E}$. We have

$$K_S^2 = (\varphi^* H)^2 = 0$$

and

$$(K_Z + H(\mathcal{E})_Z)^3 = (K_M + 2H(\mathcal{E}))^3_Z = (p^*\pi^* H)^3_Z = 0.$$

Let us compute $\chi(\mathcal{O}_Z), \chi(\mathcal{O}_S)$ and $K_S H(\mathcal{E})_S$. First, since Z is the blow-up of X along C, Z is birational to X. Thus

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_X) = \chi(\mathcal{O}_B) = 1 - q.$$

Next, since $K_S = \varphi^* H$ and φ has connected fibers, we have

$$h^0(K_S) = h^0(H).$$

Furthermore, by the Kodaira vanishing theorem,

$$0 = h^1(K_M + 2H(\mathcal{E})) = h^1(H),$$

since $K_M + 2H(\mathcal{E}) = p^*\pi^* H$. Moreover, from the exact sequence

$$0 \to \mathcal{O}_Z(-S) \to \mathcal{O}_Z \to \mathcal{O}_S \to 0,$$

we see that

$$h^1(\mathcal{O}_S) = h^1(\mathcal{O}_Z) = q.$$

Therefore, the Riemann-Roch theorem applied to (B, H) gives

$$\begin{aligned} 2q - 2 + \rho = \delta = \deg H &= h^0(H) + q - 1 \\ &= h^0(K_S) + q - 1 = h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S) + 2q - 2 \\ &= \chi(\mathcal{O}_S) + 2q - 2. \end{aligned}$$

Consequently

$$\chi(\mathcal{O}_S) = \rho.$$

Finally, we have

$$\begin{aligned} K_S H(\mathcal{E})_S &= (\varphi^* H) H(\mathcal{E})_S = (p^*\pi^* H)_Z H(\mathcal{E})^2_Z = (p^*\pi^* H) H(\mathcal{E})^3 \\ &= \delta \mathbb{P}_F(\mathcal{E}_F) H(\mathcal{E})^3 = \delta H(\mathcal{E}_F)^3 \\ &= \delta(c_1(\mathcal{E}_F)^2 - c_2(\mathcal{E}_F)) = \delta(c_1(\mathcal{E}_F)^2 - d) \\ &= (2q - 2 + \rho)(c_1(\mathcal{E}_F)^2 - d). \end{aligned}$$

Thus, in sum,

$$(1.0)\qquad e(Z)-48(1-q)+84\rho-17(2q-2+\rho)(c_1(\mathcal{E}_F)^2-d)+D(D-20)\geq 0.$$

We proceed now by cases.

(1.1) *Case* (a). In this case we have

$$e(Z)=e(X)+e(C)=e(\mathbb{P}^2)e(B)+e(C)=3(2-2q)+2-2g.$$

Combining this with (0.2) gives $e(Z)=6(1-q)-d(2q-2+\rho)$. Since $c_1(\mathcal{E}_F)^2=9$ and $d=2$, (1.0) tells us that

$$6(1-q)-2(2q-2+\rho)-48(1-q)+84\rho-119(2q-2+\rho)+D(D-20)\geq 0.$$

An easy calculation shows that

$$D(D-20)\geq 200(q-1)+37\rho,$$

and the result is proved.

(1.2) *Case* (b). Here we have $c_1(\mathcal{E}_F)^2=9$ and $d=3$. By the same argument as that in (1.1), we get

$$6(1-q)-3(2q-2+\rho)-48(1-q)+84\rho-102(2q-2+\rho)+D(D-20)\geq 0.$$

Consequently

$$D(D-20)\geq 168(q-1)+21\rho.$$

(1.3) *Case* (c). Before proceeding with the proof, we present the following

Lemma 4. *Let* $(X,\mathcal{E})$ *be as in case* (c). *Then every fiber of* π *is an irreducible and reduced quadric surface in* $\mathbb{P}^3$.

Proof. We set $L=\det\mathcal{E}$. Then, as we pointed out, $K_X+L=\pi^*H$.

We first claim that every fiber F of π is irreducible and reduced. Suppose to the contrary that F is not irreducible and reduced for some F. Then we can write $F=n_1F_1+\cdots+n_rF_r$ for distinct integral surfaces $F_1,\dots,F_r$ and positive integers $n_1,\dots,n_r$ with $n_1+\cdots+n_r\geq 2$. Hence a general element $T\in|L|$ must meet F in a curve $f=n_1f_1+\cdots+n_rf_r$ for distinct integral curves $f_1,\dots,f_r$. Now we know that $Lf_i\geq 2$ for any i from [LM1, Corollary 1]. If $Lf_i=2$ for some i, then $f_i\cong\mathbb{P}^1$ by [LM1, Corollary 2]. Since $f_i\in|L_{F_i}|$, we see that $f_i\cap\mathrm{Sing}(F_i)=\emptyset$, where $\mathrm{Sing}(F_i)$ is the singular locus of F_i. Thus $\mathrm{Sing}(F_i)\subset F_i-f_i$. Since $\mathrm{Sing}(F_i)$ is a compact algebraic set and F_i-f_i is affine, we conclude that F_i has at most isolated singularities. This implies that F_i is normal, and the classification of the polarized surfaces of sectional genus zero applies to (F_i,L_{F_i}) (see for example [BS2, Corollary 3.2.10]). However, since $L_{F_i}E\geq 2$ for any rational curve E on F_i, (F_i,L_{F_i}) must be $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2))$. Therefore $L_{F_i}^2 = 4$. On the other hand, $L_{F_i}^2 = L_{F_i} f_i = 2$. This is a contradiction. Consequently $Lf_i = L_{F_i}^2 \geq 3$ for any i. We should note that $Lf = L_F^2 = 8$ because $\mathcal{E}_G \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)^{\oplus 2}$ for a general fiber G of π. Moreover, since $K_X + L = \pi^* H$, we have $K_T = \pi_T^* H$, where π_T is the restriction of π to T. Thus T is a properly elliptic minimal surface, so that [S, Lemma 0.5.1] tells us that $\pi_T : T \to B$ has no multiple fibers. As a direct result of this observation, we obtain $r = 2$ and $n_1 = n_2 = 1$. Since $K_T = \pi_T^* H$, we have $(K_T + f_i) f_i = f_i^2$ for $i = 1, 2$. Moreover, since $f = f_1 + f_2$, we get $f_i(f_1 + f_2) = 0$. Thus $f_1^2 = -f_1 f_2 < 0$, since π_T has connected fibers. Similarly $f_2^2 < 0$. Hence $f_1 \cong \mathbb{P}^1$ and $f_2 \cong \mathbb{P}^1$, and so by the same argument as above we have $F_1 \cong \mathbb{P}^2$ and $F_2 \cong \mathbb{P}^2$. Now we know that $F = F_1 + F_2$, so that $\mathcal{O}_{F_1}(F_1 + F_2) \cong \mathcal{O}_{F_1}$. Since F_1 meets F_2 in a curve, the normal bundle $N_{F_1/X} = \mathcal{O}_{F_1}(F_1)$ to F_1 in X is negative. By a well-known theorem of Grauert (see for example [BS2, Theorem 3.2.7]) there exists a holomorphic map $p : X \to Y$ onto a normal analytic variety Y such that $p(F_1)$ is a point, y, and p induces a biholomorphism $X - F_1 \cong Y - \{y\}$. In particular, $F_1 \cap F_2$ is contracted. However, this is absurd because F_1 has no curves with negative self-intersection. Consequently every fiber F of π is irreducible and reduced.

If F is smooth, then $(F, \mathcal{E}_F) \cong (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus 2})$, where $\mathbb{Q}^2$ is a smooth quadric surface in $\mathbb{P}^3$. Let F be a singular fiber of π. We claim that F is a singular quadric surface in $\mathbb{P}^3$. To see this, take a general element $T \in |L|$. Then T meets F in an irreducible and reduced curve f. We note that the arithmetic genus of f is one because $(K_T + f)f = 0$. If f is not smooth, then f has a single singular point, so that $\mathrm{Sing}(F)$ contains a curve γ with $L\gamma = 1$. This contradicts the fact that $L\gamma \geq 2$. Hence f is smooth, and the same argument as above again shows that F is normal. Since $K_X + L = \pi^* H$, we have $K_F + L_F = \mathcal{O}_F$, and we conclude that F is a normal Gorenstein Del Pezzo surface with $K_F^2 = 8$. According to [Br, Theorem 1], F is one of the following:

(i) F is a singular quadric surface in $\mathbb{P}^3$;
(ii) F is the space obtained by blowing down the zero section Δ of a $\mathbb{P}^1$-bundle P on a smooth elliptic curve Γ;
(iii) F is a rational surface with only rational double points as singularities, obtained from $\mathbb{P}^2$ by blowing up some number $\alpha \leq 8$ points (iterations allowed) then blowing down some number $\beta \leq \alpha$ smooth rational curves, each with self-intersection -2.

In case (ii) we can write $K_P + \mathcal{O}_P(k\Delta) = \sigma^* K_F$ for some integer k, where σ is the blowing-down of Δ. Since $(K_P + \mathcal{O}_P(\Delta))_\Delta = K_\Delta = \mathcal{O}_\Delta$, we obtain

$$\begin{aligned} \mathcal{O}_\Delta = (\sigma^* K_F)_\Delta &= (K_P + \mathcal{O}_P(k\Delta))_\Delta \\ &= (K_P + \mathcal{O}_P(\Delta) + \mathcal{O}_P((k-1)\Delta))_\Delta = \mathcal{O}_\Delta((k-1)\Delta). \end{aligned}$$

This implies that $k = 1$. Hence $K_P + \mathcal{O}_P(\Delta) = \sigma^* K_F$. Set $\Delta^2 = -r$ for some positive integer r. Then $K_P \equiv -2\Delta - r\rho$, where ρ is a fiber of the bundle projection $P \to \Gamma$. Thus $-\sigma^* K_F \equiv \Delta + r\rho$, so that

$$L_F \sigma(\rho) = -K_F \sigma(\rho) = (-\sigma^* K_F)\rho = (\Delta + r\rho)\rho = 1.$$

This contradicts $L_F\sigma(\rho) \geq 2$. In case (iii), let $\varphi : G \to \mathbb{P}^2$ be the composite of α blowing-up morphisms, and let $\tau : G \to F$ be the composite of β blowing-down morphisms. Then $K_F^2 = K_G^2 = K_{\mathbb{P}}^2 - \alpha = 9 - \alpha$. Hence $\alpha = 1$, since $K_F^2 = 8$. Consequently G is the first Hirzebruch surface. Moreover, either $\beta = 0$ or $\beta = 1$. If $\beta = 0$, then $F = G$. This is impossible because F is singular. However, the case $\beta = 1$ is also impossible because G has no (-2)-curves. Therefore case (iii) does not occur. Consequently F must be a singular quadric surface in $\mathbb{P}^3$, and the result is proved. □

We return to the proof of Theorem 1. In case (c), let F' be a singular fiber of π, and let k denote the number of singular fibers of π. Then, since F' is a singular quadric with an isolated singularity by Lemma 4, we obtain

$$\begin{aligned} e(Z) = e(X) + e(C) &= e(X - kF') + ke(F') + e(C) \\ &= 4(2 - 2q - k) + 3k + 2 - 2g = 8(1 - q) - k - d(2q - 2 + \rho). \end{aligned}$$

Since $c_1(\mathcal{E}_F)^2 = 8$ and $d = 2$, it follows from (1.0) that

$$8(1 - q) - k - 2(2q - 2 + \rho) - 48(1 - q) + 84\rho - 102(2q - 2 + \rho) + D(D - 20) \geq 0.$$

Therefore

$$D(D - 20) \geq 168(q - 1) + 20\rho + k,$$

and we have thus proved Theorem 1.

2. The case of a hyperelliptic curve

Let $\mathcal{E}$ be a very ample vector bundle of rank two on a smooth projective 3-fold X, and assume that $(X, \mathcal{E})$ is one of (a), (b) or (c). In this section we set up the following condition:

(∗) There exists a global section $s \in \Gamma(\mathcal{E})$ whose zero locus $C = (s)_0$ is a smooth hyperelliptic curve of genus $g \geq 2$, and the base curve B has genus $q > 0$.

Theorem 5. *Under the assumption* (∗)*, case* (a) *does not occur.*

Proof. The proof is by contradiction. Let $(X, \mathcal{E})$ be as in case (a). Then we can write $X = \mathbb{P}_B(\mathcal{V})$ for some vector bundle $\mathcal{V}$ of rank three on B. Since $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)$ for any fiber F of π, $\mathcal{F} := \pi_*(\mathcal{E} \otimes (-2H(\mathcal{V})))$ is a line bundle on B, and we have an exact sequence

$$0 \to 2H(\mathcal{V}) + \pi^*\mathcal{F} \to \mathcal{E} \to Q \to 0$$

for some line bundle Q on X. Then $Q_F = \det \mathcal{E}_F - 2H(\mathcal{V})_F \cong \mathcal{O}_{\mathbb{P}}(1)$. Hence $Q = H(\mathcal{V}) + \pi^*L$ for some line bundle L on B. We note that Q is very ample, because $\mathcal{E}$ is very ample. Let $\mathcal{G} = \pi_*Q$. Then, since $Q_F \cong \mathcal{O}_{\mathbb{P}}(1)$ for any F, $\mathcal{G}$ is a vector bundle of rank three on B such that $(X, Q) \cong (\mathbb{P}_B(\mathcal{G}), H(\mathcal{G}))$. Thus $\mathcal{G}$ is very ample. We can write numerically $2H(\mathcal{V})+\pi^*\mathcal{F}\equiv 2H(\mathcal{G})+\mu F$ for some integer μ. Thus $\det\mathcal{E}=2H(\mathcal{V})+\pi^*\mathcal{F}+Q\equiv 3H(\mathcal{G})+\mu F$. Since $K_X+\det\mathcal{E}=\pi^*H$ for some ample line bundle H on B, it follows from (0.1) that $K_X+\det\mathcal{E}\equiv(2q-2+\rho)F$. Therefore

$$K_X\equiv-\det\mathcal{E}+(2q-2+\rho)F\equiv-3H(\mathcal{G})+(2q-2+\rho-\mu)F.$$

On the other hand, from the basic relation $K_X=-3H(\mathcal{G})+\pi^*(K_B+\det\mathcal{G})$, we have

$$K_X\equiv-3H(\mathcal{G})+(2q-2+x)F,$$

where $x=\deg\mathcal{G}$. Consequently $\rho-\mu=x$, i.e., $\mu=\rho-x$.

Now, from the above exact sequence, $c_2(\mathcal{E})=(2H(\mathcal{V})+\pi^*\mathcal{F})Q\equiv 2H(\mathcal{G})^2+\mu H(\mathcal{G})F$. As observed in Section 1, $D=c_1(\mathcal{E})^3-2c_1(\mathcal{E})c_2(\mathcal{E})$. Hence

$$\begin{aligned} D&=(3H(\mathcal{G})+\mu F)^3-2(3H(\mathcal{G})+\mu F)(2H(\mathcal{G})^2+\mu H(\mathcal{G})F)\\ &=15H(\mathcal{G})^3+17\mu H(\mathcal{G})^2F=15x+17\mu\\ &=17\rho-2x. \end{aligned}\tag{2.1}$$

From Theorem 1, we have $D(D-20)\geq 200(q-1)+37\rho$. Moreover, under the assumption $(*)$, by Lemma 3 ρ is either 1 or 2. Hence $D(D-20)\geq 37$. This implies that $D\geq 22$. Combining this with (2.1) gives $22\leq 17\rho-2x<17\rho$, since $x>0$. Thus $\rho=2$. We claim that $q=1$. To see this, suppose that $q\geq 2$. Since $\rho=2$, we obtain $D(D-20)\geq 274$. Thus $D\geq 30$. By using (2.1) again, we have $x=17-(1/2)D\leq 2$. Since $x=H(\mathcal{G})^3$ and $H(\mathcal{G})$ is very ample, we have a contradiction. Therefore $q=1$, and $D(D-20)\geq 74$. Hence $D\geq 24$, and by (2.1) we get $x=17-(1/2)D\leq 5$. On the other hand, applying [IT, Proposition 1] to $\mathcal{G}$ gives $x\geq 7$. This is also absurd. □

Theorem 6. *Under the assumption* $(*)$*, let* $(X,\mathcal{E})$ *be as in case* (b)*. Then* $g=4$ *and* $q=1$*.*

Proof. We can write $X=\mathbb{P}_B(\mathcal{G})$ for some vector bundle $\mathcal{G}$ of rank three on B. Let $x=\deg\mathcal{G}$. Then, since $K_X=-3H(\mathcal{G})+\pi^*(K_B+\det\mathcal{G})$, we have

$$K_X\equiv-3H(\mathcal{G})+(2q-2+x)F.\tag{2.2}$$

Let $\mathcal{V}=\pi_*(\mathcal{E}\otimes(-H(\mathcal{G})))$. Then, since $\mathcal{E}_F\cong T_{\mathbb{P}}$ for any F, $\mathcal{V}$ is a vector bundle of rank three on B, and we have an exact sequence

$$0\to\pi^*L\to(\pi^*\mathcal{V})\otimes H(\mathcal{G})\to\mathcal{E}\to 0$$

for some line bundle L on B. Let $l=\deg L$ and let $v=\deg\mathcal{V}$. Then

$$\det\mathcal{E}=3H(\mathcal{G})+\pi^*\det\mathcal{V}-\pi^*L\equiv 3H(\mathcal{G})+(v-l)F.\tag{2.3}$$

Thus, by (2.2),

$$K_X + \det \mathcal{E} \equiv (2q - 2 + x + v - l)F.$$

On the other hand, since $K_X + \det \mathcal{E} = \pi^* H$, it follows from (0.1) that

$$K_X + \det \mathcal{E} \equiv (2q - 2 + \rho)F.$$

Hence

$$\rho = x + v - l. \tag{2.4}$$

From the above exact sequence, we get

$$c_2(\mathcal{E}) + (\pi^* L) \det \mathcal{E} = c_2((\pi^* \mathcal{V}) \otimes H(\mathcal{G})) = 3H(\mathcal{G})^2 + 2(\pi^* \det \mathcal{V})H(\mathcal{G}).$$

Thus, by (2.3), we see that

$$c_2(\mathcal{E}) \equiv 3H(\mathcal{G})^2 + 2vH(\mathcal{G})F - lF(3H(\mathcal{G}) + (v - l)F) = 3H(\mathcal{G})^2 + (2v - 3l)H(\mathcal{G})F.$$

Using this and the above exact sequence again, we obtain

$$3l = c_2(\mathcal{E})\pi^* L = c_3((\pi^* \mathcal{V}) \otimes H(\mathcal{G})) = H(\mathcal{G})^3 + (\pi^* \det \mathcal{V})H(\mathcal{G})^2 = x + v.$$

Therefore by (2.4), we have $\rho = x + v - l = 3l - l = 2l$, so that ρ is even. We know from Lemma 3 that ρ is either 1 or 2 under the assumption $(*)$. Thus $\rho = 2$ and $l = 1$. Let us compute $c_1(\mathcal{E})^3$ and $c_1(\mathcal{E})c_2(\mathcal{E})$. First, by (2.3) and (2.4),

$$\begin{aligned} c_1(\mathcal{E})^3 = (3H(\mathcal{G}) + (v - l)F)^3 &= 27H(\mathcal{G})^3 + 27(v - l)H(\mathcal{G})^2 F = 27x + 27(v - l) \\ &= 27(x + v - l) = 27\rho = 54. \end{aligned}$$

Next,

$$\begin{aligned} c_1(\mathcal{E})c_2(\mathcal{E}) &= (3H(\mathcal{G}) + (v - l)F)(3H(\mathcal{G})^2 + (2v - 3l)H(\mathcal{G})F) \\ &= 9H(\mathcal{G})^3 + (9v - 12l)H(\mathcal{G})^2 F = 9(x + v - l) - 3l = 9\rho - 3l = 15. \end{aligned}$$

Since $D = c_1(\mathcal{E})^3 - 2c_1(\mathcal{E})c_2(\mathcal{E})$, we have $D = 24$. On the other hand, Theorem 1 tells us that $D(D - 20) \geq 168(q - 1) + 42$, because $\rho = 2$. If $q \geq 2$, then $D(D-20) \geq 210$, which implies that $D \geq 28$, a contradiction. Consequently $q = 1$. Recalling that $d = 3$, we conclude with the aid of (0.2) that $g = 4$. □

Especially for 3-folds, Theorem 5 allows us to improve [MS, Theorem 6] as follows:

Theorem 7. *Let $\mathcal{E}$ be a very ample vector bundle of rank two on a smooth projective 3-fold X. Assume that $g(X, \mathcal{E}) = 2$. Then $(X, \mathcal{E})$ is one of the following:*

(1) *X is a $\mathbb{P}^2$-bundle over a smooth curve B of genus 2, and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$ for any fiber F of the projection $X \to B$;*
(2) *X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)$ for any fiber F of the projection $\pi : X \to \mathbb{P}^1$, and $K_X + \det \mathcal{E}$ is the pullback of a line bundle of degree 1 on $\mathbb{P}^1$ by π;*
(3) *there exists a surjective morphism $\pi : X \to B$ onto a smooth curve B of genus ≤ 1 such that a general fiber F of π is a smooth quadric surface $\mathbb{Q}^2$ in $\mathbb{P}^3$ with $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus 2}$, and $K_X + \det \mathcal{E}$ is the pullback of a line bundle of degree 1 on B by π.*

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