On the birationality of the bicanonical map of a surface section of a threefold *

M.C. Beltrametti, C. Ciliberto, A. Lanteri, and A.J. Sommese

Abstract

Let $(\mathcal{M}, \mathcal{L})$ be a polarized threefold of log-general type. The birationality of the bicanonical map of a smooth surface $S \in |\mathcal{L}|$ is studied. This problem was previously considered and partially solved by the first and fourth author, who gave a satisfactory classification unless $h^1(\mathcal{O}_{\mathcal{M}}) = 0$ and $p_g(S) = 3, 4, 5$. This paper focuses on the remaining cases which are the hardest, settling the problem.

Introduction and statement of the result

Let $\mathcal{M}$ be a smooth projective threefold and let $\mathcal{L}$ be a very ample line bundle on $\mathcal{M}$. In this paper we consider the problem of describing the pair $(\mathcal{M}, \mathcal{L})$ in the case when there exists at least one smooth surface $\widehat{S} \in |\mathcal{L}|$ such that $\widehat{S}$ is of general type, and the bicanonical map associated to $|2K_{\widehat{S}}|$ is not birational. If a surface of general type $\widehat{S}$ is a very ample (or even merely, an ample) divisor on a smooth threefold $\mathcal{M}$, then it follows from [3, (7.9.1)] that either:

1. $\mathcal{M}$ is a $\mathbb{P}^1$-bundle over a smooth surface Y with $\widehat{S}$ a meromorphic section; or
2. there exists a fibering $p : \mathcal{M} \to Y$ of $\mathcal{M}$ onto a smooth surface Y with the general fiber isomorphic to $\mathbb{P}^1$ and $p_{|\widehat{S}}$ a generically two-to-one morphism; or
3. $(\mathcal{M}, \mathcal{L})$ is of log-general type, i.e., for sufficiently large N, the linear system $|N(K_{\mathcal{M}} + \mathcal{L})|$ gives a birational map.

The first class is trivial in the sense that every smooth $\widehat{S} \in |\mathcal{L}|$ is birational to the base surface Y, and conversely, given any smooth surface Y, some surface birational to it is a very ample divisor as in this class. The second class contains very special varieties, while the third class contains the overwhelming majority of threefolds with

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a general type surface as an ample divisor. Some results for the second class are contained in [18, §3.2].

Let now $(\mathcal{M}, \mathcal{L})$ be as in the third class with the property that there is at least one smooth surface $\widehat{S} \in |\mathcal{L}|$ such that $\widehat{S}$ is of general type, and the bicanonical map associated to $|2K_{\widehat{S}}|$ is not birational. In [7] the first and the fourth author give a satisfactory classification unless

$$q(\widehat{S}) := h^1(\mathcal{O}_{\widehat{S}})(= h^1(\mathcal{O}_{\mathcal{M}})) = 0 \quad \text{and} \quad 3 \leq h^2(\mathcal{O}_{\widehat{S}}) \leq 5. \tag{1}$$

Note that these cases fall in Du Val's list [11] of surfaces of general type having not birational bicanonical map, and they are in fact the hardest to be treated.

In this paper we deal with $(\mathcal{M}, \mathcal{L})$, $\widehat{S} \in |\mathcal{L}|$ as above satisfying conditions (1), and we complete the results of [7]. We need in fact some extra assumptions, which, however, turn out to be quite natural in our framework. In particular we assume that the linear system $|K_S|$ associated to the minimal model of $\widehat{S}$ contains a smooth (irreducible) element. Note that this is essentially Du Val's assumption [11] (compare also with [9, (3.2.1)]).

We prove the following theorem. Let us emphasize the fact that case 2) below really occurs and, under the reasonable assumption that the canonical bundle K_S is ample, it leads to only one possible case described in §5.

Theorem 0.1 *Let $\mathcal{L}$ be a very ample line bundle on a smooth threefold $\mathcal{M}$. Assume that $\kappa(K_{\mathcal{M}} + \mathcal{L}) = 3$. Let (M, L) be the first reduction of $(\mathcal{M}, \mathcal{L})$, let $\pi : \mathcal{M} \to M$ be the corresponding morphism* (*see* (1.2) *below*)*, and suppose that the linear system $|K_M + L|$ contains a smooth member A. For a smooth $\widehat{S} \in |\mathcal{L}|$ let $S := \pi(\widehat{S}) \in |L|$. Assume that $(\mathcal{M}, \mathcal{L})$, $\widehat{S}$ satisfy conditions* (1) *and that the canonical system $|K_S|$ contains a smooth member. Then either:*

1. *The bicanonical map $\Phi_2 : S \to \mathbb{P}^N$ associated to $|2K_S|$ is birational; or*

2. *$(K_M + L)^3 = 1$, $(K_M + L)^2 \cdot L = 3$, $(K_M + L) \cdot L^2 = 9$ and A is a Del Pezzo surface with $K_A^2 = 1$.*

Note. For a general $\widehat{S} \in |\mathcal{L}|$ the corresponding surface $S = \pi(\widehat{S})$ on M meets the smooth divisor A transversely and therefore the assumption in Theorem (0.1) that $|K_S|$ contains a smooth member is satisfied. But this could not be true for the given surface S we are considering.

The paper is organized as follows. In §1 we recall some preliminaries we need. In §2 we show that, under the assumption of Theorem (0.1), the case when there is a morphism $p : S \to B$ over a smooth curve B whose general fiber is a smooth curve of genus 2 (the *standard case*) does not occur. Lemma (1.5) plays here a key-role. In §3 we reduce the possibilities for (M, L), and an S as in Theorem (0.1) with the bicanonical map Φ_2 failing to be birational, to three cases only, corresponding to the values $K_S^2 = 6, 7, 8$ (unless we fall in case 2) of (0.1)). In §4 we complete the proof of Theorem (0.1) by a detailed analysis of the base loci of $|K_S|$ and $|K_M + L|$, which leads us to rule out the above mentioned cases $K_S^2 = 6, 7, 8$. Key-results here are Proposition (4.1) and Lemmas (4.2) and (4.6). In §5 we give an explicit example as in case 2) of (0.1). We also show that whenever K_S is ample, this is the only possible case.

The main tools used through the paper are classification results for log-general type polarized threefolds $(\mathcal{M}, \mathcal{L})$ based on adjunction theory ([2], [4], [5], [6]), and refined results on minimal general type regular surfaces to be applied to surfaces occurring as hyperplane sections of $(\mathcal{M}, \mathcal{L})$ ([8], [9], [10]). The leit-motiv of the paper is the interaction between the adjunction theoretic approach and these results.

1 Background material

We work over the complex field $\mathbb{C}$. Throughout the paper we deal with projective varieties V. We denote by $\mathcal{O}_V$ the structure sheaf of V. For any coherent sheaf $\mathcal{F}$ on V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V, \mathcal{F})$.

Let L be a line bundle on V. The line bundle L is said to be *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on V. L is said to be *big* if $\kappa(L) = \dim V$, where $\kappa(L)$ denotes the Kodaira dimension of L. If L is nef then this is equivalent to $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L and $n = \dim V$. The pull-back $\iota^* L$ of a line bundle L on V by an embedding $\iota : W \hookrightarrow V$ is denoted by L_W. We denote by K_V the canonical bundle of a smooth variety V.

1.1 Notation. We use standard notation from algebraic geometry, among which we recall the following ones:

$\approx$, the linear equivalence of line bundles; $\sim$, the numerical equivalence of line bundles;

$\chi(L) = \sum_i (-1)^i h^i(L)$, the Euler characteristic of a line bundle L;

$|L|$, the complete linear system associated to a line bundle L;

$\kappa(D)$, the Kodaira dimension of the line bundle associated to a $\mathbb{Q}$-Cartier divisor D on V; and $\kappa(V) := \kappa(K_V)$, the Kodaira dimension of V, for V smooth.

Line bundles and divisors are used with little (or no) distinction. We almost always use the additive notation. We say that a line bundle L is *spanned* if it is spanned, i.e., globally generated, at all points of V by $H^0(L)$.

For a line bundle L on an irreducible normal variety V of dimension n the sectional genus, $g(L) = g(V, L)$, of (V, L) is defined by $2g(L) - 2 = (K_V + (n-1)L) \cdot L^{n-1}$.

We freely use Castelnuovo's bound [3, (1.4.9)] for the genus of an irreducible curve in projective space. See [15, Theorem 3.7] for a discussion and proof of the result.

1.2 Assumptions and general set up. From this point on in the article we make the assumptions that

a) $(\mathcal{M}, \mathcal{L})$ is a smooth projective variety of dimension $n = 3$ polarized with a very ample line bundle $\mathcal{L}$; and

b) $(\mathcal{M}, \mathcal{L})$ is of log-general type, i.e., $\kappa(K_{\mathcal{M}} + \mathcal{L}) = 3$.

Under these assumptions, there exists a smooth polarized threefold (M, L), which is called the *first reduction* of $(\mathcal{M}, \mathcal{L})$ and satisfies (see e.g., [3, Chapters 7, 12 and (13.2.5)]):

1. there exists a morphism $\pi : \mathcal{M} \to M$ expressing $\mathcal{M}$ as the blowing up of M at a finite set of points, B, such that $L := (\pi_*\mathcal{L})^{**}$ is ample and $\mathcal{L} \approx \pi^*L - [\pi^{-1}(B)]$ or, equivalently, $K_{\mathcal{M}} + 2\mathcal{L} \approx \pi^*(K_M + 2L)$;

2. $K_M + 2L$ is very ample;

3. $K_M + L$ is nef and big, with $2(K_M + L)$ spanned by global sections.

Note that there is a one-to-one correspondence between smooth divisors of $|L|$ which contain the set B and smooth divisors of $|\mathcal{L}|$.

Moreover for $m \gg 0$, $|m(K_M + L)|$ gives rise to a morphism $\varphi : M \to X$ with connected fibers and normal image. Thus there is an ample line bundle $\mathcal{H}$ on X such that $K_M + L \approx \varphi^*\mathcal{H}$. The morphism φ is very well behaved (see e.g., [3, §§7.5, 7.6, 7.7 and Chapter 12]). Furthermore X has terminal, 2-Gorenstein (i.e., $2K_X$ is a line bundle) isolated singularities and $\mathcal{H} \approx K_X + \mathcal{D}$, where $\mathcal{D} := (\varphi_*L)^{**}$ is a 2-Cartier divisor such that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ for some effective divisor Δ on M which is φ-exceptional, i.e., $\dim \varphi(\Delta) \leq 1$ (see [3, (7.5.7)]). The pair $(X, \mathcal{D})$ is known as the *second reduction* of $(\mathcal{M}, \mathcal{L})$. For definition and properties of terminal singularities we also refer to [16].

Since $K_M + L$ is nef and big, the canonical bundle K_S of a smooth $S \in |L|$ is nef and big, so that S is a minimal surface of general type (see also [3, (7.6.10)]). Notice that Miyaoka's inequality holds in the stronger form [3, Remark (5.1.7)]

$$K_S^2 < 9\chi(\mathcal{O}_S). \tag{2}$$

1.3 Pluridegrees. Let $(\mathcal{M}, \mathcal{L})$, (M, L) be as in (1.2). Define the *pluridegrees*, for $0 \leq j \leq 3$, by $\widehat{d_j} := (K_{\mathcal{M}} + \mathcal{L})^j \cdot \mathcal{L}^{3-j}$ and $d_j := (K_M + L)^j \cdot L^{3-j}$. If γ denotes the number of points blown up under $\pi : \mathcal{M} \to M$, then because $K_{\mathcal{M}} + \mathcal{L} \approx \pi^*(K_M + L) + \sum_i E_i$, E_i the exceptional divisors, the invariants $\widehat{d_j}$, d_j are related by $\widehat{d_j} = d_j - (-1)^j\gamma$. We put $\widehat{d} := \widehat{d_0}$, $d := d_0$. If $K_M + L$ is nef, by the generalized Hodge index theorem (see e.g., [3, (2.5.1), (13.1)]) one has

$$d_j^2 \geq d_{j-1}d_{j+1}, \quad j = 1, 2, \tag{3}$$

and the parity relations [3, Lemma (13.1.1)] say that

$$d \equiv d_1 \bmod (2), \quad d_2 \equiv d_3 \bmod (2).$$

Moreover since $K_M + L$ is nef and big, the d_j's are positive. Using the notation from (1.2), defining

$$d'_j := \mathcal{H}^j \cdot \mathcal{D}^{3-j}, \ \ 0 \le j \le 3, \ d' := d'_0,$$

we have [6, §1.4] $d_j = d'_j$ for $j \ge 2$.

Note also that, for $j = 1$, we have $d'_1 \ge d_1$. Indeed from the argument of [1, Chapter 2, Formula (2.1)] we get $d'_1 \ge \widehat{d_1}$. Since $\widehat{d_1} = d_1 + \gamma$ we are done.

We freely use the following three consequences of the log-general type assumption.

First, a consequence of the Tsuji inequality (see [19, (5.2)]), the log version of the usual Yau inequality (see also [3, (13.1.7), (13.1.8)]):

$$(0 <)\ d_3 + \frac{8}{3} d_1 \le 32(2h^0(K_M + L) - \chi(\mathcal{O}_S)). \tag{4}$$

Second, a lower bound for the degree that follows immediately from [6, Lemma (1.5)]. If $\kappa(K_{\mathcal{M}} + \mathcal{L}) = 3$, but $K_{\mathcal{M}} + \mathcal{L}$ is not very ample, then

$$d \ge \widehat{d} := \mathcal{L}^3 \ge 8. \tag{5}$$

Finally, by [2, Theorem (1.2)] we know that

$$h := h^0(K_M + L) \ge 2. \tag{6}$$

From this it follows that for a general $S \in |L|$ and hence for all smooth $S \in |L|$, one has

$$h^2(\mathcal{O}_S) := h^0(K_S) \ge 2. \tag{7}$$

For further properties of log-general type polarized pairs see, e.g., [3, §13.2] and [2, (0.10)].

Let us state some general results.

Lemma 1.4 *Let $(\mathcal{M}, \mathcal{L})$, (M, L), $\pi : \mathcal{M} \to M$ be as in* (1.2). *For a smooth $\widehat{S} \in |\mathcal{L}|$, let $S = \pi(\widehat{S})$ and assume that there exists a morphism $p : S \to B$ over a smooth curve B whose general fiber f is a smooth curve of genus $g(f) = 2$. Then $B \cong \mathbb{P}^1$.*

Proof. Assume that B has positive genus. Then it is a general fact that there exists a morphism $\widetilde{p} : M \to B$ which extends the morphism $p : S \to B$ (see e.g., [3, (5.2.3)]). Let F be a general fiber of $\widetilde{p}$. Recall that L comes from a very ample line bundle $\mathcal{L}$ on $\mathcal{M}$, and that its base locus on M (hence on S) is a finite set. Since f and F are general fibers, this implies that the fiber f of p is a very ample divisor on F of genus $g(f) = 2$. Note that $L_F \cong \mathcal{O}_F(f)$. Then from the classification of polarized surfaces of sectional genus two (see e.g., [3, (8.7.1), (10.2.7)]) we know that (F, L_F) is either a conic fibration over $\mathbb{P}^1$ or a scroll over a smooth curve of genus two. In the former case the restriction $(K_M + L)_F \cong K_F + L_F$ is not big since $|m(K_F + L_F)|$, for $m \gg 0$, defines the conic fibration map; but this contradicts the bigness assumption on $K_M + L$ (see [13, (1.4)]). In the latter case the restriction of $K_F + L_F$ to any fiber of the scroll projection has degree -1, contradicting the nefness assumption on $K_M + L$. Q.E.D.

Lemma 1.5 *Let* $(\mathcal{M}, \mathcal{L})$, (M, L), $\pi : \mathcal{M} \to M$ *be as in* (1.2). *For a smooth* $\widehat{S} \in |\mathcal{L}|$, *let* $S = \pi(\widehat{S})$ *and assume that* $|K_S|$ *contains a smooth hyperelliptic curve* C. *Let* d_j, $0 \le j \le 3$, *be the pluridegrees of* (M, L). *Then* $d_1 \ge 4 + d_2$.

Proof. Let $\widehat{C}$ be the proper transform of C under π. Note that $\widehat{C} (\cong C)$ is a hyperelliptic curve of genus $g(\widehat{C}) = g(C) = d_2 + 1$.

Note also that $h^1(\mathcal{L}_{\widehat{C}}) = h^0(K_{\widehat{C}} - \mathcal{L}_{\widehat{C}}) = 0$. Otherwise, let $\delta \in |K_{\widehat{C}} - \mathcal{L}_{\widehat{C}}|$; then $\delta + |\mathcal{L}_{\widehat{C}}| \subset |K_{\widehat{C}}|$. Hence $|K_{\widehat{C}}|$ would be very ample since $\mathcal{L}_{\widehat{C}}$ is very ample, contradicting the fact that $\widehat{C}$ is hyperelliptic. Thus

$$h^0(\mathcal{L}_{\widehat{C}}) = \deg(\mathcal{L}_{\widehat{C}}) + 1 - g(\widehat{C}) = \deg(\mathcal{L}_{\widehat{C}}) - d_2. \tag{8}$$

Let E_i be the exceptional divisors of π and compute

$$\deg(\mathcal{L}_{\widehat{C}}) = \mathcal{L} \cdot \widehat{C} = (\pi^* L - \sum_i E_i) \cdot \widehat{C} = L \cdot C - \sum_i E_i \cdot \widehat{C} \le L \cdot C = d_1.$$

Thus, since there are no smooth hyperelliptic plane curves, (8) gives $4 \le h^0(\mathcal{L}_{\widehat{C}}) \le d_1 - d_2$. Q.E.D.

Proposition 1.6 ([7, (2.3)]) *Let* $(\mathcal{M}, \mathcal{L})$, (M, L), $\pi : \mathcal{M} \to M$ *be as in* (1.2). *For a smooth* $\widehat{S} \in |\mathcal{L}|$, *let* $S = \pi(\widehat{S})$ *and assume that the bicanonical map* Φ_2 *associated to* $|2K_S|$ *is not birational. If* $h^1(\mathcal{O}_M) = 0$, *then* $\kappa(M) < 0$ *and* $4 \le \chi(\mathcal{O}_S) \le 6$.

1.7 A surface result. Assume that S is a smooth minimal surface of general type such that

(*) There is a morphism $p : S \to B$ over a smooth curve B whose general fiber f is a smooth curve of genus 2.

Note that if $(*)$ happens then certainly the bicanonical map Φ_2 associated to $|2K_S|$ is not birational (see [9, (1.2)]). If $(*)$ holds we say that S presents the *standard case* for the non-birationality of Φ_2. Note also that if $S \in |L|$ where (M, L) is the first reduction of a log-general type threefold $(\mathcal{M}, \mathcal{L})$ and S presents the standard case, then $B \cong \mathbb{P}^1$ by Lemma (1.4).

Proposition 1.8 *Let* $p : S \to \mathbb{P}^1$ *be a morphism whose general fiber* f *is a smooth curve of genus* 2. *Assume that* $p_g(S) \ge 3$ *and that* $|K_S|$ *is not composed with the pencil* $|f|$. *Then the image of* S *under the rational map associated to* $|K_S|$ *is two dimensional.*

Proof. Note that $K_S \cdot f = 2$ for any general fiber f of p. Since $p_g(S) \geq 3$, we have $\dim |K_S| \geq 2$, and hence we see that the different curves $C \in |K_S|$ must cut out a nontrivial linear system $|C|_f$ on f of dimension at least one, i.e., the rational map ψ_S associated to $|K_S|$ is nonconstant on the general fiber f of p. Indeed, otherwise, each curve $C \in |K_S|$ would contain the two points cut out on f by K_S. This is not possible for a general f, unless $|K_S|$ contains a fixed component Γ such that $\Gamma \cdot f = 2$. But in this case we can easily see that $|K_S| = \Gamma + (p_g(S) - 1)|f|$, which contradicts the assumption. On the other hand, from the exact sequence

$$0 \to K_S - f \to K_S \to K_{S|f} \cong K_f \to 0$$

we get the inequality $h^0(K_S - f) \geq h^0(K_S) - h^0(K_f) \geq 3 - 2 = 1$ (in fact, $h^0(K_S - f) = h^0(K_S) - h^0(K_f)$ by the above). Thus we conclude that K_S is the sum of an effective divisor plus f. But f is the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$. From this the result follows. Q.E.D.

For any further background material we refer to [7].

2 The standard case as hyperplane section

The goal of this section is to prove the following general result. It will be used in §3 to show that under the assumptions as in Theorem (0.1) the standard case cannot occur.

Theorem 2.1 *Let $(\mathcal{M}, \mathcal{L})$, (M, L), $\pi : \mathcal{M} \to M$ be as in* (1.2). *Assume that $h^1(\mathcal{O}_{\mathcal{M}}) = 0$. Further assume that there is a smooth $S \in |L|$ with a morphism $p : S \to \mathbb{P}^1$ whose general fiber is a smooth curve of genus* 2. *Let $\mathcal{K} := K_M + L$. If the map Φ associated to $|2\mathcal{K}|$ is birational, then $|\mathcal{K}|$ has one dimensional image and there is no smooth $C \in |K_S|$.*

Let us first show some preliminary facts we need. As the first thing, note that

$$h^1(\mathcal{O}_S) = 0. \tag{9}$$

This follows from the assumption $h^1(\mathcal{O}_{\mathcal{M}}) = h^1(\mathcal{O}_M) = 0$ combined with Lefschetz theorem.

With the assumptions and notation as in (2.1), let f be a smooth fiber of $p : S \to \mathbb{P}^1$. Then we have $\mathcal{K}_f \cong K_{S|f} \cong K_f$, and hence in particular $\mathcal{K} \cdot f = 2$. Since f is of positive genus, and L comes from a very ample line bundle, this implies $K_M \cdot f < 0$ and therefore $\kappa(M) < 0$ (so the conclusion $\kappa(M) < 0$ as in Proposition (1.6) is clear when S presents the standard case.) Thus from the exact sequence

$$0 \to K_M \to \mathcal{K} \to K_S \to 0 \tag{10}$$

we infer that $h \leq h^0(K_S)$. Equivalently, in view of (9),

$$\chi(\mathcal{O}_S) \geq h + 1. \tag{11}$$

Let $Z \subset P \times M$ be the family of smooth deformations of smooth fibers of p. Consider the flat morphism $q_1 : Z \to P$, and the projection $q_2 : Z \to M$.

Lemma 2.2 *The image $q_2(Z)$ is dense in M and given a general point $x \in M$, one has* $\dim q_2^{-1}(x) \geq 1$.

Proof. Note that given a smooth fiber f of p there is an exact sequence

$$0 \to \mathcal{O}_f \to N_{f/M} \to L_f \to 0,$$

where $N_{f/M}$ denotes the normal bundle of f in M. Since $\mathcal{L}$ is very ample, we easily see that $L \cdot f \geq 5$. Thus $\chi(N_{f/M}) = \chi(\mathcal{O}_f) + \chi(L_f) \geq 3$, since $\chi(\mathcal{O}_f) = -1$ and $\chi(L_f) = L \cdot f + 1 - g(f) \geq 4$. This proves that the deformations of f are Zariski dense in M. Moreover since the degree of $N_{f/M}$ is at least 5 for any f in the family Z, we see for any x on any f in the family that $\chi(N_{f/M}(-x)) \geq 1$. This implies $\dim q_2^{-1}(x) \geq 1$. Q.E.D.

Let $\phi : M \to \mathbb{P}^n$ denote the rational map associated to $\mathcal{K} = K_M + L$. Let B denote the base locus of $|\mathcal{K}|$. Recall that $h = h^0(\mathcal{K}) \geq 2$ by (6), and so $\dim \phi(M) \geq 1$. The following lemma proves the first assertion of (2.1).

Lemma 2.3 *We have* $\dim \phi(M) = 1$.

Proof. We claim that for a general f in the family Z, the image of $H^0(\mathcal{K})$ in $H^0(\mathcal{K}_f)$ is one dimensional. To see this assume otherwise. Then $\mathcal{K}_f$ is a degree 2 divisor on a smooth genus 2 curve with $h^0(\mathcal{K}_f) \geq 2$ (note that the image of $H^0(\mathcal{K})$ in $H^0(\mathcal{K}_f)$ cannot be 0-dimensional since $\mathcal{K}$ is effective). Thus we conclude that $\mathcal{K}_f$ must be the canonical bundle of f. But this implies that the map Φ associated to $|2\mathcal{K}|$ is not birational when restricted to the general f. Since the f are dense in M, this implies that Φ itself is not birational, contrary to the assumption in (2.1).

By the above claim, we get that all sections $s \in H^0(\mathcal{K})$ are constant on f and therefore we conclude that $\dim \phi(f) = 0$ for the general f in the family Z. Thus $\dim \phi(M) \leq 2$. As we already observed, $\dim \phi(M) \geq 1$. Thus it suffices to show that $\dim \phi(M)$ is not two. But note that if f is a general fiber and x is a general point of f, then since $x \notin B$, we conclude from Lemma (2.2) that there is a one parameter family of members of the family passing through x. They each go to a point under ϕ and since x is a point where ϕ is defined we conclude they all go to the same point. Thus the general fiber is at least two dimensional. Q.E.D.

If the family of deformations of smooth fibers f of the morphism p contains more than one family, choose one that contains the general f of p, and satisfies the conditions of Lemma (2.2). Since ϕ takes a general element of this family to a point, we conclude that any element of the family not contained in the base locus B must map to a point. Therefore the general fiber of p is mapped to a point. Thus the restriction of ϕ to S factors through p, and we have a commutative diagram

$$\begin{array}{ccc} S & \hookrightarrow & M \\ p \downarrow & & \downarrow \phi \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^n. \end{array}$$

Remark 2.4 Note that the rational map ϕ cannot be a morphism. Indeed, if it was, the nefness and bigness assumption on $\mathcal{K}$ would contradict Lemma (2.3). It thus follows that

$$h = h^0(\mathcal{K}) \leq 6, \tag{12}$$

since in [4] it is shown that ϕ is a morphism if $h \geq 7$. We also conclude from [4, (2.4)] that

$$d = L^3 \geq 9. \tag{13}$$

Let now ℓ denote the transversal intersection of a general fiber F of ϕ with a general $S \in |L|$. As noted in [4, §5], we know that $\ell \cdot \ell = 0$ if and only if ϕ is a morphism. Thus by the above we can assume $\ell \cdot \ell \geq 1$.

Arguing by contradiction, let's now assume that there is a smooth $C \in |K_S|$. Recall that in our present assumptions the map Φ_2 is not birational as noted in (1.7). Moreover, recalling (9), [9, 3.4 (a)] applies to conclude that C is a hyperelliptic curve. Thus Lemma (1.5) yields

$$d_1 \geq d_2 + 4. \tag{14}$$

Since Φ_2 is not birational, we also know from [7, (2.1)] that

$$6h + d_3 = 4\chi(\mathcal{O}_S) + d_2. \tag{15}$$

Claim 2.5 *$h \geq 3$.*

Proof. From (6) we know that $h \geq 2$. Assume $h = 2$. Then (15) gives $d_2 + 4\chi(\mathcal{O}_S) = 12 + d_3$ and hence from (11) we find $d_3 \geq d_2$. By the Hodge index inequalities we have $d_2^2 \geq d_1 d_3$, and therefore $d_3 \geq d_1$. Thus (14) leads to a contradiction. □

From Claim (2.5) we conclude that $\chi(\mathcal{O}_S) \geq h + 1 \geq 4$.

Claim 2.6 *$h^2(\mathcal{O}_M) > 0$.*

Proof. To see this assume that $h^2(\mathcal{O}_M) = 0$, and consider the exact sequence (10). Since $h^1(K_M) = h^2(\mathcal{O}_M) = 0$, the map ϕ restricted to S is the map associated to $|K_S|$. Since by the above $\chi(\mathcal{O}_S) \geq 4$, we get $p_g(S) \geq 3$ by (9) and hence Proposition (1.8) applies to say that the map associated to $|K_S|$ has a two-dimensional image. Thus ϕ has a two-dimensional image, contradicting Lemma (2.3). □

Conclusion of the proof of (2.1). Recall that Φ_2 is not birational and $h^1(\mathcal{O}_M) = 0$. Recalling that $\kappa(M) < 0$, we have $h^3(\mathcal{O}_M) = 0$ and, since $h^2(\mathcal{O}_M) > 0$, sequence (10) gives

$$\chi(\mathcal{O}_S) = \chi(\mathcal{K}) + \chi(\mathcal{O}_M) = h + 1 + h^2(\mathcal{O}_M) \geq h + 2$$

Therefore Tsuji inequality (4) yields

$$2h \geq h + 2 + \frac{d_1}{12} + \frac{d_3}{32}. \tag{16}$$

We know that $\ell \cdot \ell \geq 1$. First assume $\ell \cdot \ell \geq 2$. Thus from relation iv) of [4, (3.1)] we have

$$d_2 \geq (\ell \cdot \ell)(h-1)^2 \geq 2(h-1)^2. \tag{17}$$

Since $\chi(\mathcal{O}_S) \geq h+2$, equality (15) gives $d_3 \geq d_2 - 2h + 8$. Then (17) yields

$$d_3 \geq 2(h-1)^2 - 2h + 8.$$

Combining (14) with (17) gives

$$d_1 \geq d_2 + 4 \geq 2(h-1)^2 + 4.$$

Substituting these inequalities in (16) we get

$$h \geq 2 + \frac{2(h-1)^2 + 4}{12} + \frac{2(h-1)^2 - 2h + 8}{32}.$$

A direct numerical check shows that this is equivalent to $11h^2 - 73h + 135 \leq 0$, which is not possible for $h \geq 2$.

Assume now $\ell \cdot \ell = 1$. Now relation iii) in [4, (3.1)] gives

$$d_2 \geq (h-1)(2h-1). \tag{18}$$

From the inequality $d_3 \geq d_2 - 2h + 8$ noted above we have now

$$d_3 \geq (h-1)(2h-1) - 2h + 8.$$

Moreover, combining (14) with (18) gives

$$d_1 \geq d_2 + 4 \geq (h-1)(2h-1) + 4.$$

Substituting these inequalities in (16) we find

$$h \geq 2 + \frac{(h-1)(2h-1) + 4}{12} + \frac{(h-1)(2h-1) - 2h + 8}{32}.$$

A direct numerical check shows that this is equivalent to $22h^2 - 135h + 259 \leq 0$, which is not possible for $h \geq 2$.

This completes the proof of (2.1).

3 Proof of the result, I

In this section we reduce the possibilities for (M, L), and an S as in Theorem (0.1), with Φ_2 failing to be birational, to three cases only.

Proposition 3.1 *Let* $(\mathcal{M}, \mathcal{L})$, (M, L), $\pi : \mathcal{M} \to M$ *be as in* (1.2). *Suppose that* $|K_M + L|$ *contains a smooth member* A. *For a smooth* $\widehat{S} \in |\mathcal{L}|$ *let* $S := \pi(\widehat{S}) \in |L|$. *Assume that* $(\mathcal{M}, \mathcal{L})$, $\widehat{S}$ *satisfy conditions* (1) *and that the canonical system* $|K_S|$ *contains a smooth member. Further assume that the bicanonical map* $\Phi_2 : S \to \mathbb{P}^N$ *associated to* $|2K_S|$ *is not birational. Then either* (M, L), A *are as in case* 2) *of* (0.1), *or* (M, L), S *and the invariants* $p_g := p_g(S)$, d_3, d_2, d_1, $h := h^0(K_M + L)$ *are as in the following table* (* *means "ancestor case" in the sense of* [9, (3.5)]).

p_g	d_2	d_3	h	d_1
	6	2	4	$10 \le d_1 \le 18$
4	7	3	4	$11 \le d_1 \le 16$
	8^*	4	4	$12 \le d_1 \le 16$

Proof. By [5, (2.1)], either (M, L), A are as in case 2) of (0.1), or the map Φ associated to $|2(K_M + L)|$ is birational. In the latter case by Theorem (2.1) and our present assumption that $|K_S|$ contains a smooth member, we can assume that the standard case does not occur. Then the result follows from Du Val's list in [9, (3.5)], by combining Hodge index inequalities as in (1.3) with Tsuji inequality (4), the assumption $h^1(\mathcal{O}_S) = 0$, the facts that $\kappa(M) < 0$ and $4 \le \chi(\mathcal{O}_S) \le 6$ from Proposition (1.6), and relation (15). We also need to recall that, due to the assumption on $|K_S|$, any smooth member $C \in |K_S|$ is a hyperelliptic curve, since Φ_2 is not birational [9, (3.4)(a)].

First of all, recalling (1), the exact cohomology sequence of (10) shows that

$$h \le p_g = 3, 4, 5.$$

Note also that relation (4) yields $h \ge 3$ (since otherwise the right hand side would be ≤ 0); and, moreover, $h = 3$ implies $\chi(\mathcal{O}_S) \ne 6$, i.e., $p_g \ne 5$. To give an idea of how the argument runs producing further restrictions, consider Du Val's list as in [9, (3.5)] and let us confine to show that the case $p_g = 5$ and the ancestor case $p_g = 3$, $d_2 = 2$ are ruled out.

(a) $p_g = 5$, $7 \le d_2 \le 8$.

First of all note that $2h - \chi(\mathcal{O}_S) \ge 1$ by (4). Hence $h \ge 4$, since $\chi(\mathcal{O}_S) = 1 + p_g = 6$.

Now assume $d_2 = 7$. In this case (15) gives $6h + d_3 = 31$, hence $h \le 5$, as noted before. If $h = 4$, then $d_3 = 7$ and so $7d_1 = d_1 d_3 \le d_2^2$ implies $d_1 \le 7$. But, recalling that a general curve $C \in |K_S|$ is hyperelliptic, this contradicts the inequality $d_1 \ge d_2 + 4$ established in Lemma (1.5). Therefore $(d_3, h) = (1, 5)$ is the only possibility.

As $d_3 = 1$, we know from [6, (6.2)] that $(\mathcal{M}, \mathcal{L})$ has $(X, \mathcal{D}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$ as second reduction. This implies that the surface S is birational to a smooth quintic surface S' in $\mathbb{P}^3$. But then the smooth curves of the canonical system $|K_{S'}|$ are plane quintics, hence they cannot be hyperelliptic, a contradiction.

Let $d_2 = 8$. In a similar way we see that either $d_3 = d_1 = 8$, $h = 4$, or $d_3 = 2$, $h = 5$.

In the former case, recalling that $d \geq 8$ by (5), the inequality $d_1^2 \geq dd_2 (\geq 8d_2)$ turns out to be an equality. In other words, $(K_S \cdot L_S)^2 = L_S^2 \cdot K_S^2$. This implies $K_S \approx \lambda L_S$, $\lambda \in \mathbb{Q}$, and then $d_2 = K_S \cdot K_S = \lambda L_S \cdot K_S = \lambda d_1$ yields $\lambda = 1$. Therefore $K_S \approx L_S$, or, equivalently, $(K_M + L)_S \approx L_S$. From [2, (0.9)] we thus conclude that $K_M \approx \mathcal{O}_M$, but this contradicts the inequality $\kappa(M) < 0$ shown in Proposition (1.6).

In the latter case consider again the second reduction $(X, \mathcal{D})$ of $(\mathcal{M}, \mathcal{L})$. We can assume that $K_X + 2\mathcal{H}$ is nef, where $\mathcal{H} \approx K_X + \mathcal{D}$ as in (1.2). Otherwise, in view of the exceptions listed in [6, §6] either $(X, \mathcal{D}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, a smooth quadric in $\mathbb{P}^4$, which is impossible since S would be birational to a smooth quadric surface, or X is a $\mathbb{P}^2$-bundle over a smooth curve of genus 1, since $d_3 = 2$ (see [6, (6.1)]), but this contradicts our assumption that $h^1(\mathcal{O}_M) = 0$. On the other hand, the nefness of $K_X + 2\mathcal{H}$ implies that

$$(K_X + 2\mathcal{H}) \cdot \mathcal{H} \cdot \mathcal{H} = (3\mathcal{H} - \mathcal{D}) \cdot \mathcal{H} \cdot \mathcal{H} \geq 0.$$

Therefore $3d_3 \geq d_2$, recalling that $d_j = \mathcal{H}^j \cdot \mathcal{D}^{3-j}$, $0 \leq j \leq 3$ (see (1.3)). But this cannot happen, since $d_2 = 8$, $d_3 = 2$.

(b) $p_g = 3$, $d_2 = 2^*$.

In this case $4 = d_2^2 \geq d_1 d_3$ gives $d_1 \leq 4$ and then $16 \geq d_1^2 \geq d_2 d \geq 8d_2 = 16$ shows that $d_1 = 4$. Hence the previous inequality yields $d_3 = 1$, which contradicts (15).

In conclusion, the above arguments show that $p_g \leq 4$ and $d_2 \geq 3$ if $p_g = 3$. This leads to the following provisional list of possible cases:

p_g	d_2	d_3	h	d_1
4	6	2	4	$7 \leq d_1 \leq 18$
	7	3	4	$8 \leq d_1 \leq 16$
	8^*	4	4	$8 \leq d_1 \leq 16$
3	8^*	6	3	9
	7	5	3	8, 9
	6	4	3	8, 9
	5	3	3	7, 8

Bounds for d_1 follow from the Hodge index inequalities $d_1^2 \geq dd_2 (\geq 8d_2)$, $d_2^2 \geq d_1 d_3$. Now, by applying the inequality $d_1 \geq d_2 + 4$ from Lemma (1.5), we succeed to rule out all cases with $p_g = 3$ and get further refinements, leading to the table in the statement. Q.E.D.

We will use the following simple fact in the sequel.

Remark 3.2 Notation and assumptions as in (3.1). Note that in all surviving cases we have $p_g(S) = h$. Looking at the exact sequence

$$0 \to K_M \to K_M + L \to K_S \to 0$$

and recalling that $h^0(K_M) = 0$ by Proposition (1.6), this means that the restriction to S induces an isomorphism $H^0(K_M + L) \cong H^0(K_S)$. Then Kodaira's vanishing theorem yields $h^1(K_M) = 0$. Therefore the same exact sequence as above written for any smooth surface $S' \in |L|$ gives the isomorphism

$$H^0(K_M + L) \cong H^0(K_{S'}). \tag{19}$$

It thus follows that, for any smooth surface $S' \in |L|$ (in particular for $S' = S$),

$$\mathrm{Bs}|K_{S'}| = \mathrm{Bs}|K_M + L| \cap S'. \tag{20}$$

Recalling our assumption that $|K_S|$ contains a smooth element (see Theorem (0.1)), (19) and (20) imply that the linear system $|K_{S'}|$ has no fixed components for any smooth $S' \in |L|$. Moreover

$$\dim \mathrm{Bs}|K_M + L| \leq 1.$$

4 Proof of the result, II

In this section we conclude the proof of Theorem (0.1) showing that the three cases in the table in Proposition (3.1) do not occur.

We deal with the three cases $K_S^2 = 6, 7, 8$ at the same time. According to [10, (2.1), (3.2)] the canonical map $\varphi_{|K_S|} : S \to Y_0 \subset \mathbb{P}^3$ is generically finite of degree 2 onto a surface Y_0, which is a quadric cone for $K_S^2 = 6, 7$ and a smooth quadric for $K_S^2 = 8$. By assumption, the base locus of $|K_S|$ is finite. Let $\#$ be its scheme-theoretic degree. Then, from the equality

$$\deg \varphi_{|K_S|} \deg Y_0 + \# = K_S^2 \tag{21}$$

we infer that $\# = 2, 3, 4$ according to the three values of K_S^2, that is, $\mathrm{Bs}|K_S| = \{p_1, \ldots, p_\#\}$ consists of $\#$, possibly infinitely near, points. Note that $\# = d_3 = (K_M + L)^3$ by Proposition (3.1). For $K_S^2 = 6, 7$, one of the base points of $|K_S|$ corresponds to the vertex of Y_0, which does not lie on the remaining part of the branch locus of the map $\varphi_{|K_S|}$. Moreover, for each value of K_S^2, the remaining base points of $|K_S|$ correspond to triple points of type $[3, 3]$ of the branch locus ([10, (2.1)], [9, (3.2)]). Note that $|K_S|$ cannot have fixed components under our assumptions, hence $\mathrm{Bs}|K_S|$ consists of 4 simple distinct points for $K_S^2 = 8$, according to [10, (3.2)]. In fact, regardless the value of K_S^2, we have the following

Proposition 4.1 Bs$|K_S|$ *does not contain infinitely near points.*

Proof. This is a consequence of a result of Calabri and Ferraro [8]. For reader's convenience let us sketch the argument. We may assume that $K_S^2 = 6, 7$ by what we said before. Suppose that $|K_S|$ has two infinitely near points. We restrict to the case of exactly two infinitely near points. This corresponds to requiring that the branch curve $\mathcal{B}_0 \subset Y_0$ has one singular point as in [8, Example (13.2)] (with $g = 1$, $k = 2$ in their notation). Let $\widetilde{\varphi} : \widetilde{S} \to \mathbb{F}_2$ be the double cover induced by $\varphi_{|K_S|}$. Note that $\widetilde{S}$ is birational to S and consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \tau\ } & \widetilde{S} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\widetilde{\varphi}} \\ Y & \longrightarrow & \mathbb{F}_2, \end{array}$$

where $\alpha : X \to Y$ is the canonical resolution of the double cover $\widetilde{\varphi}$ and τ is a birational morphism. As $\mathbb{F}_2$ is the desingularization of a quadric cone, the ramification formula and the projection formula show that $|K_X| = \pi^*|K_Y + \mathcal{B}/2|$, where $\mathcal{B}$ is the branch locus of α. So, the fixed part of $|K_X|$ is simply the pull-back of the fixed part $\overline{E}$ of $|K_Y + \mathcal{B}/2|$. According to [8, (14.10)], we have $\overline{E} = E_1 + E_2 + E_3$, where the E_i's are (-2)-curves on Y arising in the resolution process of the branch curve and such that $E_1 \cdot E_2 = E_2 \cdot E_3 = 1$, $E_1 \cdot E_3 = 0$. Moreover, E_1, E_3 are in the branch locus, while E_2 does not. Let $C_i = \pi^{-1}(E_i)$ for $i = 1, 2, 3$. Since E_1, E_3 are in $\mathcal{B}$ we have $\pi^*E_i = 2C_i$ for $i = 1, 3$. Therefore $(2C_i)^2 = 2E_i^2 = -4$, so that C_1, C_3 are (-1)-curves on X. On the other hand, $C_2 = \pi^*E_2$, hence $C_2^2 = 2E_2^2 = -4$. Moreover, $C_1 \cdot C_2 = C_2 \cdot C_3 = 1$. Since S is smooth and minimal, τ is the contraction of C_1 and C_3, and then $\tau(C_2)$, which is a (-2)-curve on $\widetilde{S}$, maps to a fixed component of $|K_S|$. But this contradicts our assumptions. Q.E.D.

With the notation as in (1.2), let $\mathfrak{D} := |L - B|$ be the linear system of divisors of $|L|$ which contain the finite set of points blown up under the first reduction map $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$. Recall that there is a bijection between smooth divisors of $\mathfrak{D}$ and smooth divisors of $\mathcal{L}$.

Let $S' \in |L|$ be any smooth element. By Remark (3.2) the base locus of $|K_{S'}|$ is finite; let $\#'$ be its scheme theoretic degree. We need the following key result.

Lemma 4.2 *Notation as above.*

1. *For any smooth member $S' \in |L|$, either $\#' = \#$, or $\#' \leq \# - 2$;*

2. *There exists a smooth element $S' \in \mathfrak{D}$ such that $\#' < \#$ (and hence $\#' \leq \# - 2$ by 1)).*

Proof. 1) For any smooth $S' \in |L|$ we have $p_g(S') = 4$ by (3.2). Consider the canonical map $\varphi' = \varphi_{|K_{S'}|} : S \to Z$, where Z is a surface of degree ≥ 2 in $\mathbb{P}^3$ and

$$\deg \varphi' \deg Z + \#' (= K_{S'}^2) = K_S^2 = 4 + \#. \tag{22}$$

Note that since S' is of general type, it must be $\deg \varphi' \deg Z \geq 4$. Therefore $\#' \leq \#$.

Assume $\#' = \# - 1$ for some smooth $S' \in |L|$. Then for such S' we get from (22) the equality $\deg \varphi' \deg Z = 5$. Thus φ' is a birational map and Z is a quintic surface in $\mathbb{P}^3$. Therefore φ' maps birationally a general curve $C \in |K_{S'}|$ to a plane quintic, a hyperplane section of Z. By the genus formula for C we get the numerical contradiction

$$7 \leq 1 + K_S^2 = 1 + K_{S'}^2 = g(C) \leq 6.$$

2) Assume by contradiction that $\#' = \#$ for each smooth $S' \in \mathfrak{D}$. Then (22) gives $\deg \varphi' \deg Z = 4$, hence $\deg \varphi' = \deg Z = 2$. This implies that the general curve $C \in |K_{S'}|$ is hyperelliptic. Recall that by adjunction $|2K_{S'}|_C = |K_C|$. Therefore the bicanonical map Φ_2' of S' is not birational for each smooth $S' \in \mathfrak{D}$. On the other hand, we know from [5, §2] that the map Φ associated to $|2\mathcal{K}|$ is a morphism and its restriction to S' (in fact to each smooth $S' \in |L|$) gives the bicanonical map Φ_2' (also, Φ maps M to a threefold since $\mathcal{K}$ is nef and big). Thus the restriction of Φ to S' is not birational for each smooth $S' \in \mathfrak{D}$ and therefore Φ is not birational. Let $t := \deg \Phi$.

Claim 4.3 *The morphism Φ induces an involution i on M.*

Proof. Let $\Phi^{-1}(\Phi(x)) = \{x, x_1, \ldots, x_{t-1}\}$ for $x \in M$. For any point $x \in M$, let us denote $\mathfrak{D} - x := |L - B - x|$. On each smooth surface $S' \in \mathfrak{D} - x$ there is a unique point $x_{i(S')}$ (depending on S') which corresponds to x under the involution defined on S' by $\Phi_2' = \Phi_{|S'}$. This defines a continuous map

$$\mathcal{U}_x := \{S' \in \mathfrak{D} - x \, ; S' \text{ smooth}\} \to \{1, \ldots, t-1\}.$$

Note that, since there is a one-to-one correspondence between smooth divisors of $\mathfrak{D}$ and smooth divisors of $\mathcal{L}$, $\mathcal{U}_x$ is a non-empty Zariski open subset of the linear system $\mathfrak{D} - x$. Therefore it is connected. We thus conclude that the map above is the constant map. Therefore for each smooth $S' \in \mathfrak{D} - x$ a point $y = x_{i(S')}$ is uniquely defined, and mapping $x \mapsto y$ defines an involution

$$i : M \to M.$$

Note also that for each $x \in M$ there exists a smooth $S' \in \mathfrak{D} - x$, since L comes from a very ample line bundle $\mathcal{L}$. Therefore i is defined on the whole M. □

The above argument shows that $\mathfrak{D}$ does not separate the conjugate points under the involution i, or, in other words, that the linear system $\mathfrak{D}$ is not birationally very ample. Since (M, L) is the first reduction of $(\mathcal{M}, \mathcal{L})$ with $\mathcal{L}$ very ample, this leads to a contradiction. Q.E.D.

According to Remark (3.2), we have to consider the following possible cases.

(i) $\mathrm{Bs}|K_M + L|$ is of pure dimension 0; or

(ii) $\mathrm{Bs}|K_M + L|$ is of pure dimension 1; or

(iii) $\mathrm{Bs}|K_M + L|$ has dimension 1, but it is not pure dimensional.

Case (i). Let ν be the scheme theoretic degree of $\mathrm{Bs}|K_M + L|$. Note that $\nu \geq \#$ by Remark (3.2). On the other hand, we clearly have $\nu \leq (K_M + L)^3 = d_3$. Then, recalling that $\# = d_3$ by Proposition (3.1), we also have $\# \geq \nu$. We thus conclude that $\# = \nu$, which means that $\mathrm{Bs}|K_M + L| = \mathrm{Bs}|K_S|$. In particular $\mathrm{Bs}|K_M + L| = \{p_1, \ldots, p_\#\}$ consists of $\#$ distinct points by Proposition (4.1). Let

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\ \sigma\ } & M \\ \widetilde{\phi}\downarrow & & \downarrow\phi \\ \mathbb{P}^n & \xrightarrow{\ \mathrm{id}_{\mathbb{P}^n}\ } & \mathbb{P}^n \end{array} \tag{23}$$

be the resolution of the base locus of $|K_M + L|$. Let $D \in |K_M + L|$ and let $\overline{D}$ be the proper transform of D under σ. Note that $\overline{D}$ is spanned and $\widetilde{\phi}$ is the morphism associated to the linear system $|\overline{D}|$.

Lemma 4.4 *We have* $\overline{D}^3 = 0$.

Proof. Since $p_1, \ldots, p_\#$ are distinct points, one has $\sigma^* D = \overline{D} + m_1 E_1 + \cdots + m_\# E_\#$, where $E_i = \sigma^{-1}(p_i)$ are the exceptional divisors and $m_i = \mathrm{mult}_{p_i}(D)$ are the multiplicities of D at p_i, $i = 1, \ldots, \#$. Note that the isomorphism $H^0(D) \cong H^0(K_S)$ pointed out in Remark (3.2) implies $m_1 = \cdots = m_\# = 1$, since, otherwise, $|K_S|$ would not have simple base points at $p_1, \ldots, p_\#$.

Recall that $D^3 = (K_M + L)^3 = d_3 = \#$ and compute

$$\begin{aligned} \# = D^3 = (\sigma^* D)^3 = (\sigma^* D)^2 \cdot (\overline{D} + E_1 + \cdots + E_\#) = (\sigma^* D)^2 \cdot \overline{D} = \\ = \sigma^* D \cdot (\overline{D} + E_1 + \cdots + E_\#) \cdot \overline{D} = (\sigma^* D) \cdot (\overline{D})^2 = \\ = (\overline{D} + E_1 + \cdots + E_\#) \cdot \overline{D}^2 = \overline{D}^3 + E_1 \cdot \overline{D}^2 + \cdots + E_\# \cdot \overline{D}^2 \geq \overline{D}^3 + \#. \end{aligned}$$

Thus $\overline{D}^3 = 0$ since $\overline{D}$ is nef. Q.E.D.

Now, come back to diagram (23). By the previous lemma we know that $\dim(\mathrm{Im}\widetilde{\phi}) \leq 2$. Since $h^1(\mathcal{O}_S) = 0$, one has $|K_S| = |K_M + L|_{|S}$, i.e., $\phi_{|S} = \varphi_{|K_S|}$ (this is also a consequence of (3.2)). Then we conclude that $\phi(M) = \widetilde{\phi}(\widetilde{M}) = \varphi_{|K_S|}(S) = \mathcal{Q}$, where $\mathcal{Q}$ is a quadric in $\mathbb{P}^3$. It thus follows that the exceptional divisors $E_i \cong \mathbb{P}^2$, $i = 1, \ldots, \#$, dominate $\mathcal{Q}$ under $\widetilde{\phi}$. This is not possible for degree reasons. Indeed, σ is the morphism associated to the line bundle $\sigma^*(K_M + L) - \sum_{i=1}^{\#} E_i$. Its restriction to E_i is $\mathcal{O}_{E_i}(-E_i) \cong \mathcal{O}_{\mathbb{P}^2}(1)$, $i = 1, \ldots, \#$, a line bundle on $\mathbb{P}^2$ of self-intersection one.

Case (ii). This case does not occur. Indeed, if $\mathrm{Bs}|K_M + L|$ is of pure dimension 1, then $\#' = \#$ for every smooth $S' \in |L|$. This contradicts Lemma (4.2), 2).

Case (iii). Let $\mathrm{Bs}|K_M + L| = \Sigma_1 \cup \Sigma_0$, where Σ_i is the i-dimensional part, $i = 0, 1$. By the discussion above, we can assume $\deg \Sigma_1 := \Sigma_1 \cdot L \geq 1$ and $\Sigma_0 \neq \emptyset$. Note that equality (20) in Remark (3.2) reads

$$\mathrm{Bs}|K_{S'}| = (\Sigma_1 \cap S') \cup (\Sigma_0 \cap S') \tag{24}$$

for each smooth $S' \in |L|$ (hence for S as well). In particular, recalling Proposition (4.1), we know that $\Sigma_0 \cap S = \{p_1, \ldots, p_{\#-\deg \Sigma_1}\}$ consists of $\# - \deg \Sigma_1$ distinct points.

Claim 4.5 *For a general member $S' \in \mathfrak{D} = |L - B|$, the equality $\Sigma_0 \cap S' = \Sigma_0 \cap B$ holds true.*

Proof. Clearly $\Sigma_0 \cap B \subseteq \Sigma_0 \cap S'$. Consider a point $p \in \Sigma_0$, $p \notin B$. Then $\pi^{-1}(p) = w$ is a point on $\mathcal{M}$ and since $\mathcal{L}$ is very ample the general element $T \in |\mathcal{L}|$ is a smooth surface not containing w. Let $S' := \pi(T)$. Therefore S' is a smooth element of $\mathfrak{D}$ missing the point p. This shows that for the general $S' \in \mathfrak{D}$ one has $\Sigma_0 \cap S' \subseteq \Sigma_0 \cap B$. □

Lemma 4.6 $\deg \Sigma_1 \leq \# - 3$.

Proof. Note first that $\deg \Sigma_1 \leq \#$ since $\Sigma_1 \cdot L = \Sigma_1 \cdot S$ and $\Sigma_1 \cap S \subseteq \mathrm{Bs}|K_S|$.

Assume that $\deg \Sigma_1 = \#$. Then there exists a point $p \in \Sigma_0$, $p \notin S$. Let $w \in \mathcal{M}$ be a point such that $\pi(w) = p$ under the first reduction map $\pi : \mathcal{M} \to M$. Since $\mathcal{L}$ is very ample, there exists a smooth element $T \in |\mathcal{L}|$ passing through w. Then $S' := \pi(T)$ is a smooth element of $\mathfrak{D}$ which contains p. Therefore $\#' \geq \# + 1$, which contradicts Lemma (4.2), 1).

Assume now that $\deg \Sigma_1 = \# - 1$. Notice that for each smooth $S' \in |L|$, it must be $\Sigma_0 \cap S' \neq \emptyset$; otherwise, we would have $\mathrm{Bs}|K_{S'}| = \Sigma_1 \cap S'$, and hence $\#' = \deg \Sigma_1 = \# - 1$, contradicting Lemma (4.2), 1). On the other hand, for each smooth $S' \in |L|$, $\Sigma_0 \cap S'$ cannot consists of (at least) two distinct points, since otherwise we would have $\#' \geq \# - 1 + 2 = \# + 1$. This contradicts again Lemma (4.2), 1). Thus, for each smooth $S' \in |L|$, $\Sigma_0 \cap S' = \{p'\}$ consists of a single point p'. Therefore equality (24) implies $\#' = \#$, which contradicts Lemma (4.2), 2).

Next, assume $\deg \Sigma_1 = \# - 2$. Let $\nu_0(S')$, $\nu_0(B)$ be the scheme theoretic degree of $\Sigma_0 \cap S'$, $\Sigma_0 \cap B$ respectively. In this case (24) gives , for each smooth $S' \in \mathfrak{D}$,

$$\# \geq \#' = \# - 2 + \nu_0(S') \geq \# - 2 + \nu_0(B), \tag{25}$$

where the last inequality is clear since $B = \mathrm{Bs}\mathfrak{D} \subset S'$. Therefore $\nu_0(B) \leq 2$.

Let $\nu_0(B) = 2$. Then $\nu_0(B) = \nu_0(S')$ for each smooth $S' \in \mathfrak{D}$, so that in particular $\nu_0(B) = \nu_0(S)$. Since $\Sigma_0 \cap B \subseteq \Sigma_0 \cap S$, it follows that $\Sigma_0 \cap B = \Sigma_0 \cap S = \{p_1, p_2\}$ consists of two distinct points. Thus (25) gives $\#' = \#$ for each smooth $S \in \mathfrak{D}$, which contradicts again Lemma (4.2), 2).

Let $\nu_0(B) = 1$. Then $\Sigma_0 \cap B = \{p_1\}$ consists of a single point, say p_1. Hence, recalling Claim (4.5), relation (25) yields $\#' = \# - 1$ for a general $S' \in \mathfrak{D}$. This contradicts Lemma (4.2), 1).

Let $\nu_0(B) = 0$, that is $\Sigma_0 \cap B = \emptyset$. Therefore $\pi^{-1}(t_j) = w_j$ is a single point on $\mathcal{M}$, for each point $t_j \in \Sigma_0$. Since $\mathcal{L}$ is very ample, there exists a smooth element $T \in |\mathcal{L}|$ such that $w_1 \in T$, $w_j \notin T$ for $j \neq 1$. Let $S' := \pi(T)$. Then S' contains t_1 and misses each point t_j, $j \neq 1$. For such an S', the equality $\#' = \# - 2 + \nu_0(S')$ from (25) gives $\#' = \# - 1$. Once again, this contradicts Lemma (4.2), 1). Q.E.D.

Let us consider the case when $\deg \Sigma_1 = 1$. Let $\sigma_1 : M_1 \to M$ be the blowing up of M along Σ_1. Let $E_1 = \sigma_1^{-1}(\Sigma_1)$ be the exceptional divisor. Let $\overline{D}$ be the proper transform of $D \in |K_M + L|$ under σ_1.

Lemma 4.7 *If* $\deg \Sigma_1 = 1$, *then* Σ_1 *is a smooth* $\mathbb{P}^1$ *and* $\overline{D}$ *is nef.*

Proof. For semplicity, write $\ell = \Sigma_1$, $E = E_1$, $\sigma = \sigma_1$. Note that $\ell \cong \mathbb{P}^1$ since $\ell \cdot L = \deg \Sigma_1 = 1$ and L comes from a very ample line bundle.

Note also that E is a rational $\mathbb{P}^1$-bundle. Let $\rho = \sigma_{|E} : E \to \ell$ be the projection. We have $\sigma^* D = \overline{D} + mE$ for some positive integer m. However, in view of the isomorphism $H^0(D) \cong H^0(K_S)$ from (3.2), we conclude as at the beginning of the proof of Lemma (4.4) that $m = 1$.

Let C be any irreducible curve in M_1. If C is not contained in E it is easy to see that $\overline{D} \cdot C \geq 0$.

Let C_0 and f be a minimal section of E ($C_0^2 = -e$, $e \geq 0$) and a fiber, respectively. Recall that C_0 is the only irreducible curve of E with negative self-intersection for $e > 0$. Note also that the restricted divisor $\overline{D}_{|E}$ is a section of ρ.

We claim that $\overline{D}_{|E} \neq C_0$ if $e > 0$. Let $\overline{S}$ be the proper transform of S under σ and denote $f_0 = \overline{S} \cap E$. Then f is the fiber of ρ parameterizing the tangent directions to S at $x := \ell \cap S$. Since x is a simple base point of $|K_S|$ we can find two curves $C_1, C_2 \in |K_S| = |K_S - x|$, smooth at x, with distinct tangent directions at x. Due to the isomorphism $H^0(D) \cong H^0(K_S)$ mentioned above, there exist $D_1, D_2 \in |K_M + L| = |K_M + L - \ell|$ such that $\overline{D}_{1|E}$, $\overline{D}_{2|E}$ cut the fiber f_0 in two distinct points. It thus follows that $\overline{D}_{|E}$ moves, hence $\overline{D}_{|E} \neq C_0$ if $e > 0$.

Since all curves C contained in E are linearly equivalent to positive linear combinations of C_0 and f, to show that $\overline{D}$ is nef it is enough to show that $\overline{D} \cdot f \geq 0$ and $\overline{D} \cdot C_0 \geq 0$. Of course $\overline{D} \cdot f = \overline{D}_{|E} \cdot f = 1$. On the other hand $\overline{D} \cdot C_0 = \overline{D}_{|E} \cdot C_0 \geq 0$ due to the claim. Q.E.D.

Lemma 4.8 *If* $\deg \Sigma_1 = 1$, *then* $\Sigma_0 = \Sigma_0 \cap S = \{p_1, \ldots, p_{\#-1}\}$ *consists of* $\# - 1$ *distinct points.*

Proof. Notation as in the proof of (4.7). Let ν_0 be the scheme theoretic degree of Σ_0. We have $\sigma^* D = \overline{D} + E$ as shown in the proof of (4.7). Note that $\sigma^* D_{|E} (= \rho^*(D_\ell))$

and $\overline{D}_{|E}$ are union of fibers and a section of the $\mathbb{P}^1$-bundle $\rho : E \to \ell$ respectively. Then in particular $\sigma^* D^2 \cdot E = 0$. Recall that $\# = d_3$. Thus compute

$$\# = D^3 = (\sigma^* D)^3 = (\sigma^* D)^2 \cdot (\overline{D} + E) = \sigma^* D^2 \cdot \overline{D} = \sigma^* D \cdot \overline{D} \cdot (\overline{D} + E) = \\ = \sigma^* D \cdot \overline{D}^2 + \sigma^* D_{|E} \cdot \overline{D}_{|E} \geq \sigma^* D \cdot \overline{D}^2 + 1. \quad (26)$$

Now,

$$\sigma^* D \cdot \overline{D}^2 = (\overline{D} + E) \cdot \overline{D}^2 = \overline{D}^3 + E \cdot \overline{D}^2 \geq \nu_0$$

since $\overline{D}^3 \geq \nu_0$ and $\overline{D}$ is nef. Thus (26) yields $\# \geq \nu_0 + 1$. On the other hand, (24) yields

$$\mathrm{Bs}|K_S| = (\ell \cap S) \cup (\Sigma_0 \cap S) \subseteq (\ell \cap S) \cup \Sigma_0,$$

whence $\# \leq 1 + \nu_0$. Thus we conclude that $\# = 1 + \nu_0$. Since $\deg \Sigma_1 = 1$, we have, as noted above, $\Sigma_0 \cap S = \{p_1, \ldots, p_{\#-1}\}$, where $p_1, \ldots, p_{\#-1}$ are distinct points. Therefore the numerical equality $\nu_0 = \# - 1$ gives $\Sigma_0 = \Sigma_0 \cap S$. Q.E.D.

Conclusion of the proof of (0.1). We proceed by cases according to the possible values of $d_2 = K_S^2$.

If $d_2 = 6, 7$ then $\# = 2, 3$ respectively. Therefore Lemma (4.6) leads to a contradiction since $\deg \Sigma_1 \geq 1$.

Thus it remains to consider the case $d_2 = 8$, $\# = 4$, and $\deg \Sigma_1 = 1$. In view of Lemmas (4.7), (4.8) we know that $\Sigma_1 := \ell$ is a linear $\mathbb{P}^1$ and $\mathrm{Bs}|K_M + L| = \{\ell, p_1, p_2, p_3\}$, with p_1, p_2, p_3 distinct points not belonging to ℓ.

Let $\sigma_1 : M_1 \to M$ be the blowing up of M along ℓ, and let $\sigma_0' : \widetilde{M} \to M_1$ be the blowing up of M_1 at p_1', p_2', p_3', with $\sigma_1(p_i') = p_i$, $i = 1, 2, 3$. Let $\sigma = \sigma_0' \circ \sigma_1$ and let $E_1 = \sigma_1^{-1}(\ell)$ be the exceptional divisor of σ_1. Then E_1 is a $\mathbb{P}^1$-bundle on ℓ. By Proposition (4.1) we know that $p_i' \notin E_1$, $i = 1, 2, 3$.

Let $\sigma_0 : M_0 \to M$ be the blowing up of M at p_1, p_2, p_3 and let $\overline{D}_0$ be the proper transform of D under σ_0. The same argument as in the proof of Lemma (4.4) gives $\overline{D}_0^3 = 1$. Note that σ factors through σ_0 and the blowing up of M_0 along $\sigma_0^{-1}(\ell)(\cong \ell)$. Therefore, up to replacing the new pair $(M_0, \overline{D}_0)$ with the original pair (M, D), we can assume that σ is simply the blowing up of M along ℓ. Let $\overline{D}$ be the proper transform of D under σ. By Lemma (4.7) we know that $\overline{D}$ is nef.

Claim 4.9 *We have $\overline{D}^3 = 0$.*

The same argument as at the beginning of the proof of Lemma (4.7) shows that $\sigma^* D = \overline{D} + E$ and, by the above, $D^3 = 1$. Following the proof of Lemma (4.8), compute

$$1 = D^3 = (\sigma^* D)^3 = (\sigma^* D)^2 \cdot (\overline{D} + E) = \sigma^* D^2 \cdot \overline{D} = \sigma^* D \cdot \overline{D} \cdot (\overline{D} + E) = \\ = \sigma^* D \cdot \overline{D}^2 + \sigma^* D_{|E} \cdot \overline{D}_{|E} \geq \sigma^* D \cdot \overline{D}^2 + 1. \quad (27)$$

Now,
$$\sigma^* D \cdot \overline{D}^2 = (\overline{D} + E) \cdot \overline{D}^2 = \overline{D}^3 + E \cdot \overline{D}^2 \geq 0$$
since $\overline{D}$ is nef. Thus (27) yields $1 \geq 1 + \overline{D}^3$, whence $\overline{D}^3 = 0$. $\square$

Finally, Claim (4.9) leads to the same contradiction as at the end of the discussion of Case (i). Combining this with the previous discussion, we rule out Case (iii). This completes the proof of Theorem (0.1).

5 An example

According to Theorem (2.1), if $|2\mathcal{K}|$ gives a birational map and there is a smooth curve $C \in |K_S|$, then S cannot present the standard case. In this section we produce an example where $|2\mathcal{K}|$ gives a morphism Φ of degree 2 and there is a hyperplane $\mathcal{S}$ of $|L|$ such that for the general element $S \in \mathcal{S}$:

(a) S presents the standard case, and

(b) the general element $C \in |K_S|$ is a smooth curve.

As in [17, §1] (see also [5, Example (2.2)]), consider a Del Pezzo threefold (M, H) of degree $H^3 = 1$, and let $L = 3H$. The following facts are known:

(i) H is an ample line bundle with $h^0(H) = 3$, whose complete linear system has a single base point $x := \mathrm{Bs}|H|$;

(ii) $2H$ is spanned and defines a morphism $\psi : M \to \Gamma \subset \mathbb{P}^6$ of degree 2 onto a cone over the Veronese surface $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ with vertex $v := \psi(x)$, branched at v and along a smooth surface not containing v [12, §14] (see also [14, (4.2), (6.10)]); and

(iii) L is very ample [17, Theorem (1.2)].

Recall that $K_M = -2H$. Hence $\mathcal{K} = K_M + L = H$. Then the morphism Φ associated to $|2\mathcal{K}|$ coincides with ψ. Fix a general element $A \in |\mathcal{K}|$. Note that, due to (i), Bertini's theorem and the fact that $A^3 = 1$, A is a smooth surface passing through x. Moreover, $K_A^2 = H^3 = 1$, hence A is a Del Pezzo surface of degree 1 and so $\psi_{|A}$, which turns out to be the morphism associated with $-2K_A = 2H_A$, exhibits A as a double cover $\psi_{|A} : A \to \gamma \subset \mathbb{P}^3$ of a quadric cone γ of vertex v. Now consider any smooth surface $S \in |L|$. From the exact sequence
$$0 \to -H \to 2H = 2(K_M + L) \to 2K_S \to 0,$$
we see that $\psi_{|S}$ is the bicanonical morphism Φ_2 of S. Let $\mathcal{J}_\gamma \subset \mathcal{O}_{\mathbb{P}^6}$ be the ideal sheaf of $\gamma \subset \mathbb{P}^3 \subset \mathbb{P}^6$. Due to the equality $2H = \psi^*\mathcal{O}_\Gamma(1)$, we have that
$$|L| = |4H - A| = |\psi^*(\mathcal{O}_\Gamma(2) \otimes \mathcal{J}_\gamma)| \supseteq \psi^*|\mathcal{O}_\Gamma(2) \otimes \mathcal{J}_\gamma|.$$
Consider the linear subsystem
$$\mathcal{S} := \psi^*|\mathcal{O}_\Gamma(2) \otimes \mathcal{J}_\gamma|.$$

Lemma 5.1 *The linear system $\mathcal{S}$ is a hyperplane of $|L| = \mathbb{P}^{13}$. Moreover, its general element S is a smooth surface doubly covering via ψ a nodal Del Pezzo sextic surface $\Sigma \subset \mathbb{P}^6$, i.e., if $\nu : \widetilde{\Sigma} \to \Sigma$ is the minimal desingularization, then $\widetilde{\Sigma}$ is the blow-up of $\mathbb{P}^2$ at three collinear points. Furthermore $2K_S = \psi^*\mathcal{O}_\Sigma(1)$.*

Proof. First of all consider the exact sequence

$$0 \to 2H \to L \to L_A = -3K_A \to 0.$$

Noting that $h^1(2H) = h^1(K_M + 4H) = 0$ due to the Kodaira vanishing theorem, we immediately get $h^0(L) = h^0(2H) + h^0(-3K_A) = 14$. So $|L| = \mathbb{P}^{13}$. By an appropriate choice of coordinates in $\mathbb{P}^6$ we can assume that $v = (1 : 0 : \cdots : 0)$, the cone Γ is defined by

$$\operatorname{rk} \begin{pmatrix} x_1 & x_2 & x_4 \\ x_2 & x_3 & x_5 \\ x_4 & x_5 & x_6 \end{pmatrix} < 2, \tag{28}$$

and the equations of the quadric cone $\gamma \subset \mathbb{P}^6$ are

$$f(x_1, x_2, x_3) = x_4 = x_5 = x_6 = 0,$$

where f is a suitable homogeneous polynomial of degree 2. We have

$$h^0(\mathcal{O}_\Gamma(2) \otimes \mathcal{J}_\gamma) = h^0(\mathcal{O}_\Gamma(2)) - t, \tag{29}$$

where t is the number of linearly independent linear conditions that a quadric has to satisfy in order to contain γ. Due to our choice of coordinates, any quadric hypersurface $Q \subset \mathbb{P}^6$ containing γ has an equation of the form $\lambda f + x_4 L_1 + x_5 L_2 + x_6 L_3 = 0$, where λ is a constant and L_i are linear forms in $x_0, \ldots, x_6$ for $i = 1, 2, 3$. Taking into account the obvious repetitions of the terms $x_i x_j$, for $i, j \in \{4, 5, 6\}$, $i \neq j$, this gives $h^0(\mathcal{O}_{\mathbb{P}^6}(2) \otimes \mathcal{J}_\gamma) = 19$. Since $h^0(\mathcal{O}_{\mathbb{P}^6}(2)) = 28$, we get $t = 28 - 19 = 9$. Now consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^6}(2) \otimes \mathcal{J}_\Gamma \to \mathcal{O}_{\mathbb{P}^6}(2) \to \mathcal{O}_\Gamma(2) \to 0.$$

Note that $H^0(\mathcal{O}_{\mathbb{P}^6}(2) \otimes \mathcal{J}_\Gamma)$ is generated by the 2×2 minors of the symmetric matrix in (28); hence $h^0(\mathcal{O}_{\mathbb{P}^6}(2) \otimes \mathcal{J}_\Gamma) = 6$. Therefore

$$h^0(\mathcal{O}_\Gamma(2)) \geq h^0(\mathcal{O}_{\mathbb{P}^6}(2)) - 6 = 22.$$

On the other hand, let h be a general hyperplane section of Γ. Then $(h, \mathcal{O}_h(1)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. So, from the exact sequence

$$0 \to \mathcal{O}_\Gamma(1) \to \mathcal{O}_\Gamma(2) \to \mathcal{O}_h(2) \to 0$$

we get $h^0(\mathcal{O}_\Gamma(2)) \leq 7 + h^0(\mathcal{O}_{\mathbb{P}^2}(4)) = 22$. In conclusion, $h^0(\mathcal{O}_\Gamma(2)) = 22$ and then (29) gives $h^0(\mathcal{O}_\Gamma(2) \otimes \mathcal{J}_\gamma) = 13$. This proves the first assertion.

Since L is very ample its discriminant locus cannot be a hyperplane, hence the general element $S \in \mathcal{S}$ is a smooth surface. Moreover ψ maps S two–to–one onto a surface $\Sigma \subset \mathbb{P}^6$, linked to γ in the complete intersection $Q \cap \Gamma$, where Q is a general element of $|\mathcal{O}_{\mathbb{P}^6}(2) \otimes \mathcal{J}_\gamma|$. Thus Σ has degree 6. Moreover, since any such a Q is singular at v we see that Σ has a double point at v. Thus Σ is a nodal Del Pezzo surface. This however does not affect the smoothness of S, because v is in the branch locus of ψ. Finally, note that $2K_S = \psi^*\mathcal{O}_\Sigma(1)$, by construction. Q.E.D.

Note that every $S \in \mathcal{S}$ contains the base point x of $|H|$. So Lemma (5.1) identifies $\mathcal{S}$ with the hyperplane $|L - x| \subset |L|$. Let us look at the double cover $\psi_{|S} = \Phi_2 : S \to \Sigma$ more closely. For semplicity of notation, write in the sequel ψ for $\psi_{|S}$.

Let $\nu : \widetilde{\Sigma} \to \Sigma$ be the minimal desingularization, and let $\sigma : \widetilde{\Sigma} \to \mathbb{P}^2$ be the birational morphism expressing $\widetilde{\Sigma}$ as the plane blown-up at three points p_1, p_2, p_3 lying on a line ℓ_0. Note that $\sigma^*\mathcal{O}_{\mathbb{P}^2}(-3) + \sum_{i=1}^3 \sigma^{-1}(p_i) = K_{\widetilde{\Sigma}} = \nu^*\mathcal{O}_\Sigma(-1)$. Let $C = \sigma^{-1}(\ell_0)$. We have $C^2 = (\sigma^*\ell_0 - \sum_{i=1}^3 \sigma^{-1}(p_i))^2$. Then C is a (-2)-curve, and $\nu(C) = v$. The morphism $\psi : S \to \Sigma$ induces a double cover $\widetilde{\psi} : \widehat{S} \to \widetilde{\Sigma}$ via the blowing-up $\eta : \widehat{S} \to S$ of the point $x = \psi^{-1}(v)$. The situation is summarized in the following commutative diagram

$$\begin{array}{ccc} \widehat{S} & \xrightarrow{\ \eta\ } & S \\ {\scriptstyle\widetilde{\psi}}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{\Sigma} & \xrightarrow{\ \nu\ } & \Sigma \\ {\scriptstyle\sigma}\downarrow & & \\ \mathbb{P}^2. & & \end{array}$$

Let $E = \eta^{-1}(x)$ be the exceptional curve corresponding to x; then $K_{\widehat{S}} = \eta^*K_S + E$. Since v is in the branch locus of ψ, the branch divisor of $\widetilde{\psi}$ contains C, so it has the form $C + D$, where $D \subset \widetilde{\Sigma}$ is a smooth curve such that $C \cdot D = 0$. Moreover, $C + D \in |2\mathcal{B}|$, for some line bundle $\mathcal{B} \in \mathrm{Pic}(\widetilde{\Sigma})$. Since $\widetilde{\psi}^*C = 2E$, the ramification formula gives

$$2\eta^*K_S = 2K_{\widehat{S}} - 2E = \widetilde{\psi}^*(2K_{\widetilde{\Sigma}} + 2\mathcal{B}) - 2E = \widetilde{\psi}^*(2K_{\widetilde{\Sigma}} + D).$$

On the other hand, by Lemma (5.1) and the commutativity of the diagram above we know that

$$2\eta^*K_S = \eta^*\psi^*\mathcal{O}_\Sigma(1) = \widetilde{\psi}^*\nu^*\mathcal{O}_\Sigma(1) = \widetilde{\psi}^*(-K_{\widetilde{\Sigma}}).$$

By comparison we thus get that $D \in |3(-K_{\widetilde{\Sigma}})|$, i.e., $D \in |\sigma^* \mathcal{O}_{\mathbb{P}^2}(9) - 3\sum_{i=1}^3 \sigma^{-1}(p_i)|$ In other words, the curve $\Delta := \sigma(D)$ is a plane curve of degree 9 having triple points at p_1, p_2, p_3. Now, let $\mathcal{P}$ be the pencil of lines of $\mathbb{P}^2$ passing through p_1, let $\ell \in \mathcal{P}$ be a general line (transverse to Δ outside p_1), and set $\widetilde{f} = \widetilde{\psi}^*(\sigma^{-1}(\ell))$. Note that $\widetilde{f}$ is a double cover of ℓ branched at the six points where Δ meets ℓ outside p_1. So $\widetilde{f}$ is a smooth curve of genus two. On the other hand

$$\widetilde{f}^2 = (\widetilde{\psi}^*(\sigma^{-1}(\ell)))^2 = 2(\sigma^{-1}(\ell))^2 = 0.$$

Therefore the pencil $\mathcal{P}$ defines a fibration $\widetilde{\pi} : \widehat{S} \to \mathbb{P}^1$ in curves of genus two. Note that

$$\widetilde{f} \cdot E = \frac{1}{2}\widetilde{\psi}^*(\sigma^{-1}(\ell)) \cdot \widetilde{\psi}^* C = \sigma^{-1}(\ell) \cdot C = 0.$$

So, letting $f = \eta(\widetilde{f})$, we have that $\widetilde{f} = \eta^* f$, hence f itself is a smooth curve of genus two on S; moreover $f^2 = 0$. Thus $\widetilde{\pi}$ induces a fibration $\pi : S \to \mathbb{P}^1$ in curves of genus two. This shows that S presents the standard case.

Finally we show that the general element $C \in |K_S|$ is a smooth curve. We have $K_S = (K_M + L)_S = H_S$, by adjunction. In particular, $K_S^2 = H_S^2 = H^2 \cdot L = 3H^3 = 3$. Moreover we have the following exact sequence

$$0 \to -2H \to H \to H_S = K_S \to 0,$$

which gives an isomorphism $H^0(H) \cong H^0(K_S)$. Recalling that S contains x we thus get $\mathrm{Bs}|K_S| = \mathrm{Bs}|H| \cap S = \{x\}$. In conclusion $|K_S|$ is an ample linear system with a single base point at x and $K_S^2 = 3$. This implies that its general element is a smooth irreducible curve.

5.2 It is worth noting that under the further assumption that K_S is ample, the example discussed above is the only possible case. This follows from the following partial improvement of [5, (2.1)].

Proposition 5.3 *Let $(\mathcal{M}, \mathcal{L})$, (M, L), $\pi : \mathcal{M} \to M$ be as in* (1.2)*. Suppose that the linear system $|\mathcal{K}| := |K_M + L|$ contains a smooth member A. For a smooth $\widehat{S} \in |\mathcal{L}|$ let $S := \pi(\widehat{S}) \in |L|$. Further assume that K_S is ample. Then either:*

1. *The morphism Φ associated to $|2\mathcal{K}|$ is birational; or*
2. *$(M, \mathcal{K})$ is a Del Pezzo threefold of degree* 1 *and $L = 3\mathcal{K}$.*

Proof. From [5, (2.1)] we know that either Φ is birational or $d_3 = 1$, $d_2 = 3$, $d_1 = 9$ and A is a Del Pezzo surface with $K_A^2 = 1$. Then it is enough to show that the latter case gives 2) above. Note that $9 = d_2^2 = d_1 d_3$. Since K_S is ample, we can apply [17, (0.1)] to conclude that $\mathcal{K}$ is ample and $3\mathcal{K} \sim L$. Now, by definition, $\mathcal{K} = K_M + L \sim K_M + 3\mathcal{K}$. Hence $-K_M$ is numerically equivalent to $2\mathcal{K}$. This shows that M is Fano and therefore $-K_M = 2\mathcal{K}$. Thus $(M, \mathcal{K})$ is a Del Pezzo threefold of degree $\mathcal{K}^3 = d_3 = 1$. Q.E.D.

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M.C. Beltrametti
Dipartimento di Matematica
Via Dodecaneso 35
I-16146 Genova, Italy
beltrame@dima.unige.it

C. Ciliberto
Dipartimento di Matematica
Seconda Università di Roma “Tor Vergata”
I-00133 Roma, Italy
cilibert@mat.uniroma2.it

A. Lanteri
Dipartimento di Matematica “F. Enriques”
Università di Milano
Via C. Saldini 50
I-20133 Milano, Italy
lanteri@mat.unimi.it

A.J. Sommese
Department of Mathematics
Notre Dame, Indiana, 46556, U.S.A.
sommese@nd.edu
http://www.nd.edu/∼sommese