Reducible hyperplane sections, II

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Abstract

Let $\hat{X}$ be a smooth connected subvariety of complex projective space $\mathbb{P}^n$. The question was raised in [2] of how to characterize $\hat{X}$ if it admits a reducible hyperplane section $\hat{L}$. In the case in which $\hat{L}$ is the union of $r \geq 2$ smooth normal crossing divisors, each of sectional genus zero, classification theorems were given for $\dim \hat{X} \geq 5$ or $\dim X = 4$ and $r = 2$.

This paper restricts attention to the case of two divisors on a threefold, whose sum is ample, and which meet transversely in a smooth curve of genus at least 2. A finiteness theorem and some general results are proven when the two divisors are in a restricted class including $\mathbb{P}^1$-bundles over curves of genus less than two and surfaces with nef and big anticanonical bundle. Next, we give results on the case of a projective threefold $\hat{X}$ with hyperplane section $\hat{L}$ that is the union of two transverse divisors, each of which is either $\mathbb{P}^2$, a Hirzebruch surface $\mathbb{F}_r$, or $\mathbb{F}_2$.

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Introduction

This paper is a sequel of [2], which initiated the study of a connected submanifold $\hat{X}$ of complex projective space, which has a reducible hyperplane section $\hat{L}$. As $\dim \hat{X}$ increases so does the simplicity of the characterization. In [2] a description is given of $(\hat{X}, \hat{L})$ for which $\hat{L}$ decomposes as $\hat{A}_1 + \cdots + \hat{A}_r$ into $r \geq 2$ smooth components with normal crossings under the hypothesis that $h^1(\mathcal{O}_{\hat{A}_i})$ is equal to the sectional genus of $\hat{A}_i$ for each $i$. A complete result for the cases $n = 4$ and $r = 2$; and for $n \geq 5$ was obtained. Further, in the case of $n = 3$ and $r = 2$ the situation in which the curve $A_1 \cap A_2$ has genus at most 1 was thoroughly analyzed. Here we investigate the more delicate issues presented by the following specialization of the question.

Problem. Let $\hat{L}$ be a very ample line bundle on a projective threefold $\hat{X}$. Suppose that $\hat{L}$ decomposes as a divisor into a sum $\hat{L} = \hat{A} + \hat{B}$, where $\hat{A}$ and $\hat{B}$ are smooth connected surfaces meeting transversely along a smooth curve $h = \hat{A} \cap \hat{B}$. Assume that each of $\hat{A}$, $\hat{B}$ is either $\mathbb{P}^2$ or $\mathbb{F}_r$. Then describe $(\hat{X}, \hat{L})$.

The curve $h$ is connected [2, Corollary (2.3)]. We also call $h$ the hinge curve.
In this paper we shall deal only with the situation when $h$ has genus $g(h) \geq 2$: we refer to [2, Theorems (3.10), (3.11)] for the cases when $g(h) \leq 1$. We also refer to [2, 5, 7] for related results.

The organization of the paper is as follows. In §2, we present a general finiteness theorem for a threefold $\hat{X}$ with an ample divisor $\hat{L}$ of the form $\hat{A} + \hat{B}$, where $\hat{A}$, $\hat{B}$ are in a restricted class $\mathcal{C}$ of surfaces and meet transversely in a smooth curve of genus $\geq 2$. The class $\mathcal{C}$ includes surfaces with nef and big anticanonical bundle; and $\mathbb{P}^1$-bundles over either $\mathbb{P}^1$ or and elliptic curve. The finiteness theorem asserts that there is an $\epsilon > 0$ such that the Kodaira dimension of $K_{\hat{X}} + \left(\frac{1}{2} + \epsilon\right)\hat{L}$ is $-\infty$. By a result of Fujita, this implies that $(\hat{X}, \hat{L})$ is a birational transform of members of an explicit list of very special pairs.

In §3, it was shown that if the divisors $\hat{A}$, $\hat{B}$ are $\mathbb{P}^2$ or scrolls over $\mathbb{P}^1$, then the restriction of the bundle $K_{\hat{X}} + \hat{L}$ to the divisors in big.

In §4, the Hodge Index type theorem for reducible divisors leads to the elimination the cases in which both $\hat{A}$ and $\hat{B}$ are among $\mathbb{P}^2$ and the singular quadric with an isolated singularity $\mathbb{F}_2$.

Finally, in §5 we study the case when $\hat{A}$ is $\mathbb{P}^2$, the Hirzebruch surface $\mathbb{F}_r$, or the singular quadric with isolated singularity $\mathbb{F}_2$; and $\hat{B} = \mathbb{F}_s$, under the extra assumption that $(\hat{A}, L_{\hat{A}})$, $(\hat{B}, L_{\hat{B}})$ are scrolls.

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1 Background Material

We work over the complex field $\mathbb{C}$. Throughout the paper we deal with projective varieties $V$, and follow the usual notation of algebraic geometry. The book [1] is a good reference for standard results and notation of adjunction theory.

For a line bundle $L$ on an irreducible normal variety $V$ of dimension $n$ the sectional genus, $g(L) = g(V, L)$, of $(V, L)$ is defined by $2g(L) - 2 = (K_V + (n - 1)L) \cdot L^{n-1}$.

By $\mathbb{F}_r$ with $r \geq 0$ we denote the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a section $E$ taking on the minimal self intersection $E^2 = -r$ on the surface. By $\mathbb{F}_2$ we denote $\mathbb{F}_2$ with the section, which has self intersection $-2$, blown down. Note that $\mathbb{F}_2$ is isomorphic to any quadric hypersurface $Q \subset \mathbb{P}^3$, which has a single isolated singularity.
Let \( V \) be a normal \( r \)-Gorenstein (i.e., \( rK_V \) is a Cartier divisor) projective variety of dimension \( n \) and let \( D \) be a \( \mathbb{Q} \)-Cartier divisor on \( V \) such that \( \kappa(D) = n \). We define the unnormalized spectral value of the pair \((V, D)\) as

\[
u(V, D) := \sup \{ t \in \mathbb{Q} \mid \kappa(K_V + tD) = -\infty \}.
\]

We refer to [1, §7.1] for details.

The following result follows immediately from [2, §2].

**Lemma 1.1** Let \( \tilde{L} \) be an ample line bundle on a smooth projective 3-fold \( \tilde{X} \). Assume that there are two smooth connected divisors \( \hat{A}, \hat{B} \) on \( \tilde{X} \). Assume that \( \hat{A} + \hat{B} \in [\tilde{L}] \), that \( \hat{A} \) and \( \hat{B} \) are rational, and that \( \hat{A}, \hat{B} \) intersect transversely in a smooth curve \( \hat{h} \). Then \( \hat{h} \) is connected, and \( h^1(O_{\tilde{X}}) = h^2(O_{\tilde{X}}) = 0 \).

**2 A Finiteness Theorem**

In this section we prove a general finiteness theorem for pairs \((\tilde{X}, \tilde{L})\) consisting of an ample line bundle on a smooth projective threefold \( \tilde{X} \), with \([\tilde{L}]\) containing a divisor \( D = \hat{A} + \hat{B} \), having two irreducible components from a large class \( \mathcal{C} \) of negative Kodaira dimension surfaces. The class \( \mathcal{C} \) consists of the normal connected Gorenstein projective surfaces \( S \) with the property, that given any smooth connected Cartier divisor \( C \) on \( S \), it follows that either \( h^1(O_C) \leq 1 \) or \( K_S \cdot C \leq -1 \).

**Lemma 2.1** The class \( \mathcal{C} \) includes:

1. normal Gorenstein surfaces with \(-K_S \) nef and big; or
2. \( \mathbb{F}_r \), \( r \geq 0 \), the \( r \)-th Hirzebruch surface; or
3. a \( \mathbb{P}^1 \) bundle over an elliptic curve.

In cases 1) and 2), smooth connected Cartier divisors \( C \) with \( h^1(O_C) \geq 2 \) satisfy \( K_S \cdot C \leq -3 \).

**Proof.** Let \( C \) be a smooth connected Cartier divisor of \( S \), i.e., let \( C \) be a curve on \( S \) with \( C \) contained in \( S_{\text{reg}} \), the smooth points of \( S \). We assume that we are in the situation that \( h^1(O_C) \geq 2 \), since otherwise there is nothing to show.

First assume that \(-K_S \) is nef and big, and that the result is false, i.e., that \(-K_S \cdot C \leq 2 \). We know that \(-K_S \cdot C = 0, 1, 2 \). If \(-K_S \cdot C = 0 \), then we conclude, using the Hodge Index Theorem, that \( C^2 \leq 0 \), which contradicts \( h^1(O_C) \geq 2 \). If \(-K_S \cdot C = 1 \), then we conclude that \( C^2 \geq 3 \), which contradicts the Hodge Index Theorem, i.e., \( C^2 \leq C^2 K_S^2 \leq 1 \). If \(-K_S \cdot C = 2 \), then we conclude that \( C^2 \geq 4 \), which gives equality in the Hodge Index Theorem, i.e., \( 4 \leq C^2 \leq C^2 K_S^2 \leq 4 \). This implies that numerically \( C \sim -K_S \), which implies the contradiction \( K_S \cdot C \sim 0 \).

For \( S \) a Hirzebruch surface the result is a straightforward check.

Assume finally that \( S \) is a \( \mathbb{P}^1 \)-bundle over an elliptic curve \( Y \). In this case the section \( \sigma \) of minimal self-intersection satisfies \( e := -\sigma^2 \geq -1 \), and \( K_S \) is numerically equal to \(-2\sigma - ef \) for a fiber of the induced projection \( \pi : S \to Y \). Since we are assuming that
\( h^1(\mathcal{O}_C) \geq 2 \), we know that numerically \( C = ka + tf \) where \( k \geq 2 \). Moreover \( K_S \cdot C \geq 0 \) gives \( ke - 2t = 2ke - ek - 2t \geq 0 \). Since \( C^2 = -ek^2 + 2kt \), we have the absurdity that
\[
2 \leq 2g(C) - 2 = K_S \cdot C + C^2 = (1 - k)(ke - 2t) \leq 0.
\]
Q.E.D.

One main result of the paper is the Finiteness Theorem (2.2). This theorem shows that, if the hinging curve \( h \) has genus \( g(h) \geq 2 \), the pair \( (\hat{X}, \hat{L}) \) belongs to an explicit list of very special cases described by Fujita (see [3, 4] and also [1, (7.8.1)]).

Note in the following that the hypothesis that \( h \) is connected is automatically satisfied if \( \hat{A} \) and \( \hat{B} \) are connected [2, Corollary (2.3)].

**Theorem 2.2 (Finiteness theorem)** Let \( \hat{L} \) be an ample line bundle on a smooth projective 3-fold \( \hat{X} \). Assume that there are two divisors \( \hat{A}, \hat{B} \) on \( \hat{X} \) from the class \( \mathcal{C} \). Assume that \( \hat{A} + \hat{B} \in \hat{L} \) and that \( \hat{A}, \hat{B} \) intersect transversely in a smooth connected curve \( h \) of genus \( g(h) \geq 2 \). Then \( u(\hat{X}, \hat{L}) > \frac{1}{2} \). In particular, \( \hat{X} \) is of Kodaira dimension \( -\infty \), and thus satisfies \( h^3(\mathcal{O}_{\hat{X}}) = 0 \).

**Proof.** For simplicity of notation, we omit \( \sim \)’s in this proof. The genus formula yields
\[
(K_X + L) \cdot h = (K_X + A + B) \cdot A \cdot B = 2g(h) - 2,
\]
or \( (K_A + B_A) \cdot B_A = 2g(h) - 2 \), and therefore, by definition of class \( \mathcal{C} \), one has \( B_A \cdot B_A \geq 2g(h) - 1 \), and similarly \( A_B \cdot A_B \geq 2g(h) - 1 \). Then (1) gives
\[
K_X \cdot h \leq -2g(h).
\]
Now compute, for any real number \( \varepsilon, 0 < \varepsilon < \frac{1}{4g(h) - 2} \),
\[
(K_X + \left( \frac{1}{2} + \varepsilon \right) L) \cdot h = (K_X + L - \left( \frac{1}{2} - \varepsilon \right) L) \cdot h
\]
\[
= 2g(h) - 2 - \left( \frac{1}{2} - \varepsilon \right) L \cdot h
\]
\[
\leq 2g(h) - 2 - \left( \frac{1}{2} - \varepsilon \right) (4g(h) - 2)
\]
\[
= -1 + \varepsilon(4g(h) - 2) < 0.
\]
Finally, for \( h = A \cap B \) on \( X \), we have the normal bundle decomposition \( N_{h/X} = N_{h/A} \oplus N_{h/B} \) and \( \deg(N_{h/A}) = B^2 \cdot A = B_A \cdot B_A \geq 2g(h) - 1 \) by the above. It follows that \( h^1(N_{h/A}) = 0 \) and \( N_{h/A} \) has not identically zero sections. Similarly for \( N_{h/B} \). Then \( N_{h/X} \) is generically spanned by its global sections and \( h^1(N_{h/X}) = 0 \). Thus general deformation theory implies that the union of the deformations of \( h \) on \( X \) contains an open set. Therefore the inequality \( (K_X + \left( \frac{1}{2} + \varepsilon \right)L) \cdot h < 0 \) proved above, shows that \( u(X, L) > \frac{1}{2} \), cf., [1, (7.6.4)].

Q.E.D.

A little more can be said on the case of \( \mathbb{P}^1 \)-bundles over \( \mathbb{P}^1 \) or surfaces with nef and big anticanonical bundle.
Proposition 2.3 Let \( \hat{\mathcal{L}} \) be an ample line bundle on a smooth projective 3-fold \( \hat{X} \). Assume that there are two smooth divisors \( \hat{A}, \hat{B} \) on \( \hat{X} \) which are either \( \mathbb{P}^1 \)-bundles over \( \mathbb{P}^1 \) or surfaces with nef and big anticanonical bundle. Assume that \( \hat{A} + \hat{B} \in [\hat{\mathcal{L}}] \) and that \( \hat{A}, \hat{B} \) intersect transversely in a smooth connected curve \( h \). Then \( H^0(K_{\hat{\mathcal{L}}} + \hat{\mathcal{L}}) \to H^0(K_h) \to 0 \).

Proof. Tensor the Koszul complex

\[
0 \to \mathcal{O}_{\hat{X}} \to \hat{A} \oplus \hat{B} \to \hat{\mathcal{L}} \to \hat{\mathcal{L}}_h \to 0
\]

with \( K_{\hat{\mathcal{L}}} \). Using the hypercohomology spectral sequence, we see that the desired result will follow if we show that \( H^2(K_{\hat{\mathcal{L}}} + \hat{A}) = H^1(K_{\hat{\mathcal{L}}} + \hat{B}) = 0 \).

The assertion \( H^2(K_{\hat{\mathcal{L}}} + \hat{A}) = 0 \) follows from Lemma (1.1). To see that \( H^1(K_{\hat{\mathcal{L}}} + \hat{B}) = 0 \) consider the exact sequence

\[
0 \to K_{\hat{\mathcal{L}}} \to K_{\hat{\mathcal{L}}} + \hat{A} \to K_{\hat{\mathcal{L}}} \to 0.
\]

Now use Lemma (1.1) and the fact that \( \hat{A} \) is rational. The argument for \( H^1(K_{\hat{\mathcal{L}}} + \hat{B}) = 0 \) is identical.

Q.E.D.

One consequence of Proposition (2.3) is that under the same hypotheses with the added assumption that \( g(h) \geq 2 \), it follows that the Kodaira dimension of \( K_{\hat{\mathcal{L}}} + \hat{\mathcal{L}} \) is at least one. This implies that the Kodaira dimension of \( K_{\hat{\mathcal{L}}} + 2\hat{\mathcal{L}} \) is three, and also that the restriction of \( K_{\hat{\mathcal{L}}} + 2\hat{\mathcal{L}} \) to \( \hat{\mathcal{A}} \) (or \( \hat{\mathcal{B}} \)) is nontrivial. Therefore \([2, \text{ Theorems (3.6), (3.8)}]\) specialize to the following result.

Theorem 2.4 Let \( \hat{\mathcal{L}} \) be an ample line bundle on a smooth projective 3-fold \( \hat{X} \). Assume that there are two smooth divisors \( \hat{A}, \hat{B} \) on \( \hat{X} \) which are either \( \mathbb{P}^1 \)-bundles over \( \mathbb{P}^1 \) or surfaces with nef and big anticanonical bundle. Assume that \( \hat{A} + \hat{B} \in [\hat{\mathcal{L}}] \) and that \( \hat{A}, \hat{B} \) intersect transversely in a smooth connected curve \( h \) of genus \( \geq 2 \). Then there is a surjective morphism \( \phi : \hat{X} \to X \), where \( X \) is a smooth projective 3-fold, such that:

1. \( \phi \) expresses \( \hat{X} \) as the blowup of \( X \) at a finite set \( \mathcal{F} \), and there is an ample line bundle \( \mathcal{L} \) on \( X \) such that \( \hat{\mathcal{L}} \cong \phi^* \mathcal{L} - \phi^{-1}(\mathcal{F}) \);
2. \( K_{\hat{\mathcal{L}}} + 2\hat{\mathcal{L}} \cong \phi^*(K_X + 2\mathcal{L}) \) where \( K_X + 2\mathcal{L} \) is ample;
3. \( K_X + \mathcal{L} \) is either nef and big, or \( (X, \mathcal{L}) \) is a conic fibration over a surface \( Y \) in the sense of adjunction theory \([1]\), i.e., there exists a morphism \( \nu : X \to Y \) with \( K_X + \mathcal{L} \cong \nu^* H \) for an ample line bundle \( \mathcal{H} \) on a normal surface \( Y \);
4. \( \phi \) is an embedding in a neighborhood of \( h \); and
5. \( \mathcal{L} = A + B \) where \( A := \phi(\hat{A}) \) and \( B := \phi(\hat{B}) \) are Cartier divisors meeting transversely in \( \phi(h) \) and each having at most one point contained in the set \( \mathcal{F} \).

From now on we usually abuse notation, and let \( h \) to denote \( \phi(h) \). We also write \( h_A \) (respectively \( h_B \)) to emphasize that we view \( h \) as a curve on \( A \) (respectively on \( B \)).

Lemma 2.5 Let \( (\hat{X}, \hat{\mathcal{L}}), (X, \mathcal{L}), \hat{A}, \hat{B}, A, B \) be as in (2.4). Then
1. $h^i,0(X) = 0, i = 1, 2, 3$;
2. $h^i(K_X + A) = h^i(K_X + B) = 0$ for all $i \geq 0$; and
3. the restriction map gives the following isomorphisms

\[ H^0(K_X + L) \cong H^0(K_A + h_A) \cong H^0(K_B + h_B) \cong H^0(K_B). \]

**Proof.** Noting that the first reduction morphism, $\phi$, of Lemma (2.4) is birational, the first assertion follows immediately from Lemma (1.1) and Theorem (2.2).

To prove 2), consider the exact sequence

\[ 0 \to K_X + B \to K_X + L \to K_A \to 0. \]

By the assumption on $A$, $h^0(K_A) = h^1(K_A) = 0, h^2(K_A) = 1, h^3(K_A) = 0$. Thus from the cohomology sequence associated to the sequence above we infer that $h^i(K_X + B) = 0$ (and by symmetry $h^i(K_X + A) = 0$) for all $i \geq 0$.

Item 3) follows immediately from the first two assertions. Q.E.D.

**Theorem 2.6** Let $\widehat{L}$ be an ample line bundle on a smooth projective 3-fold $\widehat{X}$. Assume that there are two smooth divisors $\widehat{A}, \widehat{B}$ on $\widehat{X}$ which are either $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ or surfaces with nef and big anticanonical bundle. Assume that $\widehat{A} + \widehat{B} \in [\widehat{L}]$ and that $\widehat{A}, \widehat{B}$ intersect transversely in a smooth connected curve $h$ of genus $g(h) \geq 2$. Let $X, A, B, L$ be as in (2.4). Then $H^0(K_X + L)$ spans $K_X + L$ in a neighborhood of $A + B$.

**Proof.** By Lemma (2.5), the desired spannedness of $K_X + L$ will follow from the spannedness of $K_A + h_A$ and $K_B + h_B$. From Theorem (2.4) we know that $K_X + L$ is nef (and hence $K_A + h_A$ and $K_B + h_B$ are also).

First assume that $\widehat{A}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Either the map $\phi$ of Theorem (2.4) is an isomorphism on $\widehat{A}$, in which case $\widehat{A}$ is also a $\mathbb{P}^1$-bundle, or, by [2, Theorem (3.6), 2], $\phi^\wedge \widehat{A}$ expresses $\widehat{A}$ as the blowup of $A$ at one point. In this latter case, $\widehat{A}$ is the Hirzebruch surface $\mathbb{F}_1$, and $A := \phi(\widehat{A}) = \mathbb{P}^2$ (note that $\mathbb{F}_1$ is the only Hirzebruch surface with a $-1$ curve). Since $X + L$ is nef, $K_A + h_A$ is nef, and for either $\mathbb{P}^2$ or $\mathbb{P}^1$-bundles over $\mathbb{P}^1$, nef line bundles are spanned.

Now assume that $-K_A^\wedge$ is nef and big. Note that $-K_A$ is also nef and big. Indeed, going to the first reduction map we have a birational morphism $\phi_A^\wedge : \widehat{A} \to A$ where some disjoint $-1$ curves are collapsed. Writing $-K_A^\wedge = K_A^\wedge + 2(-K_A^\wedge)$, we see from the basepoint free theorem that $-N K_A^\wedge$ is spanned for $N \gg 0$. Thus $-N K_A$ is spanned off the finite set equal to the image of the exceptional curves. This implies $-K_A$ is nef. Since $K_A^\wedge > K_A^\wedge$, bigness is clear.

Consider the line bundle $h_A$. We would like to show by Reider's Theorem [6] that $K_A + h_A$ is spanned. Note that $h_A^2 = 2g(h_A) - 2 - K_A \cdot h_A \geq 2 + 3 = 5$ by the hypothesis $g(h_A) \geq 2$ and Lemma (2.1). Since $h_A$ is a smooth curve of positive genus, and $K_A \cdot h_A < 0$, we conclude that $h_A$ is nef and big. Therefore by Reider’s Theorem, either $K_A + h_A$ is spanned, or there exists an effective Cartier divisor $\ell \subset A$ such that either $h_A \cdot \ell = 0$ with $\ell^2 = -1$, or $h_A \cdot \ell = 1$ with $\ell^2 = 0$.

In the former case, $K_A \cdot \ell < 0$, since $K_A \cdot \ell \leq 0$ and $K_A \cdot \ell + \ell^2$ is even. This contradicts the nefness of $K_A + h_A$. 

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Finally consider the case \( h_A \cdot \ell = 1 \) with \( \ell^2 = 0 \). Note that since \( \ell \) is effective, we cannot have \( -K_A \cdot \ell = 0 \) by the usual Hodge index relation. Thus we have \( K_A \cdot \ell \leq 0 \).

Since \( K_A \cdot \ell + \ell^2 \) is even, we have that \( K_A \cdot \ell \leq -2 \). This implies that \( (K_A + h_A) \cdot \ell \leq -1 \), which contradicts nefness of \( K_A + h_A \).

Q.E.D.

3 Some Birationality Results

3.1 (Working assumptions) Let \( \tilde{L} \) be a very ample line bundle on a 3-fold \( \tilde{X} \). Assume that there are two smooth transverse divisors \( \tilde{A}, \tilde{B} \) on \( \tilde{X} \) with \( \tilde{A} + \tilde{B} \in |L| \) and \( \tilde{A}, \tilde{B} \in \{ \mathbb{P}^2, \mathbb{F}_r \} \). Assume that the hinge curve \( h = \tilde{A} \cap \tilde{B} \) has genus \( g(h) \geq 2 \).

From Theorem (2.4), we know that there exists the first reduction \( (X, L), \phi : \tilde{X} \rightarrow X \), with \( K_X + 2L \) ample and \( K_X + L \) nef. If \( A = \phi(\tilde{A}), \ B = \phi(\tilde{B}) \), then \( A + B \in |L| \) and \( A, B \in \{ \mathbb{P}^2, \mathbb{F}_r \} \). Furthermore we know by 5) of Theorem (2.4), that neither \( \tilde{A} \) nor \( \tilde{B} \) is a fiber of \( \phi \) and that \( A, B \) meet transversely along the curve \( \phi(h) \) isomorphic to \( h \).

**Lemma 3.2** Assumptions and notation as in (3.1). The complete linear systems \( |K_A + h_A| \) and \( |K_B + h_B| \) maps \( h \) generically one-to-one. In particular, \( |K_A + h_A|, |K_B + h_B| \), and \( K_X + L \) are nef and big.

*Proof*. By Lemma (2.5) we see that showing complete linear systems \( |K_A + h_A| \) and \( |K_B + h_B| \) maps \( h \) generically one-to-one, shows that \( |K_X + L| \) maps \( h \) generically one-to-one.

We know from 3) of Theorem (2.4) that the restriction of \( K_X + L \) of one of the divisors \( A, B \) is nef and big (by ampleness of \( A + B \) either \( A \) or \( B \) surjects on the base). Assume for simplicity, that \( K_B + h_B \approx (K + L)_B \) is nef and big. If \( B = \mathbb{P}^2 \) or \( \mathbb{F}_{2r} \), the line bundle \( K_B + h_B \) is ample, and indeed very ample.

Thus we may assume \( B = \mathbb{F}_r, r \geq 1 \). Let \( E := E + rf \). Then either \( K_B + h_B = aE + bf \) is very ample or \( b = 0 \) and \( K_B + h_B = aE \). Thus \( |K_B + h_B| \) maps \( h \) generically one-to-one.

Now looking at conclusion 3) of Theorem (2.4), we see that if \( K + L \) fails to be nef and big, then using the notation of Theorem (2.4), that \( v \) maps \( h \) two-to-one onto a curve \( v(h) \) with all restrictions of elements of \( H^0(K_X + L) \) to \( h \) the pullbacks of restrictions of sections of \( H^0(K_X + L) \). This is a contradiction to the assertion that \( |K_X + L| \) maps \( h \) generically one-to-one onto its image.

Assume that \( (K_X + L)_A \approx K_A + h_A \) is not nef and big. Since the genus of the curve \( h_A \) is not 0, the map given by \( |K_A + h_A| \) cannot be generically one-to-one on \( h_A \).

Q.E.D.

The following is a corollary of the preceding lemma.

**Lemma 3.3** Assumptions and notation as in (3.1). Assume \( A = \mathbb{F}_r \) and let \( h = aE + bf \) on \( A \). Then \( a \geq 3 \).

*Proof*. Note that \( a = h \cdot f \geq 0 \), and \( a \neq 1 \) since \( g(h) > 0 \). Assume \( a = 2 \). Then \( (K_A + h_A) \cdot f = -2 + 2 = 0 \) and hence \( |(K_X + L)_A| = |(K_A + h_A)| \) collapses \( A \) along the ruling \( f \). This contradicts Lemma (3.2).

Q.E.D.
4 The Cone Cases

The main result in this section is to showing that in the situation when we have a reducible ample divisor \( L = A + B \) with both of \( A \) and \( B \) are in \( \{ \mathbb{P}^2, \mathbb{F}_2 \} \) is very restricted. The proof of this is based on the usual Hodge Index type theorem for ample divisors, which yields in our case

\[
[(A + B) \cdot A \cdot A][(A + B) \cdot B \cdot B] \leq [(A + B) \cdot A \cdot B]^2 \tag{3}
\]

with equality if and only if \( A \) is a rational multiple of \( B \) as homology class.

**Lemma 4.1** Let \( L \) be an ample line bundle on a smooth connected projective threefold \( X \). Assume that \( A, B \) are two reduced divisors on \( X \) which meet transversely in a smooth curve \( h \) of genus \( g(h) \). Assume that \( A + B \in [L] \), and that \( A, B \in \{ \mathbb{P}^2, \mathbb{F}_2 \} \). Then \( g(h) \leq 1 \).

**Proof.** Assume without loss of generality that \( g := g(h) \geq 2 \). Note that in this case the degree of \( h \) on \( A, B \) are uniquely determined by \( g \).

First let us do the case of \( A = B = \mathbb{P}^2 \). Then \( h_A \in |\mathcal{O}_{\mathbb{P}^2}(d)| \) and \( h_B \in |\mathcal{O}_{\mathbb{P}^2}(d)| \) where \( 2g - 2 = d(d - 3) \). Note that \( d^2 = h_A^2(B) = B \cdot B \cdot A = h_B \cdot N_{B/X} \). Thus \( N_{B/X} = \mathcal{O}_{\mathbb{P}^2}(d) \), and similarly \( N_{A/X} = \mathcal{O}_{\mathbb{P}^2}(d) \). Plugging into Equation (3), we get equality. Thus \( A = \lambda B \) as homology classes for some \( \lambda \in \mathbb{Q} \). Since \( A^2 \cdot B = d^2 = B^2 \cdot A \), we see that \( \lambda = 1 \). Thus since \( L \) is ample and since \( L = 2A = 2B \) in homology, it follows that \( A, B \) are ample. The Lefschetz theorem yields \( \text{Pic}(X) = \text{Pic}(A) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}^2}(1)] \). Therefore \( K_X \approx \mathcal{O}_X(c) \), \( \mathcal{O}_X(A) \approx \mathcal{O}_X(a) \), where \( a \geq 1 \) by ampleness. Then \( (K_X + A)_A \approx K_A \approx \mathcal{O}_{\mathbb{P}^2}(-3) \) gives \( K_X + A \approx \mathcal{O}_X(c + a) \approx \mathcal{O}_X(-3) \). Therefore \( 1 + c \leq a + c = -3 \), or \( c \leq -4 \). Thus \( X = \mathbb{P}^3 \) by the Kobayashi-Ochiai Theorem [1, (3.1.6)] and \( g = 0 \), contradicting the assumption \( g > 0 \).

The case of \( A = B = \mathbb{F}_2 \) proceeds in the same way, except that one of the possibilities allowed by the Kobayashi-Ochiai Theorem [1, (3.1.6)] is \( (X, L) \) is \( (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) \). In this case \( g = 1 \).

Finally, consider the case when one of \( A, B \) is \( \mathbb{P}^2 \) and the other is \( \mathbb{F}_2 \). By renaming if necessary we can assume that \( A = \mathbb{P}^2 \) and \( B = \mathbb{F}_2 \). Letting \( h_B = A_B = \mathcal{O}_A(d) \) and \( h_A = B_A = \mathcal{O}_A(d) \), we have that \( A^2 \cdot B = 2d^2, B^2 \cdot A = d^2 \). Also from \( d^2 = h_A^2(B) = B \cdot B \cdot A = h_B \cdot N_{B/X} \) we conclude that \( N_{B/X} = \mathcal{O}_B(d^2/2) \). Similarly we conclude that \( N_{A/X} = \mathcal{O}_A(2d^2/4) \). Thus \( A^3 = \frac{4d^4}{d^2} \) and \( B^3 = \frac{d^4}{4d^2} \). Plugging into Equation (3), we conclude again that \( A, B \) are positive multiples of \( L \) in homology and hence ample. Using the argument from the case when both are \( \mathbb{P}^2 \), we see that \( X = \mathbb{P}^3 \). In this case \( g = 0 \).

**Q.E.D.**

5 The Cone and Scroll Cases

We keep again our working assumption as in (3.1). In this section we consider the remaining case when both \( A \) and \( B \) are Hirzebruch surfaces, under the extra assumption that \( \widehat{A}, \widehat{L} \), \( \widehat{B}, \widehat{L} \) are scrolls, i.e., \( \widehat{A}, \widehat{B} \) are both scrolls with respect to \( L \).

We start with the following general lemma.
Lemma 5.1 Let $\hat{L}$ be a very ample line bundle on a 3-fold $\hat{X}$. Let $\hat{A} + \hat{B} \in |L|$, where $\hat{A}, \hat{B}$ are two smooth divisors on $\hat{X}$ meeting transversely in a smooth curve $h$ of genus $g(h) > 0$. Assume that $(\hat{A}, \hat{L}_A)$ and $(\hat{B}, \hat{L}_B)$ are scrolls or cones (from a vertex not contained in $h$) over smooth curves $\hat{C}_A, \hat{C}_B$ with scroll (or cone) projections $p_{\hat{A}} : \hat{A} \to \hat{C}_A$, $p_{\hat{B}} : \hat{B} \to \hat{C}_B$ respectively. Then $\pi = (p_{\hat{A}}, p_{\hat{B}}) : h \to \hat{C}_A \times \hat{C}_B$ maps $h$ isomorphically onto a smooth curve.

Proof. Let $\xi \subset h$ be a subscheme of degree 2 (i.e., a pair of distinct points, or a tangent subscheme supported at a single point). We show that $\pi$ separates $\xi$. If not, $\xi$ is contained in a fiber of $\pi$. Hence $\xi$ belongs to a fiber of $p_{\hat{A}}$ and one of $p_{\hat{B}}$, in which case the same holds true for the line $\ell$ spanned by $\xi$. But since the intersection $\hat{A} \cap \hat{B}$ is transverse and connected, it follows that $\hat{A} \cap \hat{B} = \ell$. This contradicts the hypothesis that $g(h) > 0$.

Q.E.D.

Theorem 5.2 Let $\hat{L}$ be a very ample line bundle on a smooth projective threefold $\hat{X}$. Assume that there exists two irreducible divisors $\hat{A}, \hat{B}$ on $\hat{X}$, which meet transversely in a smooth curve $h$, and such that $\hat{A} + \hat{B} \in |\hat{L}|$. Assume further that

1. $(\hat{A}, \hat{L}_A)$ is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, or $(Q, \mathcal{O}_{\mathbb{P}^3}(1)_Q)$ with $Q \subset \mathbb{P}^3$ the singular quadric $\hat{F}_2$; and

2. $(\hat{B}, \hat{L}_B)$ is a scroll over $\mathbb{P}^1$.

Then $g(h) = 0$.

Proof. We do the case where $\hat{A} = \mathbb{P}^2$. The case of $\hat{A} = \hat{F}_2$ is proved in a completely similar way. Let $N_{\hat{A}/\hat{X}} = \hat{A}_A \cong \mathcal{O}_{\mathbb{P}^2}(d)$ denote the normal bundle of $\hat{A}$ in $\hat{X}$. Let $h_A = \hat{B}_A = \mathcal{O}_{\mathbb{P}^2}(\delta)$. Letting $E$ denote a section of $\hat{B}$ with $E^2 = -r \leq 0$, and letting $E = E + rf$ for a fiber $f$ of the scroll projection, we have $\hat{B}_B = ME + Nf$ and $h_B = \hat{A}_B = aE + bf$ for integers $M, N, a, b$. Note that by Lemma (5.1) we have $g := g(h) = (a - 1)(\delta - 1)$. Note by the formulae for the genus on $\hat{A}$ and $\hat{B}$ we have the formulae $2g = (\delta - 1)(\delta - 2)$ and $2g = (a - 1)(ar + 2b - 2)$. Assuming that $g \geq 1$, and hence that $\delta \geq 3, a \geq 2$, immediately gives $\delta = 2a$ and $4a = ar + 2b$.

Note that $d\delta = \hat{A}_A \cdot \hat{B} = \hat{B} \cdot \hat{A}^2 = A_B^2 = a(ar + 2b)$. Combined with $\delta = 2a$ and $4a = ar + 2b$, we conclude that $d = \delta$. Since $\hat{L}_A = \mathcal{O}_{\mathbb{P}^2}(d + \delta)$, we get a contradiction to $\hat{L}_A \cong \mathcal{O}_{\mathbb{P}^2}(1)$.

Q.E.D.

Now let us specialize Lemma (5.1) to the case when $\hat{A} = \mathbb{F}_r, \hat{B} = \mathbb{F}_s$. Denote by $\mathcal{E}_A = E_A^\hat{A} + rf_A^\hat{A}, E_A^2 = -r, f_A^\hat{A}$ a fiber of the ruling $\hat{A} = \mathbb{F}_r \to \mathbb{P}^1$; and similarly for $\hat{B}$. Write

\[ h_A = a(E_A^\hat{A} + rf_A^\hat{A}) + bf_A^\hat{A}; \quad h_B = a(E_B + sf_B^\hat{A}) + \beta f_B^\hat{A}, \tag{4} \]

on $\hat{A}, \hat{B}$, respectively.

Note that by Lemma (3.3) we may assume $a \geq 3, \alpha \geq 3$. Furthermore, since $h$ is a positive genus curve on the Hirzebruch surfaces $\hat{A}, \hat{B}$, we may also assume $b \geq 0, \beta \geq 0$. 

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By Lemma (5.1), $O_{P^1 \times P^1}(\pi(h)) = O_{P^1 \times P^1}(a, \alpha)$, and hence
\[ g(h) = (a - 1)(\alpha - 1). \] (5)

The genus formula also yields
\[ 2g(h) - 2 = (a - 1)(ar + 2b - 2). \]
Therefore, by using (5), we deduce that
\[ 2\alpha = ar + 2b. \] (6)

Similarly we find
\[ 2a = \alpha s + 2\beta. \] (7)

Combining (6) and (7), we have $(4 - r s)\alpha \alpha = 4\beta \beta + 2(b \alpha s + \beta a r)$.

\[ 2\alpha \alpha = a(ar + 2b) = \alpha (\alpha s + 2\beta). \] (8)

Write
\[ N_{\widehat{\mathcal{A}}/\widehat{\mathcal{C}}} = -\lambda(E_{\widehat{\mathcal{A}}} + r f_{\widehat{\mathcal{A}}}) + \rho f_{\widehat{\mathcal{A}}}; \quad N_{\widehat{\mathcal{B}}/\widehat{\mathcal{C}}} = -\mu(E_{\widehat{\mathcal{B}}} + s f_{\widehat{\mathcal{B}}}) + \sigma f_{\widehat{\mathcal{B}}}, \]
for integers $\lambda, \mu, \rho, \sigma$.

Note that on $\widehat{\mathcal{A}}$ one has $h_{\widehat{\mathcal{A}}}^2 = \widehat{\mathcal{A}}^2 \cdot \widehat{\mathcal{B}} = \widehat{\mathcal{A}} \cdot \cdot \cdot \widehat{\mathcal{A}} = N_{\widehat{\mathcal{A}}/\widehat{\mathcal{C}}} \cdot h_{\widehat{\mathcal{A}}}$. Since
\[ h_{\widehat{\mathcal{B}}}^2 = \alpha^2 s + 2\alpha \beta = \alpha (\alpha s + 2\beta) \quad \text{and} \quad N_{\widehat{\mathcal{A}}/\widehat{\mathcal{C}}} \cdot h_{\widehat{\mathcal{A}}} = -a\lambda r - \lambda b + \rho a, \]
we find $\alpha (\alpha s + 2\beta) = -a\lambda r - \lambda b + \rho a$. Similarly, we find $a(ar + 2b) = -a\mu s - \mu \beta + \sigma \alpha$.
Then by (8) we have
\[ -a\lambda r - \lambda b + \rho a = -\alpha \mu s - \mu \beta + \sigma \alpha. \] (9)

Since $(\widehat{\mathcal{A}}, \widehat{\mathcal{L}})$ is a scroll, we also have $\widehat{\mathcal{L}}_{\widehat{\mathcal{A}}} = \widehat{\mathcal{A}} + \widehat{\mathcal{B}} = N_{\widehat{\mathcal{A}}/\widehat{\mathcal{C}}} + h_{\widehat{\mathcal{A}}} = E_{\widehat{\mathcal{A}}} + j f_{\widehat{\mathcal{A}}}$. On the other hand, the coefficient of $E_{\widehat{\mathcal{A}}}$ in the expression for $N_{\widehat{\mathcal{A}}/\widehat{\mathcal{C}}} + h_{\widehat{\mathcal{A}}}$ is $-\lambda + a$.
Therefore the last equality for $\widehat{\mathcal{L}}_{\widehat{\mathcal{A}}}$ implies $a - \lambda = 1$. Similarly the scroll condition for $(\widehat{\mathcal{B}}, \widehat{\mathcal{L}}_{\widehat{\mathcal{B}}})$ gives $\alpha - \mu = 1$. So from (9) we have
\[ -a(a - 1)r - (a - 1)b + \rho a = -\alpha (a - 1)s - (a - 1)\beta + \sigma \alpha. \]

Then in particular $\rho = 2\alpha + (a - 1)r + b + \frac{b}{a}$. This implies $\rho \geq 6$ (with $\rho = 6$ giving $r = b = 0$), as well as $a$ divides $b$, say, $b = ab'$. In the same way, we find $\sigma = 2a + (a - 1)s + \beta + \frac{\beta}{a}$. So $\sigma \geq 6$ (with $\sigma = 6$ giving $s = \alpha = 0$), as well as $\beta = \alpha \beta'$.

Thus formulas (6) and (7) become $2\alpha = a(r + 2b')$ and $2a = \alpha(s + 2\beta')$. From this we also find
\[ 4 = (r + 2b')(s + 2\beta'). \] (10)

Since $b', \beta' \geq 0$ it follows that $rs \leq 4$ and hence $r, s \in \{0, 1, 2, 3, 4\}$.

The following theorem summarizes the discussion above.
Theorem 5.3 Let $\hat{L}$ be a very ample line bundle on a 3-fold $\hat{X}$. Let $\hat{A} + \hat{B} \in |L|$, where $\hat{A}, \hat{B}$ are two smooth divisors on $\hat{X}$ meeting transversely in a smooth curve $h$ of genus $g(h) > 0$. Assume that $\hat{A} = \mathbb{F}_r$, $\hat{B} = \mathbb{F}_s$ are Hirzebruch surfaces. Further assume that $(\hat{A}, \hat{L}_\hat{A})$ and $(\hat{B}, \hat{L}_\hat{B})$ are scrolls over smooth curves. Then $r, s \in \{0, 1, 2, 4\}$ and the possible values of the coefficients $b = a\beta$, $\beta = \alpha\beta'$ as in the expressions (4) of $h$ as a curve of $\hat{A}, \hat{B}$ respectively are listed in the table below.

<table>
<thead>
<tr>
<th>$s$ \ $r$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\beta' = \beta' = 1$</td>
<td>$\beta' = 0, \beta' = 2$</td>
<td>$\beta' = 0, \beta' = 1$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
</tr>
<tr>
<td>1</td>
<td>$\beta' = 2, \beta' = 0$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\beta' = 1, \beta' = 0$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\beta' = \beta' = 0$</td>
</tr>
<tr>
<td>2</td>
<td>$\beta = 1, \beta' = 0$</td>
<td>$\beta = 0, \beta' = 1$</td>
<td>$\beta = \beta' = 0$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
</tr>
<tr>
<td>4</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
</tr>
</tbody>
</table>

Proof. A simply numerical check, by using (10) and the symmetry between $r$ and $s$, gives the possible values for the integers $r, s, \beta, \beta'$ in the table (the symbol “$\sqrt{\cdot}$” means that the corresponding case does not occur). E.g., if $r = 0$, equality (10) gives $2 = \beta(s + 2\beta')$. This leads to the cases $(s, \beta, \beta') = (0, 1, 1), (1, 2, 0), (2, 1, 0)$ as in the first column. Thus we may assume $r, s \geq 0$. E.g., if $r = 3$, (10) gives $4 = 3(s + 2\beta') + 2\beta(s + 2\beta')$, so that $\beta' \neq 0$ and hence $\beta' > 0$, this giving again a numerical contradiction. Q.E.D.

References


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