ON HIGHER ORDER EMBEDDINGS OF FANO THREEFOLDS BY THE ANTICANONICAL LINEAR SYSTEM

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Abstract. In this article the map given by the anticanonical bundle of a Fano manifold is investigated with respect to a number of natural notions of higher order embeddings of projective manifolds. This is of importance in the understanding of higher order embeddings of the special varieties of adjunction theory, which are usually fibered by special Fano manifolds. An analysis is carried out of the higher order embeddings of the special varieties of adjunction theory that arise in the study of the first and second reductions. Special attention is given to determining the order of the anticanonical embeddings of the three dimensional Fano manifolds which have been classified by Iskovskih, Mori, and Mukai and also of the Fano complete intersections in $\mathbb{P}^N$.

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Introduction. Let X be an n-dimensional connected complex projective manifold. There are three natural notions (see (1.5)) of the "order" of an embedding given by a line bundle L. From strongest to weakest they are k-jet ampleness [7], k-very ampleness [5], and k-spannedness [2]. For $k = 1$ (respectively $k = 0$) all three notions are equivalent to very ampleness (respectively spannedness at all points of X by global sections). We consider the most natural notion to be k-very ample, which by definition means that given any 0-dimensional subscheme $\mathcal{Z}$ of X of length $k+1$, the map $H^0(X, L) \to H^0(\mathcal{Z}, L_{\mathcal{Z}})$, is onto. There has been considerable work on deciding the order of the embeddings relative to these notions for the line bundles that come up on the standard classes of varieties.

In this article we investigate what the "order" of the embedding by $|-K_X|$ is.

In §1 we recall some known results that we will need in the sequel. We also prove some foundational results on k-very ampleness that are not in the literature, e.g., Lemma (1.8) and the useful lower bound for the degree and the number of sections of L, Proposition (1.11).

In §2 we give the k-jet ampleness, k-very ampleness, and k-spannedness of $-K_X$ for Fano manifolds $X \subset \mathbb{P}^N$ which are complete intersections of hypersurfaces in $\mathbb{P}^N$.

In §3 we work out the order of the embeddings of the special varieties that come up in the study of the first and second reductions of adjunction theory. Fano manifolds of special types come up naturally as the fibers of degenerate adjunction morphisms.

In §4 we continue the investigation of the Fano manifolds that came up in §3, those with $-K_X \cong (n-2)L$ and $n \geq 4$.

In §5, by using Fujita's classification of Del Pezzo threefolds, we completely settle the order of the embeddings of 3-dimensional Fano manifolds with very ample anticanonical bundle $-K_X$.

For further discussion and a guide to most of the published papers on k-very ampleness, we refer to the book [8] by the first and third author. We also call attention to [11]

of the second author, which do a thorough investigation for surfaces of the questions analogous to those we ask for higher dimensions.

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1. Background material. Throughout this paper we deal with complex projective manifolds V. We denote by $\mathcal{O}_V$ the structure sheaf of V and by K_V the canonical bundle. For any coherent sheaf $\mathcal{F}$ on V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V, \mathcal{F})$.

Let L be a line bundle on V. L is said to be *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on V. L is said to be *big* if $\kappa(L) = \dim V$, where $\kappa(L)$ denotes the Kodaira dimension of L. If L is nef then this is equivalent to be $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L and $n = \dim V$.

1.1 Notation. In this paper, we use the standard notation from algebraic geometry. Let us only fix the following.

$\approx$ (respectively $\sim$) denotes linear (respectively numerical) equivalence of line bundles;

$|L|$, the complete linear system associated with a line bundle L on a variety V;

, $(L) = H^0(L)$ denotes the space of the global sections of L. We say that L is spanned if it is spanned at all points of V by , (L);

$\kappa(V) := \kappa(K_V)$ denotes the Kodaira dimension, for V smooth;

$b_2(V) = \sum_{p+q=2} h^{p,q}$ denotes the second Betti number of V, for V smooth, where $h^{p,q} := h^q(\Omega_V^p)$ denotes the Hodge (p, q) number of V.

Line bundles and divisors are used with little (or no) distinction. Hence we freely use the additive notation.

1.2 Reductions. (See e.g., [8, Chapters 7, 12]) Let $(\widehat{X}, \widehat{L})$ be a smooth projective variety of dimension $n \geq 2$ polarized with a very ample line bundle $\widehat{L}$. A smooth polarized variety (X, L) is called a *reduction* of $(\widehat{X}, \widehat{L})$ if there is a morphism $r : \widehat{X} \to X$ expressing $\widehat{X}$ as the blowing up of X at a finite set of points, B, such that $L := (r_*\widehat{L})^{**}$ is an ample line bundle and $\widehat{L} \approx r^*L - [r^{-1}(B)]$ (or, equivalently, $K_{\widehat{X}} + (n-1)\widehat{L} \approx r^*(K_X + (n-1)L)$).

Note that there is a one to one correspondence between smooth divisors of $|L|$ which contain the set B and smooth divisors of $|\widehat{L}|$.

Except for an explicit list of well understood pairs $(\widehat{X}, \widehat{L})$ (see in particular [8, §§7.2, 7.3]) we can assume:

a) $K_{\widehat{X}} + (n-1)\widehat{L}$ is spanned and big, and $K_X + (n-1)L$ is very ample. Note that in this case this reduction, (X, L), is unique up to isomorphism. We will refer to it as *the first reduction* of $(\widehat{X}, \widehat{L})$.

b) $K_X + (n-2)L$ is nef and big, for $n \geq 3$.

Then from the Kawamata-Shokurov basepoint free theorem (see e.g., [8, (1.5.2)]) we know that $|m(K_X + (n-2)L|$, for $m \gg 0$, gives rise to a morphism $\varphi : X \to Z$, with connected fibers and normal image. Thus there is an ample line bundle $\mathcal{K}$ on Z such that $K_X + (n-2)L \approx \varphi^*\mathcal{K}$. Let $\mathcal{D} := (\varphi_*L)^{**}$. The pair $(Z, \mathcal{D})$, together with the morphism $\varphi : X \to Z$ is called the *second reduction* of $(\widehat{X}, \widehat{L})$. The morphism φ is very well behaved (see e.g., [8, §§7.5, 7.6 and §§12.1, 12.2]). In particular Z has terminal, 2-factorial isolated singularities and $\mathcal{K} \approx K_Z + (n-2)\mathcal{D}$. Moreover $\mathcal{D}$ is a 2-Cartier divisor such that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$, for some effective Cartier divisor Δ on X which is φ-exceptional (see [8, (7.5.6), (7.5.8)]).

1.3 Nefvalue. (See e.g., [8, §1.5]) Let V be a smooth projective variety and let L be an ample line bundle on V. Assume that K_V is not nef. Then from the Kawamata rationality theorem (see e.g., [8, (1.5.2)]) we know that there exists a rational number τ such that $K_V + \tau L$ is nef and not ample. Such a number, τ, is called the *nefvalue* of (V, L).

Since $K_V + \tau L$ is nef, it follows from the Kawamata-Shokurov base point free theorem (see e.g., [8, (1.5.1)]) that $|m(vK_V + uL)|$ is basepoint free for all $m \gg 0$, where $\tau = u/v$. Therefore, for such m, $|m(K_V + \tau L)|$ defines a morphism $f : V \to \mathbb{P}_{\mathbb{C}}$. Let $f = s \circ \Phi$ be the Remmert-Stein factorization of f, where $\Phi : V \to Y$ is a morphism with connected fibers onto a normal projective variety Y and $s : Y \to \mathbb{P}_{\mathbb{C}}$ is a finite-to-one morphism. For m large enough such a morphism, Φ, only depends on (V, L) (see [8, §1.5]). We call $\Phi : V \to Y$ the *nefvalue morphism* of (V, L).

1.4 Special varieties. (See e.g., [8, §3.3]) Let V be a smooth variety of dimension n and let L be an ample line bundle on V.

We say that V is a *Fano manifold* if $-K_V$ is ample. We say that V is a *Fano manifold of index* i if i is the largest positive integer such that $K_V \approx -iH$ for some ample line bundle H on V. Note that $i \leq n+1$ (see e.g., [8, (3.3.2)]) and $n - i + 1$ is referred to as the *co-index* of V.

We say that a Fano manifold, (V, L), is a *Del Pezzo manifold* (respectively a *Mukai manifold*) if $K_V \approx -(n-1)L$ (respectively $K_V \approx -(n-2)L$).

We also say that (V, L) is a *scroll* (respectively a *quadric fibration*; respectively a *Del Pezzo fibration*; respectively a *Mukai fibration*) over a normal variety Y of dimension m if there exists a surjective morphism with connected fibers $p : V \to Y$ such that $K_V + (n-m+1)L \approx p^*\mathcal{L}$ (respectively $K_V + (n-m)L \approx p^*\mathcal{L}$; respectively $K_V + (n-m-1)L \approx p^*\mathcal{L}$; respectively $K_V + (n-m-2)L \approx p^*\mathcal{L}$) for some ample line bundle $\mathcal{L}$ on Y.

1.5 k-th order embeddings. Let V be a smooth algebraic variety. We denote the Hilbert scheme of 0-dimensional subschemes $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of V with $\mathrm{length}(\mathcal{O}_{\mathcal{Z}}) = r$ by $V^{[r]}$. Since we are working in characteristic zero, we have $\mathrm{length}(\mathcal{O}_{\mathcal{Z}}) = h^0(\mathcal{O}_{\mathcal{Z}})$.

We say that a line bundle L on V is *k-very ample* if the restriction map , $(L) \to$, $(\mathcal{O}_{\mathcal{Z}}(L))$ is onto for any $\mathcal{Z} \in V^{[k+1]}$. Note that L is 0-very ample if and only if L is spanned by global sections, and L is 1-very ample if and only if L is very ample. Note also that for smooth surfaces with $k \leq 2$, L being k-very ample is equivalent to L being *k-spanned* in the sense of [2], i.e., , (L) surjects on , $(\mathcal{O}_{\mathcal{Z}}(L))$ for any *curvilinear* 0-cycle $\mathcal{Z} \in V^{[k+1]}$, i.e., any 0-dimensional subscheme, $\mathcal{Z} \subset V$, such that $\mathrm{length}(\mathcal{O}_{\mathcal{Z}}) = k+1$ and $\mathcal{Z} \subset C$ for some smooth curve C on V ([2, (0.4), (3.1)]).

Let $x_1, \ldots, x_r$ be r distinct points on V. Let $\mathfrak{m}_i$ be the maximal ideal sheaves of the points $x_i \in V$, $i = 1, \ldots, r$. Note that the stalk of $\mathfrak{m}_i$ at x_i is nothing but the maximal ideal, $\mathfrak{m}_i\mathcal{O}_{V,x_i}$, of the local ring $\mathcal{O}_{V,x_i}$, $i = 1, \dots, r$. Consider the 0-cycle $\mathcal{Z} = x_1 + \cdots + x_r$. We say that L is *k-jet ample at* $\mathcal{Z}$ if, for every r-tuple $(k_1, \dots, k_r)$ of positive integers such that $\sum_{i=1}^{r} k_i = k + 1$, the restriction map

$$,\ (L) \to\ ,\ (L \otimes (\mathcal{O}_V / \otimes_{i=1}^{r} \mathfrak{m}_i^{k_i})) \left(\cong \oplus_{i=1}^{r}\ ,\ (L \otimes (\mathcal{O}_V/\mathfrak{m}_i^{k_i})\right)$$

is onto. Here $\mathfrak{m}_i^{k_i}$ denotes the k_i-th tensor power of $\mathfrak{m}_i$.

We say that L is *k-jet ample* if, for any $r \geq 1$ and any 0-cycle $\mathcal{Z} = x_1 + \cdots + x_r$, where $x_1, \dots, x_r$ are r distinct points on V, the line bundle L is k-jet ample at $\mathcal{Z}$.

Note that L is 0-jet ample if and only if L is spanned by its global sections and L is 1-jet ample if and only if L is very ample.

Note also that if L is k-jet ample, then L is k-very ample (see [7, (2.2)] and compare also with (5.2)).

We will use over and over through the paper the fact [6, (1.3)], that if L is a k-very ample line bundle on V, then $L \cdot C \geq k$ for each irreducible curve C on V.

We refer to [4], [5] and [6], and [7] for more on k-spannedness, k-very ampleness and k-jet ampleness respectively.

Definition **1.6** Let $p : X \to Y$ be a holomorphic map between complex projective schemes. Let $\mathcal{Z}$ be a 0-dimensional subscheme of X defined by the ideal sheaf $\mathcal{J}_{\mathcal{Z}}$. Then the *image* $p(\mathcal{Z})$ of $\mathcal{Z}$ is the 0-dimensional subscheme of Y whose defining ideal is $\mathcal{I} = \{g \in \mathcal{O}_Y \mid g \circ p \in \mathcal{J}_{\mathcal{Z}}\}$.

We need the following general fact.

Lemma **1.7** *Let $p : X \to Y$ be a morphism of quasiprojective varieties X, Y. Let $\mathcal{Z}$ be a* 0*-dimensional subscheme of X of length k. Then $p(\mathcal{Z})$ has length $\leq k$.*

Proof. Let $x_1, \dots, x_t$ be t distinct points such that $\mathrm{Supp}(\mathcal{Z}) = \{x_1, \dots, x_t\}$. Let $\mathrm{length}(\mathcal{O}_{\mathcal{Z}}) = k$ and $\mathrm{length}(\mathcal{O}_{\mathcal{Z},x_i}) = k_i$, where $\mathcal{O}_{\mathcal{Z},x_i}$ denotes the stalk of $\mathcal{O}_{\mathcal{Z}}$ at x_i, $i = 1, \dots, t$. Then $k = \sum_{i=1}^{t} k_i$. Set $\mathcal{Z}' := p(\mathcal{Z})$ and let $\mathcal{J}_{\mathcal{Z}}$, $\mathcal{J}_{\mathcal{Z}'}$ be the ideal sheaves of $\mathcal{Z}$, $\mathcal{Z}'$ respectively.

Arguing by contradiction, assume that $\mathrm{length}(\mathcal{O}_{\mathcal{Z}'}) > k$. Then there are $k+1$ linearly independent functions, $g_0 = 1, g_1, \dots, g_k$, in $\mathcal{O}_{\mathcal{Z}'}$. Let $y_1, \dots, y_s$, $s \leq t$, be the images of the points x_i, $i = 1, \dots, t$. For each $j = 1, \dots, s$, consider the vector subspace of $\mathbb{C}^{k+1}$ defined by

$$V_j := \{(\lambda_0, \dots, \lambda_k) \in \mathbb{C}^{k+1} \mid \sum_{i=0}^{k} \lambda_i g_i \in \mathfrak{m}_j\},$$

where $\mathfrak{m}_j$ denote the ideal sheaf of y_j. Since

$$\dim V_j \geq k + 1 - k_j,\ j = 1, \dots, s, \quad \text{and} \quad \sum_{j=1}^{s} k_j \leq k < k + 1$$

we conclude that

$$\dim \cap_{j=1}^{s} V_j \geq \sum_{j=1}^{s} \dim V_j - (s-1)(k+1) \geq k + 1 - \sum_{j=1}^{s} k_j \geq 1.$$

Thus, there exist $\lambda_0, \ldots, \lambda_k \in \mathbb{C}$ such that $\sum_{i=0}^k \lambda_i g_i = 0$ at y_j for each $j = 1, \ldots, s$. Thus

$$p^*\left(\sum_{i=0}^k \lambda_i g_i\right) = \sum_{i=0}^k \lambda_i p^* g_i = 0$$

at x_i for each $i = 1, \ldots, t$. It follows that $\sum_{i=0}^k \lambda_i p^* g_i \in \mathcal{J}_{\mathcal{Z}}$ and hence $\sum_{i=0}^k \lambda_i g_i \in \mathcal{J}_{\mathcal{Z}'}$, or $\sum_{i=0}^k \lambda_i g_i = 0$ in $\mathcal{O}_{\mathcal{Z}'}$. This contradicts the assumption that $1, g_1, \ldots, g_k$ are linearly independent. Q.E.D.

If X_1, X_2 are projective schemes and $\mathcal{F}_1$, $\mathcal{F}_2$ are sheaves on X_1, X_2 respectively, we will denote

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 := p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2,$$

where p_1, p_2 are the projections on the two factors.

The following is the k-very ample version of Lemma (3.2) of [9].

LEMMA **1.8** *Let X_1, X_2 be complex projective schemes and L_1, L_2 line bundles on X_1, X_2 respectively. For $i = 1, 2$ assume that L_i is k_i-very ample and let $k := \min\{k_1, k_2\}$. Then $L_1 \boxtimes L_2$ is k-very ample on $X_1 \boxtimes X_2$. Furthermore if $L_1 \boxtimes L_2$ is k'-very ample on $X_1 \boxtimes X_2$ then L_1, L_2 are k'-very ample.*

Proof. Let $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ be a 0-dimensional subscheme of length $k+1$ on $X_1 \times X_2$. Let $p_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, the projections on the two factors. Let $\mathcal{Z}_i := p_i(\mathcal{Z})$ be the 0-dimensional subschemes of X_i obtained as images of $\mathcal{Z}$, as in (1.6), and let $\mathcal{I}_{\mathcal{Z}_i}$ be the ideal sheaves defining $\mathcal{Z}_i$, $i = 1, 2$. Let $J_i := p_i^* \mathcal{I}_{\mathcal{Z}_i}$, $i = 1, 2$. There exist generating function germs $f \in J_i$ of type $f = g \circ p_i$, $g \in \mathcal{I}_{\mathcal{Z}_i}$, $i = 1, 2$. Therefore for each point $z \in \mathcal{Z}_{\mathrm{red}}$, $f(z) = g(p_i(z)) = 0$, $i = 1, 2$. This means that z belongs to the subscheme of $X_1 \times X_2$ defined by the ideal sheaf of the image of $J_1 \boxtimes J_2$ in $\mathcal{O}_{X_1 \times X_2}$. Hence we have an inclusion of ideal sheaves $(J_1, J_2) \subset \mathcal{I}_{\mathcal{Z}}$. But (J_1, J_2) defines the subscheme $\mathcal{J}$ whose structural sheaf is

$$\mathcal{O}_{\mathcal{J}} = p_1^* \mathcal{O}_{\mathcal{Z}_1} \otimes p_2^* \mathcal{O}_{\mathcal{Z}_2} = \mathcal{O}_{\mathcal{Z}_1} \boxtimes \mathcal{O}_{\mathcal{Z}_2}.$$

Thus we have a surjection

$$\mathcal{O}_{\mathcal{Z}_1} \boxtimes \mathcal{O}_{\mathcal{Z}_2} \to \mathcal{O}_{\mathcal{Z}} \to 0. \tag{1}$$

On the other hand, by the Kunneth formula, we have

$$H^0(L_1 \boxtimes L_2) = H^0(p_1^* L_1 \otimes p_2^* L_2) = H^0(L_1) \otimes H^0(L_2), \tag{2}$$

as well as,

$$\begin{aligned} H^0(L_1 \boxtimes L_2 \otimes \mathcal{O}_{\mathcal{Z}_1} \boxtimes \mathcal{O}_{\mathcal{Z}_2}) &= H^0(\mathcal{O}_{\mathcal{Z}_1}(L_1) \boxtimes \mathcal{O}_{\mathcal{Z}_2}(L_2)) \\ &= H^0(\mathcal{O}_{\mathcal{Z}_1}(L_1)) \otimes H^0(\mathcal{O}_{\mathcal{Z}_2}(L_2)). \end{aligned}$$

Therefore, from (1) and (2), we get a surjection

$$H^0(\mathcal{O}_{\mathcal{Z}_1}(L_1)) \otimes H^0(\mathcal{O}_{\mathcal{Z}_2}(L_2)) \to H^0(\mathcal{O}_{\mathcal{Z}}(L_1 \boxtimes L_2)). \tag{3}$$

By Lemma (1.7) we have that $\mathcal{Z}_1$ is of length $\leq k+1 \leq k_1+1$. Therefore, since L_1 is k_1-very ample, the restriction map $H^0(L_1) \to H^0(\mathcal{O}_{\mathcal{Z}_1}(L_1))$ is onto. Similarly we have that $H^0(L_2) \to H^0(\mathcal{O}_{\mathcal{Z}_2}(L_2))$ is onto. Thus by using (2) and (3) we get a surjection

$$H^0(L_1 \boxtimes L_2) \to H^0(\mathcal{O}_{\mathcal{Z}}(L_1 \boxtimes L_2)) \to 0.$$

This shows that $L_1 \boxtimes L_2$ is k-very ample.

To show the last part of the statement note that $L_1 \boxtimes L_{2|X_1 \times x_2} \cong L_1$ for each $x_2 \in X_2$. Then L_1 is k'-very ample if $L_1 \boxtimes L_2$ is k'-very ample. Similarly for L_2. Q.E.D.

The following result is a useful partial generalization of [6, Lemma 1.1] (compare also with [7, (3.1)]).

LEMMA **1.9** *Let L denote a k-very ample line bundle on an n-dimensional projective manifold X. Assume that $k \geq 2$ and let $\pi : \widehat{X} \to X$ denote the blowup of X at a finite set $\{x_1, \ldots, x_{k-1}\} \subset X$ of distinct points. Let $E_i := \pi^{-1}(x_i)$ for $1 \leq i \leq k-1$. Then $\pi^*L - (E_1 + \cdots + E_{k-1})$ is very ample.*

Proof. Assume that $h^0(L) = N+1$, and we use $|L|$ to embed X into $\mathbb{P}^N$. Consider a linear subspace $\mathbb{P}^t \subset \mathbb{P}^N$, where $t \leq k-1$. Assume that $\mathbb{P}^t$ met X in a positive dimensional set. Then we can assume without loss of generality that it is a curve. If not we can choose a $\mathbb{P}^{t-1}$ contained in the $\mathbb{P}^t$ which will meet X in a set of dimension at most one less. Clearly $t > 1$ since otherwise $\mathbb{P}^t$ would be a line contained in X, contradicting the k-very ampleness assumption with $k \geq 2$. Choose a $\mathbb{P}^{t-1}$ in the $\mathbb{P}^t$ meeting X in a finite set. This is possible since $\mathbb{P}^t$ meets X in some points and a curve, say C. By using the very ampleness assumption we conclude that the $\mathbb{P}^{t-1}$ meets the curve C in at most t points. Choose a hyperplane $H \subset \mathbb{P}^N$ meeting the $\mathbb{P}^t$ in the $\mathbb{P}^{t-1}$. Then H meets the curve C in at most $t \leq k-1$ points. Since H restricts to L on X, this contradicts the fact that $L \cdot C \geq k$. Therefore we conclude that any $\mathbb{P}^t \subset \mathbb{P}^N$ with $t \leq k-1$ meets X scheme theoretically in a 0-dimensional subscheme, say $\mathcal{X}$. Furthermore length$(\mathcal{O}_{\mathcal{X}}) \leq t+1$. Indeed otherwise $X \cap \mathbb{P}^t$ would contain a 0-cycle of length $t+2 \leq k+1$ which spans a $\mathbb{P}^{t+1}$ since L is k-very ample.

Thus by taking the projective space $P := \mathbb{P}^{k-2}$ generated by $\{x_1, \ldots, x_{k-1}\}$, we see that this $\mathbb{P}^{k-2}$ meets X in precisely $\{x_1, \ldots, x_{k-1}\}$. Thus the blowup $\sigma : Z \to \mathbb{P}^N$ of $\mathbb{P}^N$ at this P has $\widehat{X}$ as the proper transform of X and the induced map from $\widehat{X}$ to X is π. Since $\sigma^*\mathcal{O}_{\mathbb{P}^N}(1) - \sigma^{-1}(P)$ is spanned by global sections and $\widehat{L} := \pi^*L - (E_1 + \cdots + E_{k-1})$ is the pullback to $\widehat{X}$ of $\sigma^*\mathcal{O}_{\mathbb{P}^N}(1) - \sigma^{-1}(P)$ we see that $\widehat{L}$ is spanned by global sections. Thus we conclude that global sections of $\widehat{L}$ give a map $\phi : \widehat{X} \to \mathbb{P}^{N-k+1}$.

Applying the fact that the $\mathbb{P}^{k-1}$ generated by the image in $\mathbb{P}^N$ of any 0-dimensional subscheme $\mathcal{Z} \subset X$ of length k meets X scheme theoretically in $\mathcal{Z}$, it follows that the rational map from $\widehat{X}$ to $\mathbb{P}^{N-k+1}$ induced by the projection of $\mathbb{P}^N$ from P is one-to-one on $\widehat{X}$.

To see that $\widehat{L}$ is very ample we consider the points $x \in \widehat{X} \setminus \cup_{i=1}^{k-1} E_i$ and the points $x \in \cup_{i=1}^{k-1} E_i$ separately.

First assume that $x \in \widehat{X} \setminus \cup_{i=1}^{k-1} E_i$. Choose a tangent vector τ_x at x. Let $u_1, \ldots, u_n$ be a choice of local coordinates defined in a neighborhood of x, all zero at x, and with τ_x tangent to the u_n axis. Let $\mathfrak{a}_x$ denote the ideal sheaf which equals $\mathcal{O}_X$ away from x, and at x is defined by $(u_1, \ldots, u_{n-1}, u_n^2)$. We can choose a zero dimensional subscheme $\mathcal{Z}$ of X of length $k+1$ that is defined by $\mathfrak{a}_x \otimes \mathfrak{m}_{x_1} \otimes \cdots \otimes \mathfrak{m}_{x_{k-1}}$. Since $H^0(L) \to H^0(L \otimes \mathcal{O}_{\mathcal{Z}})$ is onto by the definition of k-very ampleness, we conclude that the map ϕ given by global sections of $\widehat{L}$ has nonzero differential evaluated on the tangent vector τ_x. It follows that the global sections of $\widehat{L}$ embed away from $\cup_{i=1}^{k-1} E_i$.

To finish consider a point $x \in \cup_{i=1}^{k-1} E_i$. Without loss of generality we can assume, by relabeling if necessary, that $x \in E_1$. We thus have $x_1 = \pi(x)$. Choose a tangent vector τ_x at x. Note that since the line bundle $\widehat{L}$ is spanned and since $\widehat{L}_{E_1} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, the restriction ϕ_{E_1} is an embedding. Thus we can assume that τ_x is not tangent to E_1. Let $\tau = d\pi_x(\tau_x) \in \mathcal{T}_{X,x_1}$, where $d\pi_x : \mathcal{T}_{\widehat{X},x} \to \mathcal{T}_{X,x_1}$ is the differential map and $\mathcal{T}_{\widehat{X},x}$, $\mathcal{T}_{X,x_1}$ are the tangent bundles to $\widehat{X}$, X respectively. Let $u_1, \ldots, u_n$ be a choice of local coordinates defined in a neighborhood of x_1, all zero at x_1, with τ tangent to the u_n axis, and such that the proper transform of the u_n axis is tangent to τ_x. Let $\mathfrak{b}_x$ denote the ideal sheaf which equals $\mathcal{O}_X$ away from x, and at x is defined by $(u_1, \ldots, u_{n-1}, u_n^3)$. We can choose a zero dimensional subscheme $\mathcal{Z}$ of X of length $k+1$ that is defined by $\mathfrak{b}_x$ if $k = 2$ and by $\mathfrak{b}_x \otimes \mathfrak{m}_{x_2} \otimes \cdots \otimes \mathfrak{m}_{x_{k-1}}$ if $k \geq 3$. Since $H^0(L) \to H^0(L \otimes \mathcal{O}_{\mathcal{Z}})$ is onto by the definition of k-very ampleness, we conclude that global sections of $\widehat{L}$ give a map with nonzero differential evaluated on the tangent vector τ_x. It follows that the global sections of $\widehat{L}$ embed $\widehat{X}$. Q.E.D.

Remark **1.10** There is a nice interpretation of this result in terms of $X^{[k]}$, the Hilbert scheme of 0-dimensional subschemes of X of length k. There is a natural line bundle $\mathcal{L}$ on $X^{[k]}$ induced by L on X. The fact that L is k-very ample is equivalent to $\mathcal{L}$ being very ample [10]. Given a set $\{x_1, \ldots, x_{k-1}\} \subset X$ of $k-1$ distinct points of X, the subscheme of all 0-dimensional subschemes of X of length k containing $\{x_1, \ldots, x_{k-1}\}$ is naturally isomorphic to X blown up at $\{x_1, \ldots, x_{k-1}\} \subset X$. Under this identification the very ample line bundle $\mathcal{L}$ restricts to the line bundle $\pi^*L - (E_1 + \cdots + E_{k-1})$ of the above lemma.

We have the following estimate for the degree and the number of sections of L.

Proposition **1.11** *Let L denote a k-very ample line bundle on an n-dimensional projective manifold X. Assume that $k \geq 2$. Then $L^n \geq 2^n + k - 2$ and $h^0(L) \geq 2n + k - 1$. Moreover $h^0(L) = 2n + k - 1$ implies that $L^n = 2^n + k - 2$.*

Proof. Let $\pi : \widehat{X} \to X$ denote the blowup of X at a finite set $\{x_1, \ldots, x_{k-1}\} \subset X$ of distinct points. Let $E_i := \pi^{-1}(x_i)$ for $1 \leq i \leq k-1$. By Lemma (1.9), $\pi^*L - (E_1 + \cdots + E_{k-1})$ is very ample, and thus $\pi^*L - (E_1 + \cdots + E_{k-1}) - E_1$ is spanned. This implies $(\pi^*L - (2E_1 + E_2 \cdots + E_{k-1}))^n \geq 0$ and thus that $L^n \geq 2^n + k - 2$ since $E_i^n = (-1)^n$, $i = 1, \ldots, k-1$. If $h^0(L) \leq 2n + k - 1$, then since $h^0(\widehat{L}) = h^0(L) - k + 1$, we have $h^0(\widehat{L}) \leq 2n$. The argument in [3, Theorem (4.4.1), i)] shows that $h^0(\widehat{L}) = 2n$, so that

$h^0(L) = 2n + k - 1$, and $\widehat{L}^n = 2^n - 1$. Since $\widehat{L}^n = L^n - k + 1$, we are done. Q.E.D.

2. Fano complete intersections. Our first result gives the k-jet ampleness, k-very ampleness, and k-spannedness of $-K_X$ for Fano manifolds $X \subset \mathbb{P}^N$ which are scheme theoretically complete intersections of hypersurfaces in $\mathbb{P}^N$. We say that a curve ℓ on X is a line if $\mathcal{O}_X(1) \cdot \ell = 1$.

THEOREM **2.1** *Let X be a positive dimensional connected projective submanifold of $\mathbb{P}^N$, which is a complete intersection of hypersurfaces of $\mathbb{P}^N$ of degree d_i, $i = 1, \dots, r := N - \dim X$. If the anticanonical bundle, $-K_X$ is ample and if X is not a degree 2 curve, then X contains a line. In particular $-K_X$ is $(N + 1 - \sum_{i=1}^r d_i)$-jet ample, but not $(N + 2 - \sum_{i=1}^r d_i)$-spanned.*

Proof. Since the curve case of this result is trivial, assume that $\dim X \geq 2$. Since X is a complete intersection, $K_X = \mathcal{O}_{\mathbb{P}^N}(-N - 1 + \sum_{i=1}^r d_i)$, where $d_1, \dots, d_r$ are the degrees of the hypersurfaces which intersect transversely in X. Since $-K_X$ is very ample, we conclude that $\sum_{i=1}^r d_i \leq N$, and $-K_X$ is $(N + 1 - \sum_{i=1}^r d_i)$-jet ample (see [7, Corollary (2.1)]). If we show that X contains a line ℓ it will follow that $-K_X \cdot \ell = N + 1 - \sum_{i=1}^r d_i$, and thus $-K_X$ is not $(N + 2 - \sum_{i=1}^r d_i)$-spanned. Thus we must only show that X contains a line.

Let G denote the Grassmannian $\mathrm{Gr}(2, N + 1)$ of 2-dimensional complex vector subspaces of $\mathbb{C}^{N+1}$. Let $\mathcal{F}$ denote the tautological rank 2 quotient bundle of $G \times \mathbb{C}^{N+1}$. Note that $G \times \mathbb{C}^{N+1}$ is naturally identified with $G \times H^0(\mathcal{O}_{\mathbb{P}^N}(1))$. Under this identification $\mathbb{P}(\mathcal{F}) \subset G \times \mathbb{P}^N$ is identified with the universal family of linear $\mathbb{P}^1$'s contained in $\mathbb{P}^N$. The image in $s' \in H^0(\mathcal{F})$ of a section s of $\mathcal{O}_{\mathbb{P}^N}(1)$ vanishes at points of G corresponding to lines in $s^{-1}(0)$. Further a section s of $\mathcal{O}_{\mathbb{P}^N}(d)$ maps naturally to a section s' of $\mathcal{F}^{(d)}$, the d-th symmetric tensor product $\mathcal{F}$. Here s' vanishes at points of G corresponding to the lines contained in $s^{-1}(0)$. Thus if X is defined by sections $s_1, \dots, s_r$ of $\mathcal{O}_{\mathbb{P}^N}(d_1), \dots, \mathcal{O}_{\mathbb{P}^N}(d_r)$, the lines on X correspond to the common zeroes of the images, $s'_1, \dots, s'_r$, in $\mathcal{F}^{(d_1)}, \dots, \mathcal{F}^{(d_r)}$. Thus if we show that

$$c_{\mathrm{rank}\mathcal{F}^{(d_1)}}(\mathcal{F}^{(d_1)}) \wedge \cdots \wedge c_{\mathrm{rank}\mathcal{F}^{(d_r)}}(\mathcal{F}^{(d_r)})$$

is a nontrivial cohomology class, then it follows that $s'_1, \dots, s'_r$ must have common zeroes and X must contain lines.

Note that $\mathrm{rank}\mathcal{F}^{(d_i)} = d_i + 1$. For odd d_i we have

$$c_{d_i+1}(\mathcal{F}^{(d_i)}) = (d_i + 1)^2 c_2(\mathcal{F}) \prod_{t=1}^{\frac{d_i-1}{2}} \left(t(d_i - t)c_1^2(\mathcal{F}) + (d_i - 2t)^2 c_2(\mathcal{F})\right) \tag{4}$$

and for even d_i we have

$$c_{d_i+1}(\mathcal{F}^{(d_i)}) = (d_i + 1)^2 c_2(\mathcal{F}) \frac{d_i}{2} c_1(\mathcal{F}) \prod_{t=1}^{\frac{d_i}{2}-1} \left(t(d_i - t)c_1^2(\mathcal{F}) + (d_i - 2t)^2 c_2(\mathcal{F})\right). \tag{5}$$

Since $\mathcal{F}$ is spanned both $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$ are semipositive classes (see [12, Example (12.1.7)]). Therefore, since all the monomial terms in the above formulae (4) and (5) have positive coefficients, we have that $c_{d_1+1}(\mathcal{F}^{(d_1)}) \wedge \cdots \wedge c_{d_r+1}(\mathcal{F}^{(d_r)})$ is not zero if

$$\left(c_2(\mathcal{F}) \wedge c_1^{d_1-1}(\mathcal{F})\right) \wedge \cdots \wedge \left(c_2(\mathcal{F}) \wedge c_1^{d_r-1}(\mathcal{F})\right)$$

is not zero. Since the zero set of r general sections of $\mathcal{F}$ vanishes on a subgrassmannian $G' := \mathrm{Gr}\,(2, N+1-r) \subset G$ corresponding to an inclusion $\mathbb{C}^{N+1-r} \subset \mathbb{C}^{N+1}$ of vector spaces, this is the same as showing that

$$c_1(\mathcal{F}_{G'})^{\sum_{i=1}^{r}(d_i-1)}$$

is a nonzero cohomology class. Since $\det \mathcal{F}$ is the very ample line bundle on G which gives the Plücker embedding, we see that this is nonzero if $-r + \sum_{i=1}^{r} d_i \leq \dim G' = 2(N-1-r)$, i.e., if $\sum_{i=1}^{r} d_i \leq 2N-2-r$. If this is false, then since $\sum_{i=1}^{r} d_i \leq N$, we conclude that $N > 2N-2-r$ which implies that $r+1 \geq N$. Since $r = N - \dim X$ this implies that X is a curve. Q.E.D.

See [16] for some related results about hypersurfaces in $\mathbb{P}^1$-bundles over Grassmannians.

Remark **2.2** We follow the notation used in Theorem 2.1 and its proof. If $N > \sum_{i=1}^{r} d_i$ then the argument of Barth and Van de Ven [1] applies to show that there is a family of lines covering X with an $(N - \sum_{i=1}^{r} d_i - 1)$-dimensional space of lines through a general point.

3. Adjunction structure in case of k-very ampleness. Let $\widehat{X}$ be a smooth connected n-fold, $n \geq 3$, and let $\hat{L}$ be a k-very ample line bundle on $\widehat{X}$, $k \geq 2$. In this section we describe the first and the second reduction of $(\widehat{X}, \hat{L})$.

We have the following general fact.

Lemma **3.1** *Let $\widehat{X}$ be a smooth connected n-fold, $n \geq 3$, and let $\hat{L}$ be a k-very ample line bundle on $\widehat{X}$, $k \geq 2$. Let τ be the nefvalue of $(\widehat{X}, \hat{L})$. Then $\tau \leq \frac{n+1}{k}$.*

Proof. Let $\Phi : \widehat{X} \to W$ be the nefvalue morphism of $(\widehat{X}, \hat{L})$. Let C be an extremal curve contracted by Φ. Then $(K_{\widehat{X}} + \tau\hat{L}) \cdot C = 0$ yields $k\tau \leq \tau\hat{L} \cdot C = -K_{\widehat{X}} \cdot C \leq n+1$, or $\tau \leq \frac{n+1}{k}$. Q.E.D.

We can now prove the following structure result.

Theorem **3.2** *Let $\widehat{X}$ be a smooth connected n-fold, $n \geq 3$, and let $\hat{L}$ be a k-very ample line bundle on $\widehat{X}$, $k \geq 2$. Then either $(\widehat{X}, \hat{L}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ with $k = 2$, or the first reduction (X, L) of $(\widehat{X}, \hat{L})$ exists and $\widehat{X} \cong X$, $L \cong \hat{L}$. Furthermore either:*

i) *$(\widehat{X}, \hat{L}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$, $k = 2$;*
ii) *$(\widehat{X}, \hat{L}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$, $k = 3$;*
iii) *$(\widehat{X}, \hat{L}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$, $k = 2$;*
iv) *$(\widehat{X}, \hat{L}) \cong (Q, \mathcal{O}_Q(2))$, Q hyperquadric in $\mathbb{P}^4$, $k = 2$;*
v) *there exists a morphism $\psi : \widehat{X} \to C$ onto a smooth curve C such that $2K_{\widehat{X}} + 3\hat{L} \approx \psi^* H$ for some ample line bundle H on C and $(F, \hat{L}_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for any fiber F of ψ, $k = 2$;*
vi) *$(\widehat{X}, \hat{L})$ is a Mukai variety, i.e., $K_{\widehat{X}} \approx -(n-2)\hat{L}$ and either $n = 4, 5$ and $k = 2$ or $n = 3$ and $k \leq 4$;*

vii) $(\widehat{X},\widehat{L})$ *is a Del Pezzo fibration over a curve such that* $(F,\widehat{L}_F)\cong(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))$ *for each general fiber* F, $n=4$, $k=2$,

or there exists the second reduction $(Z,\mathcal{D})$, $\varphi: X\to Z$ *of* $(\widehat{X},\widehat{L})$. *In this case the following hold.*

1) *If* $n\geq 4$, *then* $X\cong Z$;
2) *If* $n=3$, *then either* $X\cong Z$ *or* $k=2$ *and* φ *only contracts divisors* $D\cong\mathbb{P}^2$ *such that* $L_D\cong\mathcal{O}_{\mathbb{P}^2}(2)$; *furthermore* $\mathcal{O}_D(D)\cong\mathcal{O}_D(-1)$ *and* Z *is smooth.*

Proof. We use general results from adjunction theory for which we refer to [8]. From [8, (9.2.2)] we know that $K_{\widehat{X}}+(n-1)\widehat{L}$ is spanned unless either $(\widehat{X},\widehat{L})\cong(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1))$, or $\widehat{X}\subset\mathbb{P}^{n+1}$ is a quadric hypersurface and $L\approx\mathcal{O}_{\mathbb{P}^{n+1}}(1)_{|\widehat{X}}$, or $(\widehat{X},\widehat{L})$ is a scroll over a curve. Since $L\cdot C\geq 2$ for each curve C on $\widehat{X}$, all the above cases are excluded.

Thus we can conclude that $K_{\widehat{X}}+(n-1)\widehat{L}$ is spanned. Then from [8, (7.3.2)] we know that $K_{\widehat{X}}+(n-1)\widehat{L}$ is big unless either $(\widehat{X},\widehat{L})$ is a Del Pezzo variety, i.e., $K_{\widehat{X}}\approx-(n-1)\widehat{L}$, or $(\widehat{X},\widehat{L})$ is a quadric fibration over a smooth curve, or $(\widehat{X},\widehat{L})$ is a scroll over a normal surface. Then, as above, the quadric fibration and the scroll cases are excluded, so that $(\widehat{X},\widehat{L})$ is a Del Pezzo variety. In this case $\tau=n-1$, so that Lemma (3.1) gives $2\leq k\leq\frac{n+1}{n-1}$. Hence $n=3$. By looking over the list of Del Pezzo 3-folds (see [14, (8.11)]) we conclude that $(\widehat{X},\widehat{L})\cong(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))$ in this case.

Thus we can assume that the first reduction, (X,L), of $(\widehat{X},\widehat{L})$ exists and in fact $\widehat{X}\cong X$, since otherwise we can find a line ℓ on $\widehat{X}$ such that $\widehat{L}\cdot\ell=1$. From [8, (7.3.4), (7.3.5), (7.5.3)]) we know that on $\widehat{X}\cong X$ the line bundle $K_X+(n-2)L$ is nef and big unless either

a) $(X,L)\cong(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(3))$;
b) $(X,L)\cong(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(2))$;
c) $(X,L)\cong(Q,\mathcal{O}_Q(2))$, Q hyperquadric in $\mathbb{P}^4$;
d) there exists a morphism $\psi: X\to C$ onto a smooth curve C such that $2K_X+3L\approx\psi^*H$ for some ample line bundle H on C and $(F,L_F)\cong(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(2))$ for any fiber F of ψ;
e) $K_X\approx-(n-2)L$, i.e., (X,L) is a Mukai variety;
f) (X,L) is a Del Pezzo fibration over a smooth curve under the morphism, Φ_L, given by $|m(K_X+(n-2)L)|$ for $m\gg 0$;
g) (X,L) is a quadric fibration over a normal surface under Φ_L; or
h) (X,L) is a scroll over a normal threefold under Φ_L.

Cases a), b), c), d), e) lead to cases ii), iii), iv), v), vi) respectively.

Case f) gives case vii). To see this, let F be a general fiber of Φ_L and let L_F be the restriction of L to F. Let τ_F be the nefvalue of (F,L_F). Then $K_F+(n-2)L_F$ is trivial, and hence $\tau_F=n-2=\dim F-1$. Therefore the same argument as above, by using again [14, (8.11)], gives $\dim F=3$ and $(F,L_F)\cong(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))$.

Note that in case e), one has $\tau=n-2$, so that Lemma (3.1) yields $2\leq k\leq\frac{n+1}{n-2}$. Thus either $n=4,5$ and $k=2$, or $n=3$ and $k\leq 4$.

In cases g), h) we can find a line ℓ on X such that $L\cdot\ell=1$, so that they are excluded.

Thus we can assume that the second reduction, $(Z, \mathcal{D})$, $\varphi : X \to Z$, of $(\widehat{X}, \widehat{L})$ exists. Use the structure results of the second reduction (see [8, (7.5.3), (12.2.1)]). If $n \geq 4$ we see that we can always find a line ℓ on X such that $L \cdot \ell = 1$. Then $X \cong Z$. If $n = 3$, either $X \cong Z$ or φ contracts divisors $D \cong \mathbb{P}^2$ such that $L_D \cong \mathcal{O}_{\mathbb{P}^2}(2)$. Then $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1)$ and Z is smooth in this case. Q.E.D.

The following is an immediate consequence of Theorem (3.2).

Corollary **3.3** *Let $\widehat{X}$ be a smooth connected n-fold, $n \geq 3$, and let $\widehat{L}$ be a k-very ample line bundle on $\widehat{X}$, $k \geq 2$. Then $K_{\widehat{X}} + (n-2)\widehat{L}$ is ample if $n \geq 4$ unless $k = 2$ and either $(\widehat{X}, \widehat{L}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$, or $n = 4, 5$ and $(\widehat{X}, \widehat{L})$ is a Mukai variety, or $n = 4$ and $(\widehat{X}, \widehat{L})$ is a Del Pezzo fibration over a curve such that $(F, \widehat{L}_F) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ for each fiber F.*

The results of this section justify the study of Mukai varieties of dimension $n = 3, 4, 5$, polarized by a k-very ample line bundle, $k \geq 2$.

4. Mukai varieties of dimension $n \geq 4$**.** Let (X, L) be a Mukai variety of dimension $n \geq 3$, i.e., $K_X \approx -(n-2)L$, polarized by a k-very ample line bundle L, $k \geq 2$ (see [18], [19] for classification results of Mukai varieties). Since the nefvalue, τ, of such pairs (X, L) is $\tau = n - 2$, we immediately see from Lemma (3.1) that either $n = 4, 5$, $k = 2$, or $n = 3$, $k \leq 4$ (compare with the proof of (3.2)). We have the following result.

Theorem **4.1** *Let (X, L) be a Mukai variety of dimension $n \geq 4$ polarized by a k-very ample line bundle L, $k \geq 2$. Then either*

1. *$n = 4$, $k = 2$, $(X, L) \cong (Q, \mathcal{O}_Q(2))$, Q hyperquadric in $\mathbb{P}^5$, or*
2. *$n = 5$, $k = 2$, $(X, L) \cong (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$.*

Proof. By the above we know that $k = 2$ and $n = 4, 5$. Let V be the 3-fold section obtained as transversal intersection of $n - 3$ general members of $|L|$. Let L_V be the restriction of L to V. Note that $K_V \approx -L_V$, so that V is a Fano 3-fold, and L_V is k-very ample.

We denote by r the index of V. Then $L_V \approx -K_V = rH$ for some ample line bundle H on V. Note that we have $r \geq 2$ since otherwise Shokurov's theorem [23] (see also [21]) applies to say that either $V = \mathbb{P}^1 \times \mathbb{P}^2$ or there exists a line ℓ on V with respect to H. In the latter case $(H \cdot \ell)_V = (L \cdot \ell)_X = 1$, which contradicts the assumption $k \geq 2$. The following argument rules out the former case $V = \mathbb{P}^1 \times \mathbb{P}^2$.

If $V = \mathbb{P}^1 \times \mathbb{P}^2$, as corollary of the extension theorem [24, Prop. III] we conclude that X is a linear $\mathbb{P}^3$-bundle over $\mathbb{P}^1$, $p : X \to \mathbb{P}^1$, with the restriction p_V giving the map $V = \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$. By taking the direct image of

$$0 \to \mathcal{O}_X \to L \to L_V \to 0$$

we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E} \to \mathcal{E}/\mathcal{O}_{\mathbb{P}^1} \to 0,$$

where $\mathcal{E} := p_* L$. Since $\mathbb{P}(\mathcal{E}/\mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}^1 \times \mathbb{P}^2$ we conclude that $\mathcal{E}/\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Thus $\deg(\mathcal{E}/\mathcal{O}_{\mathbb{P}^1}) = 3 = \deg(\mathcal{E}) < \operatorname{rank}(\mathcal{E}) = 4$. Therefore L cannot be ample.

By Lefschetz theorem we have $H \approx H'_V$ for some line bundle H' on X, as well as $L \approx rH'$. Hence in particular H' is ample. We have $K_X + r(n-2)H' \approx \mathcal{O}_X$, so that $r(n-2) \leq n+1$ by a well known result due to Maeda (see e.g., [8, (7.2.1)]). If $r = 4, 3$ we find numerical contradictions since we are assuming $n \geq 4$. Thus $r = 2$ and $n \leq 5$.

If $n = 5$ we have $K_X \approx -6H'$ and if $n = 4$ we have $K_X \approx -4H'$. By the Kobayashi-Ochiai theorem (see e.g., [8, (3.6.10)]) we get in the former case $(X, H') \cong (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$, or $(X, L) \cong (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$ as in case 1) of the statement, and in the latter case $(X, H') \cong (Q, \mathcal{O}_Q(1))$, Q hyperquadric in $\mathbb{P}^5$, or $(X, L) \cong (Q, \mathcal{O}_Q(2))$ as in case 2) of the statement.

Q.E.D.

Remark **4.2** Note that in both cases 1), 2) of Theorem (4.1) the line bundle L is in fact 2-jet ample (see [7, Corollary (2.1)]).

5. The Fano 3-fold case. In this section we classify the 3-dimensional Mukai varieties (X, L) polarized by a k-very ample line bundle L, $k \geq 2$, i.e., we classify all Fano 3-folds X such that the anticanonical divisor $-K_X$ is k-very ample, $k \geq 2$.

5.1 A special case. Let us start by studying a particular case. This case has a special interest also because it gives a simple explicit example of a line bundle which is 2-very ample but not 2-jet ample. This example is case 4) in the Iskovskih-Shokurov's list [15, Table 21] of Fano 3-folds of first species, i.e., $b_2(X) = 1$.

Proposition **5.2** *Let X be a smooth double cover of $\mathbb{P}^3$, $p : X \to \mathbb{P}^3$, branched along a quartic. Then $L := -K_X$ is 2-very ample but not 2-jet ample.*

Proof. We have $L := -K_X = p^*\mathcal{O}_{\mathbb{P}^3}(2)$. First we show that L is not 2-jet ample. Let R be the ramification divisor of p. Let $\mathcal{Z}$ be a length 3 zero dimensional subscheme of X such that $\mathrm{Supp}(\mathcal{Z}) = \{x\}$ with $x \in R$. We can assume that R is defined at x by a local coordinate, s, i.e., $R = \{s = 0\}$ at x. Consider local coordinates (s, v, w) on X at x. Let $y \in \mathbb{P}^3$ be a point, belonging to the branch locus of p, such that $y = p(x)$. We can consider local coordinates (t, v, w) on $\mathbb{P}^3$ at y, where $p^*t = s^2$. We have

$$H^0(L) \cong H^0(p_*p^*\mathcal{O}_{\mathbb{P}^3}(2)) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \oplus H^0(\mathcal{O}_{\mathbb{P}^3}). \tag{6}$$

Therefore we can find a base, $\mathcal{B}$, of $H^0(L)$ given by the pullback of sections of $\mathcal{O}_{\mathbb{P}^3}(2)$ and one more section, $\sigma \in H^0(\mathcal{O}_{\mathbb{P}^3})$, which in local coordinates around x is of the form λs with λ a holomorphic function that doesn't vanish at x. That is, recalling that $s^2 = p^*t$, $v = p^*v$, $w = p^*w$,

$$\mathcal{B} =< 1, s^2, v, w, s^4, v^2, w^2, s^2v, s^2w, vw, \sigma > .$$

On the other hand, $H^0(L/\mathfrak{m}_x^3)(= H^0(\mathcal{O}_{\mathcal{Z}}(L)))$ contains the elements sv, sw which are not images of elements of the base $\mathcal{B}$. This shows that the restriction map $H^0(L) \to H^0(\mathcal{O}_{\mathcal{Z}}(L))$ is not surjective. Thus L is not 2-jet ample.

We prove now that L is 2-very ample. Consider a 0-dimensional subscheme $\mathcal{Z}$ of X of length 3. Recalling (6), the fact that $\mathcal{O}_{\mathbb{P}^3}(2)$ is 2-very ample and Lemma (1.7), we see that the restriction map $H^0(L) \to H^0(\mathcal{O}_{\mathcal{Z}}(L))$ is always surjective except possibly in the case when $\mathrm{Supp}(\mathcal{Z})$ is a single point, x, belonging to the ramification divisor of the cyclic covering p.

Thus, let us assume $\mathrm{Supp}(\mathcal{Z}) = \{x\}$ and consider the ideals $\mathcal{J}_i := (\mathcal{J}_{\mathcal{Z}}, \mathfrak{m}_x^i)$, where $\mathcal{J}_{\mathcal{Z}}$, $\mathfrak{m}_x$ denote the ideal sheaves of $\mathcal{Z}$, x respectively, the sheaves $\mathcal{O}_i := \mathcal{O}_X/\mathcal{J}_i$, the maps $p_i : \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_i$, $i = 1, 2, 3$, and the cofiltration $\mathcal{O}_3 \to \mathcal{O}_2 \to \mathcal{O}_1 \to 0$. Note that the following hold true.

- If $H^0(\mathcal{O}_{\mathcal{Z}})$ is generated by only terms of degree ≤ 1 in the local coordinates s, v, w on X at the point x then (as observed before in the proof that L is not 2-jet ample) the image of $H^0(L)$ can generate $H^0(\mathcal{O}_{\mathcal{Z}}(L))$, so the restriction map $H^0(L) \to H^0(\mathcal{O}_{\mathcal{Z}}(L))$ is surjective in this case and we are done;
- $\mathrm{length}(\mathcal{O}_1) = 1$;
- $\mathrm{length}(\mathcal{O}_i) \neq \mathrm{length}(\mathcal{O}_{i+1})$, $i = 1, 2$. Indeed, otherwise, $(\mathcal{J}_{\mathcal{Z}}, \mathfrak{m}_x^i) = (\mathcal{J}_{\mathcal{Z}}, \mathfrak{m}_x^{i+1})$ so that $\mathfrak{m}_x^i \subset \mathcal{J}_{\mathcal{Z}}$ and therefore $H^0(\mathcal{O}_{\mathcal{Z}})$ is generated by only constant terms or linear terms. By the above we are done in this case.

Thus we are reduced to consider the case when $\mathrm{length}(\mathcal{O}_2) = 2$, $\mathrm{length}(\mathcal{O}_3) = 3$.

We claim that $H^0(\mathcal{O}_{\mathcal{Z}})$ contains at least one quadratic term in s, v, w. Indeed, if not, $\mathfrak{m}_x^2 \subset \mathcal{J}_{\mathcal{Z}}$ and hence we would have $(\mathcal{J}_{\mathcal{Z}}, \mathfrak{m}_x^2) = (\mathcal{J}_{\mathcal{Z}}, \mathfrak{m}_x^3)$, which gives the contradiction $\mathrm{length}(\mathcal{O}_2) = \mathrm{length}(\mathcal{O}_3)$.

Furthermore, since $\mathrm{length}(\mathcal{O}_X/(\mathcal{J}_{\mathcal{Z}}, \mathfrak{m}_x^2)) = 2$ and $\mathrm{length}(\mathcal{O}_X/\mathfrak{m}_x^2) = 4$, we conclude that $\mathcal{J}_{\mathcal{Z}}$ contains two independent linear terms, say f, g, not belonging to $\mathfrak{m}_x^2$. Write

$$f = as + bv + cw, \quad g = ds + ev + hw,$$

where the coefficients a, b, c, d, e, h belong to $\mathcal{O}_{X,x}$. Let $\mathcal{B}$ be the base of $H^0(L)$ constructed in the first part of the proof, where we showed that L is not 2-jet ample. Following that argument we see that L is 2-very ample as soon as we show that the elements sv, sw can be written in $\mathcal{O}_{\mathcal{Z}}$ in terms of elements of $\mathcal{B}$.

We go on by a case by case analysis. First, assume $a \neq 0$, i.e., a invertible in $\mathcal{O}_{X,x}$ and write

$$asw = w(as + bv + cw) - bvw - cw^2.$$

Since $as + bv + cw = f = 0$ in $\mathcal{O}_{\mathcal{Z}}$, up to dividing by a, we can express sw in terms of $vw, w^2 \in \mathcal{B}$ in $\mathcal{O}_{\mathcal{Z}}$. Similarly, writing

$$asv = v(as + bv + cw) - bv^2 - cvw,$$

we conclude that sv can be expressed in terms of $v^2, vw \in \mathcal{B}$ in $\mathcal{O}_{\mathcal{Z}}$. If $d \neq 0$ we get the same conclusion.

Thus it remains to consider the case when $a = d = 0$. In this case $f = bv + cw$, $g = ev + hw$ in $\mathcal{O}_{\mathcal{Z}}$. Then, solving with respect to v, w, and noting that $\begin{pmatrix} b & c \\ e & h \end{pmatrix}$ is a rank two matrix since f, g are independent, we can express v, w as linear functions of f, g in $\mathcal{O}_{\mathcal{Z}}$. Since $f, g \in \mathcal{J}_{\mathcal{Z}}$, we conclude that $v, w \in \mathcal{J}_{\mathcal{Z}}$ and hence sv, sw belong to $\mathcal{J}_{\mathcal{Z}}$. Therefore $sv = sw = 0$ in $\mathcal{O}_{\mathcal{Z}}$. Q.E.D.

The following general result is a consequence of Fujita's classification [13], [14, (8.11)] of Del Pezzo 3-folds (see also [17], [15] and [20] for a complete classification of Fano 3-folds).

Note that in each case of the theorem below the line bundle L is in fact k-very ample (see [7, Corollary (2.1)], Lemma (1.8) and Proposition (5.2).

THEOREM **5.3** *Let X be a Fano threefold. Assume that $L := -K_X$ is k-very ample, $k \geq 2$. Then either:*

1. *X is a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$, $L = \mathcal{O}_X(2,2)$, $k = 2$;*
2. *$X = \mathbb{P}^1 \times \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^2}(3)$, $k = 2$;*
3. *$X = V_7$, the blowing up of $\mathbb{P}^3$ at a point, $L = 2(q^*\mathcal{O}_{\mathbb{P}^3}(2) - E)$, $q : V_7 \to \mathbb{P}^3$, E the exceptional divisor, $k = 2$;*
4. *$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2)$, $k = 2$;*
5. *$X = \mathbb{P}^3$, $L = \mathcal{O}_{\mathbb{P}^3}(4)$, $k = 4$;*
6. *X is a hyperquadric in $\mathbb{P}^4$, $L = \mathcal{O}_X(3)$, $k = 3$;*
7. *X is a cubic hypersurface in $\mathbb{P}^4$, $L = \mathcal{O}_X(2)$, $k = 2$;*
8. *X is the complete intersection of two quadrics in $\mathbb{P}^5$, $L = \mathcal{O}_X(2)$, $k = 2$;*
9. *X is a double cover of $\mathbb{P}^3$, $p : X \to \mathbb{P}^3$, branched along a quartic; $L = p^*\mathcal{O}_{\mathbb{P}^3}(2)$ is 2-very ample but not 2-jet ample;*
10. *X is the section of the Grassmannian $\mathrm{Gr}\,(2,5)$ (of lines in $\mathbb{P}^4$) by a linear subspace of codimension 3, $L = \mathcal{O}_X(2)$, $k = 2$.*

Proof. Let r be the index of X. Then $L := -K_X = rH$ for some ample line bundle H on X. Note that we have $r \geq 2$ since otherwise Shokurov's theorem [23] applies to say that either $X = \mathbb{P}^1 \times \mathbb{P}^2$ or there exists a line ℓ with respect to H. In the former case we are in case 2) of the statement. In the latter case $H \cdot \ell = L \cdot \ell = 1$, which contradicts the assumption $k \geq 2$.

If $r = 4, 3$, by using the Kobayashi-Ochiai theorem (see e.g., [8, (3.6.1)]) we find cases 5), 6) of the statement respectively.

Thus we can assume $r = 2$. In this case (X, H) is a Del Pezzo 3-fold described as in [14, (8.11)]. A direct check shows that the cases listed in [14, (8.11)] lead to cases 1), 3), 4), 7), 8), 9), 10) of the statement. Recall that case 9) is discussed in Proposition (5.2).

Notice that the case of $X = \mathbb{P}(\mathcal{T})$, for the tangent bundle $\mathcal{T}$ of $\mathbb{P}^2$, as in [14, (8.11), 6)] gives our case 1) (see Remark (5.4) below). Note also that case 1) of [14, (8.11)], when (X, H) is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, \ldots, 1)$ with $H^3 = 1$, is ruled out since $L = 2H$ is not even very ample. To see this notice that there exist a smooth surface S in $|H|$ and a smooth curve C in $|H_S|$ (see [14, (6.1.3), (6.14)]). On S we have $K_S \approx -H_S$, so that $K_S^2 = 1$. Therefore C is a smooth elliptic curve with $H \cdot C = H^3 = 1$, i.e., $L \cdot C = 2$. Q.E.D.

REMARK **5.4** It is a standard fact that $\mathbb{P}(\mathcal{T})$, for the tangent bundle $\mathcal{T} := \mathcal{T}_{\mathbb{P}^n}$ of $\mathbb{P}^n$, is embedded in $\mathbb{P}^n \times \mathbb{P}^n$ as a divisor of bidegree $(1,1)$ (see also [22]). To see this, note that $\mathcal{T}(-1)$ is spanned with $n+1$ sections. Thus letting ξ denote the tautological line bundle of $\mathbb{P}(\mathcal{T}(-1))$, the map $f : \mathbb{P}(\mathcal{T}) \to \mathbb{P}^n$ associated to $|\xi|$ is an embedding on fibers of the bundle projection $p : \mathbb{P}(\mathcal{T}) \to \mathbb{P}^n$. The product map $(f, p) : \mathbb{P}(\mathcal{T}) \times \mathbb{P}(\mathcal{T}) \to \mathbb{P}^n \times \mathbb{P}^n$ is thus an embedding with image a divisor D such that $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(D)_{|F} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ on the fibers F of p. The fibers $(F', \xi_{F'})$ of f are $\cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$. To see this for

F', a generic fiber of f, note that $c_1(p^*\mathcal{O}_{\mathbb{P}^n}(1))^{n-1} \cdot F' = c_1(p^*\mathcal{O}_{\mathbb{P}^n}(1))^{n-1} \cdot c_1(\xi)^n$. Using the defining equation for the Chern classes of $\mathcal{T}(-1)$, we see that this equals $p^*\left(c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{n-1} \cdot c_1(\mathcal{T}(-1))\right) \cdot c_1(\xi)^{n-1} = c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{n-1} \cdot c_1(\mathcal{T}(-1)) = 1$. Since D, $f(D)$, and the generic fiber of f are all connected, it follows that all fibers of f are connected. From this it follows that all fibers of f are isomorphic if the automorphism group of D acts transitively on D. This can be seen by observing that given two nonzero tangent vectors of $\mathbb{P}^n$ there is an automorphism of $\mathbb{P}^n$ which takes one tangent vector to the other.

Remark **5.5** Note that the line bundles L in (4.1) and (5.3) are of course k-spanned and also k-jet ample, with the only exception of Case 9) in (5.3) (see (1.5), (5.2) and [7, Corollary (2.1)]). Thus (4.1) and (5.3) also give the classification of Mukai varieties (X, L) of dimension $n \geq 3$ polarized by either a k-spanned or a k-jet ample line bundle L, with the only exception, for k-jet ampleness, of Case 9) in (5.3).

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