Projections from Subvarieties

Mauro C. Beltrametti, Alan Howard, Michael Schneider and Andrew J. Sommese

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1 Introduction

Let $X \subset \mathbb{P}^N$ be an n-dimensional connected projective submanifold of projective space. Let $p : \mathbb{P}^N \to \mathbb{P}^{N-q-1}$ denote the projection from a linear $\mathbb{P}^q \subset \mathbb{P}^N$. Assuming that $X \not\subset \mathbb{P}^q$ we have the induced rational mapping $\psi := p_X : X \to \mathbb{P}^{N-q-1}$. This article started as an attempt to understand the structure of this mapping when ψ has a lower dimensional image. In this case of necessity we have $Y := X \cap \mathbb{P}^q$ is nonempty.

The special case when Y is a point is very classical: X is a linear subspace of $\mathbb{P}^N$. The case when $q = 1$ and $Y = \mathbb{P}^q = \mathbb{P}^1$ was settled for surfaces by the fourth author [18] and by Ilic [12] in general. Beyond this even the special case when $q \geq 2$ and $Y = \mathbb{P}^q$ is open.

We have found it convenient to study a closely related question, which includes many special cases including the case when the center of the projection $\mathbb{P}^q$ is contained in X.

Problem. Let Y be a proper connected k-dimensional projective submanifold of an n-dimensional projective manifold X. Assume that $k > 0$. Let L be a very ample line bundle on X such that $L \otimes \mathcal{J}_Y$ is spanned by global sections, where $\mathcal{J}_Y$ denotes the ideal sheaf of Y in X. Describe the structure of (X, Y, L) under the additional assumption that the image of X under the mapping ψ associated to $|L \otimes \mathcal{J}_Y|$ is lower dimensional.

Let us describe our progress on this problem.

In §3 we study upper and lower bounds for the dimensions of the spaces of sections of powers tL of a very ample line bundle L on a projective manifold X. The need for such bounds arises naturally when we consider line bundles which are multiples of a very ample line bundle. One general result Proposition (3.8) gives an upper bound for an integer t_0 such that for $t \geq t_0$, $h^0(tL \otimes \mathcal{J}_Y) > 0$.

In §4 we prove a number of general results. For example, Theorem (4.6) shows that $\dim\psi(X) \geq n-k-1$ with equality only if Y is a complete intersection in X. In particular Corollary (4.7), shows that if Y is a linear $\mathbb{P}^k$, then $\dim\psi(X) \geq n-k-1$ with equality only if $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Proposition (4.9) further shows that if $\mathrm{rankPic}(X) = 1$ and $\dim\psi(X) \geq n-k$ then $\dim\psi(X) \geq n-k+\dfrac{k}{n-k}-1$. Theorem (4.10) shows that if Y is a $\mathbb{P}^k$ (or more generally a projective manifold whose algebraic cohomology is the same as $\mathbb{P}^k$ up to dimension $2(n-k)$), then if $\dim\psi(X) \geq n-k$, it follows that $\dim\psi(X) \geq k$. In particular except for known examples, we have for a wide range of Y including $\mathbb{P}^k$, that $\dim\psi(X) \geq \dfrac{\dim X}{2}$.

In §5 we give a number of examples showing that the dimensions allowed by the above results do occur. Of particular interest is Example (5.2). This example consists for each positive integer n of an infinite sequence of projective n-folds in $\mathbb{P}^{2n-1}$ which contain a linear $\mathbb{P}^{n-1}$. All degrees of X that are allowed by theory occur.

In §6 we specialize to the case when Y is a divisor. We study bundles of the form $tL - Y$ where t is near $\delta := \deg Y$. One result, Theorem (6.4), implies that if $\delta > 1$ then $|\delta L - Y|$ gives a birational map, which is in fact an isomorphism if $2n \geq \dim\Gamma(L) + 1$.

In §7 we restrict to the case when Y is a linear $\mathbb{P}^k$ and show, among other things, that $\dim\psi(X) \geq n-k$ except when X is a hypersurface in $\mathbb{P}^{n+1}$. In §8 we restrict further to the special case when Y is a linear $\mathbb{P}^{n-1}$. In this case ψ is a morphism. Remmert-Stein factorize $\psi = s \circ \phi$ with $\phi : X \to Z$ a morphism with connected fibers onto a normal projective variety Z, and with s a finite morphism. We know that except for known examples, if $\dim\psi(X) < \dim X$ then $\dim\psi(X) = n-1$. We show that Z is very well behaved (Cohen-Macaulay, $\mathbb{Q}$-factorial, $\mathrm{Pic}(Z) \cong \mathbb{Z}$). Moreover we examine the possible degrees of s and use adjunction theory to classify the possible (X, L) for extreme values of this degree.

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how Castelnuovo theory gives lower bounds for the dimensions of spaces of sections of powers of very ample line bundles.

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2 Background material

We work over the complex numbers $\mathbb{C}$. Through the paper we deal with projective varieties V. We denote by $\mathcal{O}_V$ the structure sheaf of V and by K_V the canonical bundle, for V smooth. For any coherent sheaf $\mathcal{F}$ on V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V, \mathcal{F})$.

Let $\mathcal{L}$ be a line bundle on V. The line bundle $\mathcal{L}$ is said to be *numerically effective* (*nef*, for short) if $\mathcal{L} \cdot C \geq 0$ for all effective curves C on V. $\mathcal{L}$ is said to be *big* if $\kappa(\mathcal{L}) = \dim V$, where $\kappa(\mathcal{L})$ denotes the Kodaira dimension of $\mathcal{L}$. If $\mathcal{L}$ is nef then this is equivalent to $c_1(\mathcal{L})^n > 0$, where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$ and $n = \dim V$.

2.1 Notation. The notation used in this paper is standard from algebraic geometry. In particular, $\approx$ denotes linear equivalence of line bundles. For a line bundle $\mathcal{L}$ on a compact complex space V, $\chi(\mathcal{L}) := \sum_i (-1)^i h^i(\mathcal{L})$ denotes the Euler characteristic, and $|\mathcal{L}|$ denotes the complete linear system associated with a line bundle. We say that $\mathcal{L}$ is spanned if it is spanned at all points of V by $\Gamma(\mathcal{L})$.

For a compact connected projective manifold V, $h^{2j}(V, \mathbb{Q})_{\text{alg}}$ denotes the dimension of the vector subspace $H^{2j}(V, \mathbb{Q})_{\text{alg}}$ of $H^{2j}(V, \mathbb{Q})$ dual under Kronecker duality to the vector subspace of $H_{2j}(V, \mathbb{Q})$ spanned by the j-dimensional algebraic subvarieties of V.

We denote the ideal sheaf of an irreducible subvariety A of a variety V by $\mathcal{J}_{A/V}$ (or simply $\mathcal{J}_A$ when no confusion can result). For smooth A contained in the smooth locus of V, $N_{A/V}$ denotes the normal bundle of A in V.

Line bundles and divisors are used with little (or no) distinction. Hence we shall freely switch between the multiplicative and the additive notation.

2.2 Conductor formula. Let V be a connected projective manifold of dimension n. Let L be a very ample line bundle on V of degree $d := L^n$ with $|L|$ embedding V

into $\mathbb{P}^N$. Then the classical conductor formula for the canonical bundle states that

$$\Delta \in |(d-n-2)L - K_V|,$$

where Δ is the double point divisor of a projection of V from $\mathbb{P}^N$ to $\mathbb{P}^{n+1}$ (in the degenerate cases when $N = n$ or $n+1$, Δ is taken to be the empty divisor). In particular the line bundle $(d-n-2)L - K_V$ is spanned since given any point of V a generic projection can be chosen with the point not in the double point divisor of the projection (see [21], [22, p. 71], and [14]).

The following standard lemma is basic (see also [2, (3.1.8)]).

Lemma 2.3 *Let V be an irreducible normal projective variety with* $\mathrm{Pic}(V) \cong \mathbb{Z}$. *Let $g : V \to Z$ be a surjective morphism of V to a projective variety Z. Either g is a finite morphism or $g(V)$ is a point. The same conclusion holds for $V \cong \mathbb{P}^n$ and any holomorphic map to a compact complex space Z.*

Proof. Assume that g is not finite and doesn't map V to a point. If Z is projective, then the pullback of an ample line bundle cannot be ample and thus we see that $\mathrm{Pic}(V) \not\cong \mathbb{Z}$. Thus we can assume that $V \cong \mathbb{P}^n$ and Z is not necessarily projective.

Note that $\dim g(\mathbb{P}^n) = n$. If not let F denote a general fiber. Since it is smooth it would have trivial normal bundle. This contradicts the ampleness of the tangent bundle of $\mathbb{P}^n$.

Let F denote a positive dimensional fiber. There is a complex neighborhood U of F which maps generically one-to-one to a Stein space. Since $\mathbb{P}^n$ is homogeneous we have that the translates of F fill out an open set. Since the map g_U must map these positive dimensional subspaces to points we have the contradiction that $\dim g(U) = \dim g(\mathbb{P}^n) < n$. Q.E.D.

We also need the following general fact.

Lemma 2.4 *Let $f : X \to Y$ be a surjective proper map between normal varieties. Assume that X is Cohen-Macaulay and all fibers of f are equal dimensional. Then Y is Cohen-Macaulay.*

Proof. Note that a Cartier divisor on a Cohen-Macaulay variety is Cohen-Macaulay and that if we slice with $\dim X - \dim Y$ sufficiently ample divisors, then the restriction of the map to the slice is finite by a well known theorem of Hironaka [11, (2.1)]. Since a general hyperplane section of a normal variety is normal by Seidenberg's theorem, we can assume without loss of generality that f is finite. Let $n := \dim X = \dim Y$. By using [9, III, (7.6)] we are reduced to showing that for any locally free coherent sheaf $\mathcal{E}$ on Y, we have

$$h^i(\mathcal{E}(-q)) = 0 \text{ for } i < n \text{ and } q \gg 0, \tag{1}$$

where $\mathcal{F}(t)$ for a coherent sheaf $\mathcal{F}$ means $\mathcal{F} \otimes H^{\otimes t}$ for a fixed ample line bundle H on Y.

Since f is finite the pullback of an ample line bundle is ample. Given a coherent sheaf $\mathcal{G}$ on X, $\mathcal{G}(t)$ means $\mathcal{G}$ twisted by the t-th power of the pullback of H. Hence $(f^*\mathcal{E})(t) = f^*(\mathcal{E}(t))$.

Now since X is Cohen-Macaulay we have

$$h^i((f^*\mathcal{E})(-q)) = 0 \text{ for } i < n \text{ and } q \gg 0.$$

By the Leray spectral sequence, the projection formula and vanishing of higher direct images we obtain

$$h^i((f^*\mathcal{E})(-q)) = h^i(\mathcal{E}(-q) \otimes f_*\mathcal{O}_X). \tag{2}$$

Since both X and Y are normal we can use the trace mapping from $f_*\mathcal{O}_X \to \mathcal{O}_Y$ to see that the exact sequence $0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathcal{M} \to 0$ splits, where $\mathcal{M}$ denotes the quotient bundle. Thus $f_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{M}$. Thus by combining (1) and (2) we get

$$0 = h^i((f^*\mathcal{E})(-q)) = h^i(\mathcal{E}(-q)) + h^i(\mathcal{E}(-q) \otimes \mathcal{M}),$$

for $i < n$ and $q \gg 0$. Then (1) follows and we are done. Q.E.D.

If X is smooth and f is finite we can say more.

Lemma 2.5 *Let $f : X \to Y$ be a finite surjective map between projective varieties, where X is smooth and Y is normal. Then Y is Cohen-Macaulay and $(\deg f)$-factorial. Moreover, if $-K_X$ is nef and big, the induced map of $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is injective.*

Proof. The fact that Y is Cohen-Macaulay was proved in the previous lemma. To see that it is $(\deg f)$-factorial, let D be a Weil divisor on Y. Since X is smooth, f^*D is a Cartier divisor. We construct a Cartier divisor, $\mathrm{Norm}(f^*D)$, on Y as follows: in a small neighborhood U of any smooth point y in Y over which f is unramified, we define a rational function by multiplying the functions defining f^*D on the connected components of $f^{-1}(U)$; and we construct the divisor determined locally by this construction, first over all smooth points of Y over which f is unramified, and then (since Y is normal) to all of Y by Riemann extension.

From the way $\mathrm{Norm}(f^*D)$ was constructed it is obvious that $\mathrm{Norm}(f^*D) = (\deg f)D$. This shows that Y is $(\deg f)$-factorial. In addition, the same construction shows that if D is a Cartier divisor on Y for which f^*D is trivial, then $(\deg f) \cdot D$ is trivial. In particular, the kernel of the induced map of $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ consists entirely of torsion elements.

Now suppose that $-K_X$ is nef and big. Then $h^i(\mathcal{O}_X) = 0$ for $i > 0$. Using the direct sum decomposition $f_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{M}$ from the previous lemma together with the Leray spectral sequence applied to the finite map f, we see that $h^i(\mathcal{O}_Y) = 0$ for $i > 0$. Therefore, $\mathrm{Pic}(Y) \cong H^2(Y, \mathbb{Z})$, and it follows that $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is injective unless there is torsion in $H^2(Y, \mathbb{Z})$. We will show this can not occur.

By the universal coefficient theorem, torsion in $H^2(Y,\mathbb{Z})$ is equivalent to torsion in $H_1(Y,\mathbb{Z})$, which in turn implies the existence of a finite unbranched covering $Y' \to Y$. Lifting this to X gives the commutative diagram

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow f \\ Y' & \to & Y \end{array}$$

where Y' is connected, the vertical arrows are branched coverings, and the horizontal arrows are unbranched coverings. Let m be the common sheet number of both of the latter. It is easy to see that X' consists of a finite number of disjoint connected components, each mapping isomorphically onto X; for $h^i(\mathcal{O}_{X'}) = 0$ (because $-K_{X'}$ is big and nef), and $\chi(\mathcal{O}_{X'}) = m\chi(\mathcal{O}_X) = m$, where m is the sheet number. If A is any connected component of X', we thus get a finite surjective map $X \cong A \to Y'$. Arguing as before, we see that $h^i(Y') = 0$ for $i > 0$, so that $\chi(\mathcal{O}_{Y'}) = 1$. On the other hand, we have $\chi(\mathcal{O}_{Y'}) = m\chi(\mathcal{O}_Y) = m$. Q.E.D.

The following general lemma is well known and follows from the results in the introduction of [16] (see also [2, (6.6.1)]).

Lemma 2.6 *Let L be a very ample line bundle on an irreducible projective variety, X. Let $Y \subset X$ be an irreducible subvariety of degree δ relative to L, i.e., $\delta = L^{\dim Y} \cdot Y$. If either Y is smooth or $Y \subset \mathrm{reg}(X)$ and $\mathrm{cod}_X Y = 1$ then $\mathcal{J}_Y(\delta)$ is spanned by global sections, where $\mathcal{J}_Y(\delta)$ denotes the ideal sheaf $\mathcal{J}_Y$ of Y in X tensored with δL.*

The following result we need is a "folklore" result, for which we don't know references.

Proposition 2.7 *Assume that Hartshorne's conjecture* [8]*, that any connected non-degenerate n-dimensional smooth submanifold $X \subset \mathbb{P}^m$ is a complete intersection if $n > \frac{2}{3}m$, is true. Then each vector bundle $\mathcal{E}$ on $\mathbb{P}^m$ of rank $r < \frac{m}{3}$ splits into a direct sum of line bundles.*

Proof. We use induction over r. If $r = 1$ the assertion is true. So, let us assume the assertion true for $r-1$.

Since the assertion is independent of twisting, we may assume that $\mathcal{E}$ is generated by global sections. Take a general section $s \in H^0(\mathcal{E})$ and let $X := V(s)$, the zero locus of s. Then X is smooth and $\mathrm{cod}_{\mathbb{P}^m} X = r$. The assumption $r < \frac{m}{3}$ is equivalent to $\dim X = m - r > \frac{2}{3}m$ and therefore X is a complete intersection in $\mathbb{P}^m$ by Hartshorne's conjecture. Thus

$$\mathcal{E}_X \cong N_{X/\mathbb{P}^m} \cong \oplus_{i=1}^r \mathcal{O}_X(a_i),$$

where $\mathcal{E}_X$ denotes the restriction of $\mathcal{E}$ to X. Since $N_{X/\mathbb{P}^m}$ is ample, the a_i's are positive integers and we may assume $a_1 \geq a_2 \geq \cdots \geq a_r > 0$.

Claim. $\mathcal{E}(-a_1)$ has a section without zeros.

Assuming the Claim true, we get an exact sequence (given by that section)

$$0 \to \mathcal{O}_{\mathbb{P}^m} \to \mathcal{E}(-a_1) \to \mathcal{F} \to 0,$$

where the quotient $\mathcal{F}$ is a rank $r-1$ vector bundle. Then by induction $\mathcal{F}$ splits. Therefore $\mathcal{E}(-a_1)$, and hence $\mathcal{E}$, splits into a direct sum of line bundles.

Thus it remains to show the Claim. Note that

$$\mathcal{E}_X(-a_1) \cong \mathcal{O}_X \oplus \mathcal{O}_X(a_2 - a_1) \oplus \cdots \oplus \mathcal{O}_X(a_r - a_1),$$

where $a_i - a_1 \leq 0$ for $2 \leq i \leq r$. Let $\sigma \in H^0(\mathcal{O}_X)$ be a section of $\mathcal{E}_X(-a_1)$ with no zeros. The obstruction to extending σ to a formal neighborhood $\widehat{X}$ of X belongs to $H^1(X, S^t(N^*) \otimes \mathcal{E}_X(-a_1))$, where $N := N_{X/\mathbb{P}^m}$. Since $S^t(N^*) \otimes \mathcal{E}_X(-a_1)$ is a direct sum of negative line bundles, we have $H^1(S^t(N^*) \otimes \mathcal{E}_X(-a_1)) = 0$, $t \geq 1$, by Kodaira vanishing. Thus we conclude that there exists a section $\hat{\sigma} \in H^0(\widehat{X}, \mathcal{E}_{\widehat{X}}(-a_1))$ whose restriction to X coincides with σ. As soon as $\dim X \geq 2$ (which is the case since $m \geq 3$), it is a fairly standard fact, by using results of Barth [1, Proposition 4] and Griffiths [5, Theorems I, III, p. 378, 379] (see also [10, p. 226, 227]), that $\hat{\sigma}$ extends to a section $\tau \in H^0(\mathcal{E}(-a_1))$. Then the restriction τ_X has no zeros on X. We want to show that τ has no zeros on $\mathbb{P}^m$. Let $Y := V(\tau)$ be the zero locus of τ. If $Y \neq \emptyset$, then $\dim Y \geq m - \mathrm{rank}\mathcal{E} = m - r$. Since $r < \frac{m}{3}$, we have that $\dim(X \cap Y) \geq \dim X + \dim Y - m \geq m - 2r > 0$. Therefore $X \cap Y \neq \emptyset$ in $\mathbb{P}^m$. This contradicts the fact that the restriction τ_X has no zeros on X. Q.E.D.

3 Lower and upper bounds for $h^0(tL)$

We first state some general lower and upper bound formulas for the number of sections of multiples of a given line bundle L.

Lemma 3.1 *Let L be a big and spanned line bundle on an irreducible n-dimensional projective variety X. Then for $t \geq 0$ with $d := \deg_L(X) = L^n$ we have*

$$h^0(tL) \leq \binom{t+n-1}{n-1}\frac{dt+n}{n} = \frac{td+n}{t+n}\binom{t+n}{n}$$

Proof. If X is a curve then clearly the result is true, i.e., $h^0(tL) \leq \binom{t+1-1}{1-1}\frac{dt+1}{1} = dt+1$ with equality only if $X \cong \mathbb{P}^1$. Now in general let $A \in |L|$. Then by using the exact sequence $0 \to (s-1)L \to sL \to sL_A \to 0$ for $1 \leq s \leq t$ we see that $h^0(tL) \leq \sum_{j=0}^{t} h^0(jL_A)$. Thus by induction we have $h^0(tL) \leq \sum_{j=0}^{t}\binom{j+n-2}{n-2}\frac{dj+n-1}{n-1} = \binom{t+n-1}{n-1}\frac{dt+n}{n}$. Q.E.D.

Now assume that L is very ample. Then we also have the following lower bound

$$h^0(tL) \geq \binom{t+n+1}{n+1} \quad \text{for} \quad t < d := L^n. \tag{3}$$

To see this note that we can assume $h^0(L) \geq n+2$ since otherwise $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and the assertion is clearly true. Then X is embedded by $|L|$ in $\mathbb{P}^{n+r}$ with $r > 0$, so that we can project X generically one-to-one into $\mathbb{P}^{n+1}$. Now, for any positive integer t,

$$h^0(tL) = h^0(\mathcal{O}_{\mathbb{P}^{n+r}}(t)_X) \geq h^0(\mathcal{O}_{\mathbb{P}^{n+1}}(t)_{X'}),$$

where X' is the image of X in $\mathbb{P}^{n+1}$. But if $t < d := \deg_L(X)$ then $h^0(\mathcal{O}_{\mathbb{P}^{n+1}}(t)_X) \geq h^0(\mathcal{O}_{\mathbb{P}^{n+1}}(t))$ since the kernel of the restriction map has dimension $h^0(\mathcal{O}_{\mathbb{P}^{n+1}}(t-d)) = 0$. Thus we get $h^0(tL) \geq h^0(\mathcal{O}_{\mathbb{P}^{n+1}}(t))$, which is the bound as in (3).

Following Harris' presentation [7] of Castelnuovo theory we can significantly improve the above lower bound. Let us fix some notation. Let X_{n-i} be the $(n-i)$-dimensional subvariety of X obtained as transversal intersection of X with a general $\mathbb{P}^{n+r-i}$, $0 \leq i \leq n$, and in particular $X_n = X$. Let, for $0 \leq i \leq n$,

$$h_{X_{n-i}}(t) := \dim(\mathrm{Im}(H^0(\mathbb{P}^{n+r-i}, \mathcal{O}_{\mathbb{P}^{n+r-i}}(t)) \to H^0(X_{n-i}, \mathcal{O}_{X_{n-i}}(tL))),$$

$h^0(t) := h_{X_0}(t)$. By [7, Lemma (3.1)] one has, for a given integer $t \geq 0$,

$$h_X(t) \geq h_{X_{n-1}}(t) + h_X(t-1).$$

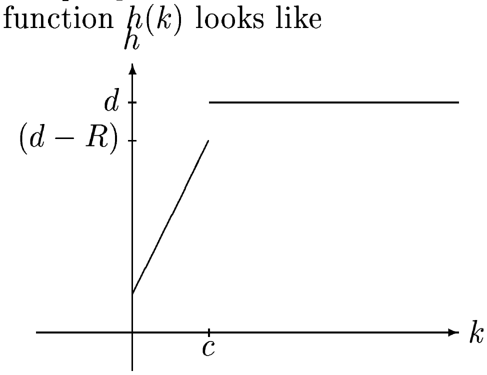
Iterating on t we get $h_{X_{n-i}}(j) \geq \sum_{k=0}^{j} h_{X_{n-i-1}}(k)$ for $0 \leq i \leq n$. Iterating on n, we get

$$h^0(tL) \geq h_X(t) \geq \sum_{k_{n-1}=0}^{t} \cdots \sum_{k_1=0}^{k_2} \sum_{k=0}^{k_1} h_{X_0}(k). \tag{4}$$

Castelnuovo theory (see [7, p. 94]) gives

$$h_{X_0}(k) \geq h(k) := \min\{d, kr+1\}. \tag{5}$$

(Note that the formula in [7, p. 94] is for a curve in $\mathbb{P}^r$, whereas we are considering a curve in $\mathbb{P}^{r+1}$.) Let $c := \left[\frac{d-1}{r}\right]$, the integral part of $\frac{d-1}{r}$, and let $R := d-1-cr$. Then the graph of the function $h(k)$ looks like



(6)

where the oblique line is the graph of the equation $h = kr + 1$.

Lemma 3.2 *Let X be an n-dimensional irreducible nondegenerate subvariety of $\mathbb{P}^{n+r}$. Let $L := \mathcal{O}_{\mathbb{P}^{n+r}}(1)_X$, and let $d = L^n$ be the degree of X in $\mathbb{P}^{n+r}$. Let $c := \left[\frac{d-1}{r}\right]$. Let R be the remainder defined as $R := d - 1 - cr$. Then for any integer $t \geq 0$ we have the lower bound*

$$\begin{aligned} h^0(tL) &\geq r\binom{n+t}{n+1} + \binom{n+t}{n} - r\binom{n+t-c-1}{n+1} + (R-r)\binom{n+t-c-1}{n} \quad (7) \\ &= \frac{tr+n+1}{t+n+1}\binom{t+n+1}{n+1} - r\binom{t+n-c}{n+1} + R\binom{t+n-c-1}{n}. \end{aligned}$$

Proof. Referring to the inequality in formula (5), we get for any positive integer a

$$\sum_{k=0}^{a} h_{X_0}(k) \geq \sum_{k=0}^{a} h(k) = \sum_{k=0}^{a} kr + 1 - \sum_{k=c+1}^{a} (kr + 1 - d)$$

(see diagram (6)). Iterating the summation as in formula (4) gives

$$\begin{aligned} h^0(tL) &\geq \sum_{k_{n-1}=0}^{t} \cdots \sum_{k_1=0}^{k_2} \sum_{k=0}^{k_1} (kr+1) - \sum_{k_{n-1}=c+1}^{t} \cdots \sum_{k_1=c+1}^{k_2} \sum_{k=c+1}^{k_1} (kr+1-d) \\ &= \sum_{k_{n-1}=0}^{t} \cdots \sum_{k_1=0}^{k_2} \sum_{k=0}^{k_1} (kr+1) - \sum_{j_{n-1}=0}^{t-c-1} \cdots \sum_{j_1=0}^{j_2} \sum_{j=0}^{j_1} ((j+c+1)r+1-d) \\ &= r \sum_{k_{n-1}=0}^{t} \cdots \sum_{k_1=0}^{k_2} \sum_{k=0}^{k_1} k + \sum_{k_{n-1}=0}^{t} \cdots \sum_{k_1=0}^{k_2} \sum_{k=0}^{k_1} 1 \\ &\quad -r \sum_{j_{n-1}=0}^{t-c-1} \cdots \sum_{j_1=0}^{j_2} \sum_{j=0}^{j_1} j - \sum_{j_{n-1}=0}^{t-c-1} \cdots \sum_{j_1=0}^{j_2} \sum_{j=0}^{j_1} ((c+1)r+1-d) \end{aligned}$$

By repeatedly using the combinatorial identity $\sum_{i=0}^{b} \binom{i+m}{q} = \binom{b+m+1}{q+1} - \binom{m}{q+1}$ for any positive integers b, m, and q, with the usual convention that $\binom{u}{v} = 0$ whenever $v > u$, we get

$$\begin{aligned} h^0(tL) &\geq r\binom{n+t}{n+1} + \binom{n+t}{n} - r\binom{n+t-c-1}{n+1} + (d-1-r(c+1))\binom{n+t-c-1}{n} \\ &= r\binom{n+t}{n+1} + \binom{n+t}{n} - r\binom{n+t-c-1}{n+1} + (R-r)\binom{n+t-c-1}{n} \\ &= \frac{tr+n+1}{t+n+1}\binom{t+n+1}{n+1} - r\binom{t+n-c}{n+1} + R\binom{t+n-c-1}{n}. \end{aligned}$$

Q.E.D.

Remark 3.3 It is easy to see that the right-hand side of this last inequality is minimized when $R=0$ and $c=1$, and therefore the bound in Lemma (3.2) yields the simpler form

$$h^0(tL) \geq \frac{tr+n+1}{t+n+1}\binom{t+n+1}{n+1} - r\binom{t+n-1}{n+1}. \tag{8}$$

Note also that if X has nonnegative Kodaira dimension, then $d \geq rn+2$ (see e.g., [2, (8.1.3)]). Thus $\frac{d-1}{r} > n$, so that $c = \left[\frac{d-1}{r}\right] \geq n$. Therefore in this case we can use the bound in (3.2), with $R=0$, $c=n$, in the form

$$h^0(tL) \geq \frac{tr+n+1}{t+n+1}\binom{t+n+1}{n+1} - r\binom{t}{n+1}.$$

We have the following general fact.

Proposition 3.4 *Let X be a nondegenerate irreducible n-dimensional subvariety of $\mathbb{P}^{n+r}$. Let $L := \mathcal{O}_{\mathbb{P}^{n+r}}(1)_X$. Let Y be a k-dimensional irreducible subvariety of X of degree $\delta := L^k \cdot Y$. Assume that $k>0$. Let $t>0$ be an integer such that*

$$\frac{tr+n+1}{t+n+1}\binom{t+n+1}{n+1} - r\binom{t+n-1}{n+1} > \frac{\delta t+k}{t+k}\binom{t+k}{k}.$$

Then $tL \otimes \mathcal{J}_Y$ has a section not identically zero on X.

Proof. Let L_Y be the restriction of L to Y. By Lemma (3.1) applied to L_Y we have

$$h^0(tL_Y) \leq \frac{\delta t+k}{t+k}\binom{t+k}{k}. \tag{9}$$

Suppose that $h^0(tL \otimes \mathcal{J}_Y) = 0$. Then $h^0(tL) \leq h^0(tL_Y)$. Thus by combining the inequalities (9) and (8) we get

$$\frac{\delta t+k}{t+k}\binom{t+k}{k} \geq \frac{tr+n+1}{t+n+1}\binom{t+n+1}{n+1} - r\binom{t+n-1}{n+1},$$

contrary to the inequality assumed in the proposition. Q.E.D.

Let us now make explicit the bound in (3.4) in the case when Y is a divisor on X.

Proposition 3.5 *Let X be a nondegenerate irreducible n-dimensional subvariety of $\mathbb{P}^{n+r}$. Let $L := \mathcal{O}_{\mathbb{P}^{n+r}}(1)_X$. Let D be an irreducible divisor of degree $\delta := L^{n-1} \cdot D > 1$. Thus, for $t \geq 1$, the inequality $t > \frac{n}{r+1}(\delta-1) - n + 1$ implies that $tL - D$ has a section not identically zero on X.*

Proof. The inequality in (3.4) becomes, in case $k = n-1$,

$$\frac{tr+n+1}{t+n+1}\binom{t+n+1}{n+1} - r\binom{t+n-1}{n+1} > \frac{\delta t+n-1}{t+n-1}\binom{t+n-1}{n-1}. \tag{10}$$

Now by a simple calculation (10) gives $-t-rt-2n+\delta n+1-rn+r<0$, or, solving in t and simplifying, $t > \frac{n}{r+1}(\delta-1)-n+1$. Q.E.D.

The following example shows that Proposition (3.5) is sharp.

Example 3.6 Let $X := \mathbb{P}^2 \times \mathbb{P}^2$, with $L := \mathcal{O}(1,1)$, and choose $D \in |\mathcal{O}(2,0)|$. Then $tL - D \approx \mathcal{O}(t-2,t)$ has a non-trivial section if and only if $t \geq 2$. Embed X in $\mathbb{P}^8$ by the Segre mapping. Then $n = r = 4$, and an easy calculation gives $\delta := L^3 \cdot D = 6$.

In this case we see that the hypothesis of (3.5) is satisfied if $t = 2$, and $2L - D$ has a non-zero section; whereas the hypothesis fails if $t = 1$ and $L - D$ has no non-trivial sections. Thus the inequality in (3.5) cannot be weakened.

Remark 3.7 We follow the notation and assumptions of (3.4). In general, it is hard to make the bound in (3.4) explicit in t. If Y has codimension two in X, then, after simplification, the condition for $h^0\left((\delta-1)L \otimes \mathcal{J}_Y\right)$ to be positive becomes

$$-rn^2-3n^2+\delta n^2-2rn\delta+9n-4\delta n+5rn-6-6r+5\delta+5r\delta-r\delta^2-\delta^2>0.$$

Let $X \subset \mathbb{P}^N$ be a nondegenerate smooth connected n-fold. Let $\deg(X) = d$ and denote by L the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to X. From Lemma (2.6) we know that $\mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{J}_X$ is spanned by global sections.

Problem. What can we say about the smallest integer $t > 0$ such that $h^0(\mathcal{O}_{\mathbb{P}^N}(t) \otimes \mathcal{J}_X) > 0$?

We define the *lower degree* in $\mathbb{P}^N$, δ_N, of a subvariety $X \subset \mathbb{P}^N$ to be the smallest positive integer t such that $h^0(\mathcal{O}_{\mathbb{P}^N}(t) \otimes \mathcal{J}_X) > 0$. One consequence of the above results is that under modest conditions there must be some form of much lower degree than d vanishing on X.

Proposition 3.8 *Let $X \subset \mathbb{P}^N$ be a nondegenerate smooth connected n-fold of degree d. Let $L = \mathcal{O}_{\mathbb{P}^N}(1)_X$ and let δ_N be the degree of the lowest-degree homogeneous form vanishing on X. Then δ_N satisfies the inequality* $\frac{1}{\delta_N - 1}\left(\frac{(\delta_N+N-1)\cdots(\delta_N+n-1)}{N\cdots(n+1)} - n\right) \leq d$.

Proof. If t is a positive integer for which $h^0(\mathcal{O}_{\mathbb{P}^N}(t) \otimes \mathcal{J}_X) = 0$ then

$$\binom{N+t}{N} = h^0(\mathcal{O}_{\mathbb{P}^N}(t)) \leq h^0(X, tL).$$

By applying Lemma (3.1) we get $\binom{N+t}{N} \leq \frac{td+n}{t+n}\binom{t+n}{n}$. An easy calculation shows that this inequality is equivalent to $\frac{1}{t}\left(\frac{(t+N)\cdots(t+n)}{N\cdots(n+1)} - n\right) \leq d$. From the definition of δ_N it follows that $h^0(\mathcal{O}_{\mathbb{P}^N}(\delta_N - 1) \otimes \mathcal{J}_X) = 0$, and substituting $t = \delta_N - 1$ in the last inequality completes the proof. Q.E.D.

For surfaces here is the explicit bound.

Corollary 3.9 *Let X be a nondegenerate smooth connected surface of degree d in $\mathbb{P}^N$. Assume $N \geq 5$. Let δ_N be the degree of the lowest-degree homogeneous form vanishing on X. Then $\delta_N^3 + 11\delta_N^2 + 46\delta_N + 96 \leq 60d$.*

Proof. We apply the bound in (3.8) with $n = 2$. Since $N \geq 5$, we get

$$\frac{1}{\delta_N - 1}\left(\frac{(\delta_N + 4)\cdots(\delta_N + 1)}{60} - 2\right) \leq d.$$

After simplifying this becomes $(\delta_N+4)(\delta_N+3)(\delta_N+2)(\delta_N+1) - 120 \leq 60d(\delta_N - 1)$. Since $\delta_N \geq 2$ we can divide both sides by $\delta_N - 1$ to obtain the desired result.Q.E.D.

For example, if X is of degree 21 then $\delta_N \leq 6$ and hence there is a form of the sixth degree vanishing on X. Or again, if X is of degree 10,000 there is a form of degree 80 vanishing on X.

Some other special cases are as follows. For threefolds in $\mathbb{P}^N$ with $N \geq 5$ the corresponding bound as in Corollary (3.9) is $\delta_N^2 + 10\delta_N + 36 \leq 20d$. For threefolds in $\mathbb{P}^N$ with $N \geq 6$ the bound becomes $\delta_N^3 + 15\delta_N^2 + 86\delta_N + 240 \leq 120d$.

4 Some general structure results for projections

In this section we discuss some general properties of projections from a k-dimensional subvariety Y of a given polarized variety X. We always assume that $k > 0$. In §6 and §7 we will present some more refined results in the cases when Y is either a divisor or a linear $\mathbb{P}^k$.

4.1 General set-up of morphisms. Let L be a very ample line bundle on X, a smooth connected variety of dimension $n \geq 2$. Let Y be a k-dimensional connected submanifold of X. *We always assume that* $k > 0$. We will denote by $\mathcal{J}_Y$ the ideal sheaf of Y in X.

Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y and set $E = \sigma^{-1}(Y)$. Let $\psi : \overline{X} \to \psi(\overline{X})$ be the surjective rational map given by $|\sigma^*L - E|$. We refer to the mapping ψ as the *projection from Y associated to L*. If $L \otimes \mathcal{J}_Y$ is spanned by its global sections, then $\sigma^*L - E$ is spanned on $\overline{X}$ and ψ is a morphism. We have the Remmert-Stein factorization $\psi = s \circ \phi$ of $\psi : \overline{X} \to \psi(\overline{X})$, where $\phi : \overline{X} \to Z$ is a morphism with connected fibers onto a normal variety Z and $s : Z \to \psi(\overline{X})$ is a finite morphism. We will refer to $\phi : \overline{X} \to Z$ as the morphism associated to $L \otimes \mathcal{J}_Y$. Note there is an ample and spanned line bundle $\mathcal{H}$ on Z such that $\sigma^* L - E \approx \phi^* \mathcal{H}$. We have the following commutative diagram

$$\begin{array}{ccccc} E & \hookrightarrow & \overline{X} & & \\ & & & \overset{\phi}{\searrow} & \\ \downarrow & & \downarrow \sigma & & \\ Y & \hookrightarrow & X & \xrightarrow{\varphi} & Z \end{array}$$

where φ is the connected part of the rational mapping associated to $|L \otimes \mathcal{J}_Y|$.

We need the following technical lemmas.

Lemma 4.2 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Let $N := N_{Y/X}$ be the normal bundle of Y in X. Let L be a line bundle on X. Assume that $L \otimes \mathcal{J}_Y$ is spanned by a vector subspace, V, of $\Gamma(L \otimes \mathcal{J}_Y)$ and $c_{n-k}(N^*(L)) = 0$. Then a general element $D \in |V|$ is smooth.*

Proof. A general $D \in |V|$ is smooth on $X \setminus Y$ by Bertini's theorem.

A given $D \in |L \otimes \mathcal{J}_Y|$ is smooth at a point $y \in Y$ if the differential in local coordinates of the defining equation of D is not zero at $y \in Y$. From the exact sequence

$$0 \to \mathcal{J}_Y^2 \otimes L \to \mathcal{J}_Y \otimes L \xrightarrow{\partial} N^*(L) \to 0$$

we see that D is smooth on Y if the image $\partial(s)$ of s defining D in $N^*(L)$ is nowhere zero. Since $c_{n-k}(N^*(L)) = 0$, a general $s \in V$ goes to a nowhere vanishing section $\partial(s)$ in $N^*(L)$. Q.E.D.

Lemma 4.3 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Let L be a very ample line bundle on X. Assume that $L \otimes \mathcal{J}_Y$ is spanned by global sections and let D be a smooth element of $|L \otimes \mathcal{J}_Y|$. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y and let $\overline{D}$ be the proper transform of D under σ. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in (4.1). Let $\phi_{\overline{D}}$ be the restriction of ϕ to $\overline{D}$. Then $\phi_{\overline{D}}$ has lower dimensional image if and only if ϕ has lower dimensional image.*

Proof. We follow the notation from (4.1). Assume that ϕ has lower dimensional image. We have $\overline{D} \in |\sigma^* L - E|$, $E = \sigma^{-1}(Y)$. Note that $\overline{D}$ is the pullback of some divisor $\mathcal{D} \subset Z$. From $\dim\phi(\overline{D}) < \dim\phi(\overline{X}) < \dim X$ we get $\dim\phi(\overline{D}) < \dim\overline{D}$.

To show the converse, note that by definition of $\overline{D}$ one has $\dim\phi(\overline{D}) = \dim\phi(\overline{X}) - 1$. Thus the assumption $\dim\phi(\overline{D}) < \dim\overline{D} = n - 1$ gives the result. Q.E.D.

Let us note some further general properties of the morphism ϕ. The notation is as in (4.1).

1. (Divisorial case) If Y is a divisor, then $X \cong \overline{X}$.

2. (Linear case) Assume that $(Y, L_Y) \cong (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$ and that $\Gamma(L)$ embeds X in $\mathbb{P}^{n+r}$. Since Y is a linear space it follows that $L\otimes\mathcal{J}_Y$ is spanned by global sections. The mapping given by $\Gamma(L \otimes \mathcal{J}_Y)$ coincides off of Y with the restriction to X of the projection of $\mathbb{P}^{n+r}$ to $\mathbb{P}^{n+r-k-1}$ from Y.

3. (Smooth case) If Y is smooth then $\delta L \otimes \mathcal{J}_Y$, $\delta = L^k \cdot Y$, is spanned by global sections by Lemma (2.6).

We have the following crude structure theorem in the case when the projection has lower dimensional image.

Theorem 4.4 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Let L be a very ample line bundle on X. Assume that $L \otimes \mathcal{J}_Y$ is spanned by its global sections. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L\otimes\mathcal{J}_Y$ as in* (4.1)*. Let $E := \sigma^{-1}(Y)$ be the exceptional divisor. Assume $n > \dim Z$. Then we have:*

1. $\phi(E) = Z$;

2. *Z is uniruled if* $\mathrm{cod}_X Y > 1$;

3. *Z is unirational if Y is unirational;*

4. *the restriction, E_F, of E to any fiber F of ϕ is an ample divisor on F.*

Proof. To show 1), assume by contradiction that the restriction, $\phi_E : E \to Z$, of ϕ to E is not surjective. Take a point $x \in Z \setminus \phi(E)$ and let $F_x = \phi^{-1}(x)$ be the fiber on x. Then the restriction $(\sigma^* L - E)_{F_x}$ is trivial. But $E_{F_x} \cong \mathcal{O}_x$ since $x \notin \phi(E)$, so that $(\sigma^* L - E)_{F_x} \cong (\sigma^* L)_{F_x} \cong L_{\sigma(F_x)}$ is ample, where the last isomorphism follows from the fact that F_x goes isomorphically to X under σ, since $F_x \cap E = \emptyset$. Thus $(\sigma^* L - E)_{F_x}$ is both ample and trivial, an absurdity that contradicts the assumption $n > \dim Z$.

To show 2), note that since $\mathrm{cod}_X Y > 1$, σ is not an isomorphism and the exceptional divisor is uniruled. This means that there exists an $(n-2)$-dimensional variety V and a rational map $V \times \mathbb{P}^1 \to E$ which is dominant. Since $\phi(E) = Z$ by 1), we get a dominant map $V \times \mathbb{P}^1 \to Z$, i.e., Z is uniruled.

To show 3), recall that E is birational to $Y \times \mathbb{P}^{n-k-1}$. Since Y is unirational we have a dominant rational map $\mathbb{P}^k \to Y$. Therefore, combining with the surjective map $\phi_E : E \to Z$, we get a dominant rational map $\mathbb{P}^k \times \mathbb{P}^{n-k-1} \to Z$. This implies that Z is unirational.

To show 4), take a fiber F of $\phi : \overline{X} \to Z$. If the restriction σ_F of the blowing up map is finite-to-one then $\sigma_F^* L$ is ample and the assertion is clear. It is easy to see that σ_F is finite. If not, then it follows that there is a positive dimensional fiber, f, of $\sigma_F : F \to \sigma(F)$. This implies that f is contained in a fiber of $E \to Y$. But

$\sigma^*L - E$ is ample on fibers of $E \to Y$. On the other hand, since $f \subset F$, the line bundle $\sigma^*L - E$ is trivial on f. Q.E.D.

We need the following lemma.

Lemma 4.5 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Let L be a very ample line bundle on X. Let $\mathcal{J}_Y$ be the ideal sheaf of Y in X and let $N := N_{Y/X}$ be the normal bundle of Y in X. Then $L \otimes \mathcal{J}_Y$ is spanned and $N^*(L)$ is trivial if and only if Y is the complete intersection of $n-k$ divisors $D_1, \ldots, D_{n-k} \in |L|$.*

Proof. The "if" part is straightforward. As to the converse, consider the exact sequence

$$0 \to L \otimes \mathcal{J}_Y^2 \to L \otimes \mathcal{J}_Y \xrightarrow{\partial} N^*(L) \to 0.$$

Set $w := n-k$. Since $N^*(L) = \oplus^w \mathcal{O}_Y$ and $L \otimes \mathcal{J}_Y$ is spanned we can find sections $s_1, \ldots, s_w \in \Gamma(L \otimes \mathcal{J}_Y)$ defining w divisors $D_1, \ldots, D_w \in |L \otimes \mathcal{J}_Y|$ on X containing Y.

For each $i = 1, \ldots, w$, D_i is smooth on $X \setminus Y$ by Bertini's theorem.

For each $i = 1, \ldots, w$, D_i is smooth at a point $y \in Y$ if the differential in local coordinates of the defining equation of D_i is not zero at y. From the exact sequence above we see that D_i is smooth on Y if the image $\partial(s_i)$ of s_i defining D_i in $N^*(L)$ is nowhere zero, $1 \leq i \leq w$. Since $N^*(L)$ is trivial we can find the sections $s_1, \ldots, s_w \in \Gamma(L \otimes \mathcal{J}_Y)$ such that $\partial(s_1), \ldots, \partial(s_w)$ are independent in $N^*(L)$. It follows that $D_1, \ldots, D_w$ are smooth as well as the intersection $D_1 \cap \ldots \cap D_w$ is smooth.

Since $\dim X \geq 2$, from the exact sequence $0 \to -L \to \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0$, we see that $h^0(\mathcal{O}_{D_i}) = 1$, so the D_i's are connected, $1 \leq i \leq w$.

Since $D_1 \cap \ldots \cap D_w$ is at least one-dimensional, we know by the Lefschetz hyperplane section theorem that $D_1 \cap \ldots \cap D_w$ is connected. Since it is also smooth of dimension $\dim Y$ and contains Y we conclude that Y is the complete intersection of $D_1, \ldots, D_w$. Q.E.D.

We can prove now the following more refined structure result, which gives a general lower bound for the dimension of the image of the projection.

Theorem 4.6 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Let L be a very ample line bundle on X. Assume that $L \otimes \mathcal{J}_Y$ is spanned by its global sections. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism from associated to $L \otimes \mathcal{J}_Y$ as in* (4.1). *Then* $\dim Z \geq n-k-1$, *with equality if and only if $Z \cong \mathbb{P}^{n-k-1}$ and Y is the complete intersection of $n-k$ divisors $D_1, \ldots, D_{n-k} \in |L|$.*

Proof. Set $w := n-k$. Let $E := \sigma^{-1}(Y)$ be the exceptional divisor. Set $\overline{L} := \sigma^*L$ and $N := N_{Y/X}$. Recall that $E \cong \mathbb{P}(N^*(L))$. Thus $\pi := \sigma_E : E \to Y$ is a $\mathbb{P}^{w-1}$-bundle. Let ξ be the tautological line bundle of $\mathbb{P}(N^*(L))$. Notice that $\xi \cong (\overline{L} - E)_E$.

Since $(\overline{L}-E)_{\mathbb{P}^{w-1}} \cong \mathcal{O}_{\mathbb{P}^{w-1}}(1)$ it follows that each fiber $\mathbb{P}^{w-1}$ of $\pi : E \to Y$ maps isomorphically under the map $\psi : \overline{X} \to \psi(\overline{X})$ given by $|\overline{L}-E|$, and hence maps isomorphically into Z under the morphism ϕ associated to $L \otimes \mathcal{J}_Y$. This shows that $\dim Z \geq w-1$. If $\dim Z = w-1$, it follows that $Z \cong \mathbb{P}^{w-1}$.

It also follows that Y is a complete intersection. For we have a surjective map of locally free sheaves $\oplus^{\dim Z+1}\mathcal{O}_{\overline{X}} \to \overline{L}-E \to 0$, and, restricting to E, we have a surjection $\oplus^{\dim Z+1}\mathcal{O}_E \to (\overline{L}-E)_E \to 0$. Consider the $\mathbb{P}^{w-1}$-bundle $\pi : E \to Y$. Notice that $(\overline{L}-E)_E \cong \xi$, the tautological line bundle of $E \cong \mathbb{P}(N^*(L))$. By pushing forward under π, we get a surjection

$$\beta : \oplus^{\dim Z+1}\mathcal{O}_Y \to N^*(L) = \pi_*\xi \to 0.$$

By comparing the ranks, since $N^*(L)$ has rank $\mathrm{cod}_X Y = w = \dim Z+1$, we conclude that β is an isomorphism, i.e., $N^*(L)$ is the trivial bundle. Thus, since $L \otimes \mathcal{J}_Y$ is spanned by global sections, Lemma (4.5) applies to give the result.

Next, we show that if $Y := D_1 \cap \ldots \cap D_w$ is the complete intersection of w divisors $D_1, \ldots, D_w \in |L|$, then $\dim Z = w-1$. We first observe that $N \cong \oplus^w L_Y$, so that $N^*(L) \cong \oplus^w \mathcal{O}_Y$ is trivial. Consider the exact sequence $0 \to L \otimes \mathcal{J}_Y^2 \to L \otimes \mathcal{J}_Y \to N^*(L) \to 0$. Since the morphism $L \otimes \mathcal{J}_Y \to N^*(L)$ is surjective at the sheaf level, $L \otimes \mathcal{J}_Y$ is spanned by global sections and $N^*(L)$ is trivial, it follows that the induced map $\alpha : \Gamma(L \otimes \mathcal{J}_Y) \to \Gamma(N^*(L)) \to 0$ is surjective.

For any integer $m \geq 2$, consider the exact sequence

$$0 \to L \otimes \mathcal{J}_Y^{m+1} \to L \otimes \mathcal{J}_Y^m \to S^m(N^*) \otimes L \to 0.$$

Since $S^m(N^*) \otimes L \cong \oplus^w {L_Y}^{-(m-1)}$ we have $h^0(S^m(N^*) \otimes L) = 0$, for $m \geq 2$, and therefore we get

$$\Gamma(L \otimes \mathcal{J}_Y^2) \cong \cdots \cong \Gamma(L \otimes \mathcal{J}_Y^m), \quad m \geq 2.$$

If $h^0(L \otimes \mathcal{J}_Y^2) \neq 0$ we thus find a section of L vanishing on Y of any given order $m \geq 2$, which is absurd. Therefore we conclude that $h^0(L \otimes \mathcal{J}_Y^2) = 0$ and hence $\Gamma(L \otimes \mathcal{J}_Y)$ injects in $\Gamma(N^*(L))$, i.e., the map α is an isomorphism. Thus $h^0(L \otimes \mathcal{J}_Y) = h^0(\overline{L}-E) = w$. Since $\overline{L}-E$ is spanned and gives the projection $\psi : \overline{X} \to \psi(\overline{X})$, we thus conclude that the $\dim Z = w-1$. Q.E.D.

There are many results from adjunction theory [2] describing all varieties with a given hyperplane section. Combining these results with Theorem (4.6) gives many consequences. By way of illustration we give two useful corollaries.

Corollary 4.7 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Let L be a very ample line bundle on X. Assume that $k > 0$ and Y is a linear $\mathbb{P}^k$ with respect to L, i.e., $L^k \cdot Y = 1$. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in* (4.1). *Then* $\dim Z = n-k-1$ *if and only if* $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. Assume $\dim Z = n-k-1$. Then, by (4.6), 2), Y is the complete intersection of $n-k$ divisors $D_1, \ldots, D_{n-k} \in |L|$. Since Y is a linear $\mathbb{P}^k$ it thus follows that $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

If $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, the projection from $Y = \mathbb{P}^k$ has an $(n-k-1)$-dimensional image. Q.E.D.

Corollary 4.8 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 3$. Assume that $k > 0$. Let L be a very ample line bundle on X. Assume that $L \otimes \mathcal{J}_Y$ is spanned by its global sections. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in* (4.1)*. Assume that Y is a $K(\pi, 1)$ and $k \geq 2$. Then* $\dim Z \geq n-k$.

Proof. By (4.6) either we are done or Y is a complete intersection of $n-k$ divisors $D_1, \ldots, D_{n-k} \in |L|$. Since Y is a $K(\pi, 1)$ with $\dim Y \geq 2$, this is not possible by a result of the fourth author [19]. Q.E.D.

Under special conditions on the cohomology of Y, we get stronger lower bounds for the image dimension of the projection. We restrict our attention to the case in which $\dim Z \geq n-k$, since the case $\dim Z = n-k-1$ was covered in (4.6).

Proposition 4.9 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Let L be a very ample line bundle on X. Assume that $L \otimes \mathcal{J}_Y$ is spanned by its global sections. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in* (4.1)*. Assume that $h^2(Y, \mathbb{Q})_{\text{alg}} = 1$ (or equivalently* $\text{Pic}(Y) \otimes \mathbb{Q} \cong \mathbb{Q}$)*. If* $\dim Z \geq n-k$*, then* $\dim Z \geq n-k+\dfrac{k}{n-k}-1$*. In particular, if* $\dim Z = \text{cod}_X Y = n-k$*, then* $n \geq 2k$.

Proof. Let $w := n-k$ and $N := N_{Y/X}$. As in the proof of (4.6), we have a surjective vector bundle map $\oplus^{\dim Z+1}\mathcal{O}_Y \to N^*(L) \to 0$. This gives a natural map $\rho : Y \to \text{Grass}(w, \dim Z+1)$ of Y in the Grassmannian of the w-dimensional quotients of $\mathbb{C}^{\dim Z+1}$. We claim that the map ρ is finite. Indeed, to see this, notice that $\det(N^*(L)) = \rho^*\mathcal{P}$, where $\mathcal{P}$ is an ample line bundle, the Plücker bundle, on $\text{Grass}(w, \dim Z+1)$. Since ρ is a not trivial map, $\det(N^*(L))$ is spanned and not trivial. Since $\text{Pic}(Y) \otimes \mathbb{Q} \cong \mathbb{Q}$, we thus conclude that $\rho^*\mathcal{P}$ is ample. Let F be a connected component of a positive dimensional fiber of ρ. Then $(\rho^*\mathcal{P})_F \cong \mathcal{O}_F$. This contradicts the ampleness of $\rho^*\mathcal{P}$. Thus

$$k = \dim Y \leq \dim\text{Grass}(w, \dim Z+1) = w(\dim Z + 1 - w)$$

gives the desired inequality.

If $\dim Z = n-k$, we have $k \leq n-k = \text{cod}_X Y$, which is the same as $2k \leq n$. Q.E.D.

Theorem 4.10 *Let Y be a connected k-dimensional proper submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Let L be a very ample line bundle on X. Assume that $L \otimes \mathcal{J}_Y$ is spanned by its global sections. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in (4.1). If $h^{2j}(Y, \mathbb{Q})_{\mathrm{alg}} = 1$ for $j \leq w := n - k$, then either $Z \cong \mathbb{P}^{n-k-1}$ with Y the complete intersection of $n-k$ divisors in $|L|$ or* $\dim Z \geq k$.

Proof. By Theorem (4.6) we can assume that if the theorem is false then

$$k - 1 \geq \dim Z \geq w. \tag{11}$$

Let $E = \sigma^{-1}(Y)$ be the exceptional divisor of $\sigma : \overline{X} \to X$ and let $\phi_E : E \to Z$ be the restriction to E of the morphism $\phi : \overline{X} \to Z$. From (4.4), 1), we know that ϕ_E is surjective. Let $N := N_{Y/X}$ be the normal bundle of Y in X. Set $\overline{L} = \sigma^* L$. Let ξ be the tautological line bundle of $\mathbb{P}(N^*(L)) \cong E$. Notice that $\xi \cong (\overline{L} - E)_E$. Therefore for each fiber $\mathbb{P}^{w-1}$ of the $\mathbb{P}^{w-1}$-bundle $\pi : E \to Y$ we have $(\overline{L} - E)_{\mathbb{P}^{w-1}} \cong \mathcal{O}_{\mathbb{P}^{w-1}}(1)$. This implies that each fiber F of ϕ_E meets $\mathbb{P}^{w-1}$ in at most one point. It thus follows that F goes isomorphically to $\pi(F)$ under π. Since $\xi_F \cong \mathcal{O}_F$, we get a surjective map $(\pi^* N^*(L))_F \to \xi_F \cong \mathcal{O}_F \to 0$. Letting $F' := \pi(F)$, we have by the above $F \cong F'$ and therefore pushing forward under π we get a surjective map

$$N^*(L)_{F'} \to \mathcal{O}_{F'} \to 0. \tag{12}$$

We claim that

$$c_w(N^*(L)) = 0. \tag{13}$$

To see this, let F be a general fiber of ϕ_E and $F' = \pi(F)$. Note that $\dim F' = \dim E - \dim Z \geq n - 1 - (k-1) = w$. In view of this and the assumption that $h^{2j}(Y, \mathbb{Q})_{\mathrm{alg}} = 1$, $j \leq w$, we see that it is enough to note that from (12) it immediately follows that $c_w(N^*(L)_{F'}) = 0$.

Thus Lemma (4.2) implies that there exists a smooth divisor $D \in |L \otimes \mathcal{J}_Y|$. Since $\dim X \geq 2$, from the exact sequence $0 \to -L \to \mathcal{O}_X \to \mathcal{O}_D \to 0$, we see that $h^0(\mathcal{O}_D) = 1$, so D is connected. Let $H \in |\overline{L} - E|$ be the divisor corresponding to D. I.e., $H = \phi^* Z_1$, where $Z_1 \in |\mathcal{H}|$ and $\overline{L} - E \approx \phi^* \mathcal{H}$ for some ample and spanned line bundle $\mathcal{H}$ on Z. Note that by the generalized Seidenberg theorem (see e.g., [2, (1.7.1)]), Z_1 is irreducible and normal since Z is irreducible and normal. Notice also that $\sigma^* D \approx H + E$.

Let $X_1 := D$. By construction, $Y \subset X_1$. Furthermore the blowing up $\sigma : \overline{X} \to X$ induces a blowing up map $\sigma_1 : \overline{X}_1 \to X_1$ of X_1 along Y. We can also consider the morphism, $\phi_1 : \overline{X}_1 \to Z_1$, associated to $L_{X_1} \otimes \mathcal{J}_Y$, where L_{X_1} is the restriction of L to X_1. Note that ϕ_1 is onto, $\dim X_1 = n-1$, $\dim Z_1 = \dim Z - 1$. Hence in particular $\dim Z_1 < k$, i.e., (11) is preserved passing from Z to Z_1.

Thus, starting from $X_1 = D$, Z_1, $\phi_1 : \overline{X}_1 \to Z_1$, $Y \subset X_1$, we proceed in such a way that from the initial data

$$(k, \dim Z, w)$$

we reach, after $w-1$ steps, the data (recall that we are working under the initial assumption that $\dim Z \geq w$)

$$(k, \dim Z - w + 1, 1).$$

I.e., Y is a divisor in X_{w-1} with $X_{w-1} \cong \overline{X}_{w-1}$, and the morphism $\phi_{w-1} : \overline{X}_{w-1} \to Z_{w-1}$ has image of dimension $\dim Z - w + 1$. In particular, since $X_{w-1} \cong \overline{X}_{w-1}$, we can restrict ϕ_{w-1} to Y, so that we get a surjective map from Y to Z_{w-1} (see (4.4), 1)). By assumption we have that $h^2(Y, \mathbb{Q})_{\text{alg}} = 1$, and therefore that $\dim Y = \dim Z_{w-1}$. Thus using (11) we have

$$k = \dim Z_{w-1} = \dim Z - w + 1 = \dim Z - n + k + 1$$

which gives that $n = \dim Z + 1$. Combined with (11) we have $n \leq k$, which contradicts the hypothesis that Y is a proper submanifold of X. Q.E.D.

Corollary 4.11 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Assume that L is a very ample line bundle on X such that $(Y, L_Y) \cong (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in* (4.1). *Then either $Z \cong \mathbb{P}^{n-k-1}$ with $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or $\dim Z \geq k$.*

Proof. It immediately follows by combining (4.7) and (4.10). Q.E.D.

Corollary 4.12 *Let Y be a connected k-dimensional submanifold of X, a connected projective manifold of dimension $n \geq 2$. Assume that $k > 0$. Let L be a very ample line bundle on X. Assume that $L \otimes \mathcal{J}_Y$ is spanned by its global sections. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in* (4.1). *Assume that Y is not a complete intersection. Further assume that $h^{2j}(Y, \mathbb{Q})_{\text{alg}} = 1$ for $j \leq n-k$. Then $\dim Z \geq n/2$.*

Proof. From (4.6) and (4.10) it follows that $\dim Z \geq n-k$ and $\dim Z \geq k$. Thus $\dim Z \geq n/2$. Q.E.D.

Let us point out some relations between the results above and Castelnuovo-Mumford regularity theory and the Castelnuovo bound conjecture (see [6]).

Remark 4.13 Let X be an n-dimensional smooth variety in $\mathbb{P}^{n+r}$ and let Y be a k-dimensional subvariety of X of degree $\delta := L^k \cdot Y$ and $L := \mathcal{O}_{\mathbb{P}^{n+r}}(1)_X$. Assume that $k > 0$. Let q be the codimension of Y in the smallest linear subspace $\mathbb{P}^{k+q} \subset \mathbb{P}^{n+r}$ containing Y. The Castelnuovo bound conjecture says that $(\delta - q + 1)L \otimes \mathcal{J}_Y$ is spanned by global sections. The conjecture is related to the question of Castelnuovo-Mumford regularity, and it is known to hold when Y has dimension 1 (see [6]) or 2 (see [15]) and when Y has dimension 3 and $\left[\frac{\delta-1}{q}\right] \geq 6$ (see [17]). Assuming the conjecture true and Y a divisor, we will show that the projection from Y associated to L is birational except in certain specific cases (see §6).

Remark 4.14 Let X be an n-dimensional smooth variety in $\mathbb{P}^{n+r}$ of degree d. Let L be a very ample line bundle on X. From the Castelnuovo-Mumford regularity theory developed in [7] it follows that in case $n=1$, for $t > \left[\frac{d-1}{r}\right]$, one has $h^1(tL)=0$ and thus that $h^0(tL)=\chi(\mathcal{O}_X(tL))$. One might hope that this extends in higher dimensions also. Unfortunately this is not true in dimension $n \geq 2$, as the following example shows.

Let C_1 be a smooth plane curve of degree d_1 with L_1 the restriction of the hyperplane section bundle of $\mathbb{P}^2$ to C_1. Let L_2 be a very ample line bundle of degree $d_2 := d' + 2g - 2$, with $d' > 0$, on a smooth curve C_2 of genus $g := g(C_2)$. Let $X := C_1 \times C_2 \subset \mathbb{P}^N$ and let $L := p_1^* L_1 \otimes p_2^* L_2$, where $p_i : X \to C_i$, $i = 1, 2$, are the projections on the two factors. Note that if $g(C_i) \geq 2$ for $i = 1, 2$, then X is a surface of general type. Since $d_2 > 2g-2$, we have $h^1(L_2) = 0$ and hence

$$h^0(L_2) = d_2 - g + 1 = d' + g - 1.$$

Thus by the Kunnëth formula we have $h^0(L) = 3(d'+g-1)$, i.e., $N = 3(d'+g-1)-1$. Therefore $r = N - 2 = 3(d'+g-2)$. Moreover $d = 2d_1 d_2 = 2d_1(d'+2g-2)$. Thus the critical value, c, is $c = \left[\frac{2d_1(d'+2g-2)-1}{3(d'+g-2)}\right]$. For a fixed g and taking $d' \gg 0$ and $d_1 \geq 10$ we have $c \sim \dfrac{2}{3} d_1 < d_1 - 3$. On the other hand, by using again Kunnëth formulas we get, for $t = d_1 - 3$,

$$h^1(tL) \geq h^1(\mathcal{O}_{C_1}(t)) = h^1(K_{C_1}) = 1. \tag{14}$$

To show that the equality $h^0(tL) = \chi(\mathcal{O}_X(tL))$ for $t > c = \left[\frac{d-1}{r}\right]$ is not true in general, consider the smooth irreducible curve and the set Γ of d distinct points obtained as transversal intersection of X with a general $\mathbb{P}^{r+1}$ and a general hyperplane $\mathbb{P}^r$ of the $\mathbb{P}^{r+1}$. Look at the exact sequence $0 \to (t-1)L_C \to L_C \to tL_\Gamma \to 0$, From [7, Theorem (3.7)] we know that $H^0(tL_C) \to H^0(tL_\Gamma)$ is surjective for $t > c$ and therefore $H^1((t-1)L_C)$ injects in $H^1(tL_C)$, for $t > c$. Since $h^1(tL_C) = 0$ for $t \gg 0$, we can conclude that $h^1((t-1)L_C) = 0$ for $t > c$. Thus from the cohomology sequence associated to the exact sequence $0 \to (t-2)L \to (t-1)L \to (t-1)L_C \to 0$ we infer that $h^2((t-2)L) = 0$ for $t > c$. From this we thus conclude that, for $t > c-2$, the equality $h^0(tL) = \chi(\mathcal{O}_X(tL))$ is equivalent to $h^1(tL) = 0$. We have just shown (see (14)) that this is not the case.

5 Examples

In this section we give some examples to illustrate the results obtained in §4. We use the same notation as in (4.1). The following example shows that the dimension of the image of the projection in Theorem (4.6) can actually reach all possible values.

Example 5.1 Let M be a smooth connected projective variety of dimension $n-s$, $s \geq 0$. Let $X := M \times \mathbb{P}^s$. Let L be a very ample line bundle on X. Let $p : X \to \mathbb{P}^s$ be the product projection and set $H := p^* \mathcal{O}_{\mathbb{P}^s}(1)$. Let Y be the k-dimensional subvariety of X obtained as transversal intersection of $n-k-1$ general members of $|L|$ and a general $D_{n-k} \in |L-H|$. Assume that $k \geq s$ and $k > 0$. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in the usual set up (4.1).

Let $N := N_{Y/X}$ be the normal bundle of Y in X. Note that $N \cong (\oplus^{n-k-1} L_Y) \oplus (L-H)_Y$. We let $V := (\oplus^{n-k-1} \mathcal{O}_X) \oplus H$ and $\mathcal{F} := (\oplus^{n-k-1} \mathcal{O}_{\mathbb{P}^s}) \oplus \mathcal{O}_{\mathbb{P}^s}(1)$. Thus $N^*(L) \cong V \cong p^*\mathcal{F}$ and $E \cong \mathbb{P}(p^*\mathcal{F})$, where E is the exceptional divisor of the blowing up, $\sigma : \overline{X} \to X$, of X along Y. Let α, β be the morphisms associated to $|\xi_{p^*\mathcal{F}}|$ and $|\xi_{\mathcal{F}}|$ respectively, where $\xi_{p^*\mathcal{F}}$ and $\xi_{\mathcal{F}}$ are the tautological line bundles of $\mathbb{P}(p^*\mathcal{F})$ and $\mathbb{P}(\mathcal{F})$. Consider the projection $p : X \to \mathbb{P}^s$. Since $\mathcal{F}$ is a spanned vector bundle on $\mathbb{P}^s$, it is a general fact that $\alpha : \mathbb{P}(p^*\mathcal{F}) \to \mathbb{P}^{N'}$ factors through $\beta : \mathbb{P}(\mathcal{F}) \to \mathbb{P}^{N'}$. Since $\mathcal{F}$ is the direct sum of a trivial bundle and a very ample line bundle, $\mathcal{O}_{\mathbb{P}^s}(1)$, $\xi_{\mathcal{F}}$ is big. This implies that $\dim(\mathrm{Im}\beta) = \dim(\mathbb{P}(\mathcal{F})) = n-k+s-1$. Since $\dim\phi(\overline{X}) \geq \dim\phi(E) = \dim Z$, it follows that $\dim\phi(\overline{X}) \geq n-k+s-1$.

Consider the Koszul complex

$$0 \to \wedge^{n-k} V \otimes (-(n-k-1)L) \to \cdots \to \wedge^2 V \otimes (-L) \to V \to \mathcal{J}_Y(L) \to 0.$$

Set $T := \oplus^{n-k-1} \mathcal{O}_X$ and note that $\wedge^m (T \oplus H) = \wedge^m T \oplus (\wedge^{m-1} T \otimes H)$ for each $m \geq 1$. Note also that $h^0(H) = s+1$ and hence $h^0(V) = n-k+s$. From the hypercohomology sequence associated to the Koszul complex above we see that $h^0(\mathcal{J}_Y(L)) = h^0(V) = n-k+s$. This is immediate if $L-H$ is assumed ample, but otherwise requires checking a few cases. Thus we conclude that the image of the morphism, $\phi : \overline{X} \to Z$, associated to $L \otimes \mathcal{J}_Y$ has dimension

$$\dim\phi(\overline{X}) = \dim Z \leq n-k+s-1 \leq \mathrm{cod}_X Y + s - 1.$$

Thus we conclude that $\dim\phi(\overline{X}) = n-k+s-1$.

Note that the complete intersection situation corresponds, in our present notation, to the case $s = 0$ with p the constant map.

We have the following three infinite sequences of examples (for one more class of examples see (8.3) in §8).

Example 5.2 (projection from a linear divisor) Let X be an n-dimensional projective submanifold of $\mathbb{P}^{2n-1}$. Assume that there is a linear $\mathbb{P}^{n-1}$, $D \subset X$. Let L denote the restriction of $\mathcal{O}_{\mathbb{P}^{2n-1}}(1)$ to X. Since the morphism, $\psi : X \to \psi(X)$ associated to $|L-D|$ agrees with the restriction of the projection of $\mathbb{P}^{2n-1}$ from D away from D, we see that $\dim\psi(X) \leq n-1$. From this we conclude that $(L-D)^n = 0$. A calculation given in Proposition (8.1) shows that $d := L^n = \dfrac{(s+1)^n - 1}{s}$ for $s \geq 1$ and n for $s = 0$, where the normal bundle of D in X is $\mathcal{O}_{\mathbb{P}^{n-1}}(-s)$. Since we have that $(L-D)_D \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s+1)$ is ample for $s \geq 0$, we conclude that if $s \geq 0$, then the morphism associated to $|L-D|$ has at least an $(n-1)$-dimensional image.

We now show that such examples occur for all integers $n > 0$ and $s \geq 0$. Fix integers $s \geq 0$ and $n > 0$. Let $\mathcal{P} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{2n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{2n-1}}(s+1))$ and let $p : \mathcal{P} \to \mathbb{P}^{2n-1}$ denote the bundle projection. Let ξ denote the tautological line bundle on $\mathcal{P}$

such that $p_*\xi \cong \mathcal{O}_{\mathbb{P}^{2n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{2n-1}}(s+1)$. Note that by counting constants we see that the transversal intersection of n general elements of $|\xi|$ is a smooth n-fold X' which maps isomorphically under p to its image X in $\mathbb{P}^{2n-1}$. Let $L := \mathcal{O}_{\mathbb{P}^{2n-1}}(1)_X$. Let $\mathcal{E} := \oplus^n \xi$. From the Koszul complex resolution of the ideal sheaf of X' we get the exact sequence

$$0 \to \det \mathcal{E}^* \to \wedge^{n-1}\mathcal{E}^* \to \cdots \to \wedge^2 \mathcal{E}^* \to \mathcal{E}^* \to \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{X'} \to 0.$$

By tensoring the sequence with $p^*\mathcal{O}_{\mathbb{P}^{2n-1}}(1)$ we see that the restriction map gives an isomorphism $H^0(\mathbb{P}^{2n-1}, \mathcal{O}_{\mathbb{P}^{2n-1}}(1)) \cong H^0(X, L)$. Moreover the intersection of X' with the section Σ corresponding to the quotient $\mathcal{O}_{\mathbb{P}^{2n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{2n-1}}(s+1) \to \mathcal{O}_{\mathbb{P}^{2n-1}}(1)$ is a linear $\mathbb{P}^{n-1}$ with respect to $\mathcal{O}_{\mathbb{P}^{2n-1}}(1)$. Thus X' contains a linear $\mathbb{P}^{n-1}$. Denote this by D. Since $N_{\Sigma/\mathcal{P}} \cong \mathcal{O}_{\mathbb{P}^{2n-1}}(-s-1) \otimes \xi_\Sigma$ and $\xi_\Sigma \cong \mathcal{O}_{\mathbb{P}^{2n-1}}(1)$, and since the normal bundle $N_{D/X}$ of D in X is isomorphic to the restriction of the normal bundle of Σ, we see that $N_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-s)$. As noted above the morphism, $\phi := p_X : X \to \mathbb{P}^{n-1}$, associated to $L \otimes \mathcal{J}_D$ has an $(n-1)$-dimensional image.

Recall that $L - D \approx \phi^*\mathcal{H}$ for some ample and spanned line bundle $\mathcal{H}$ on $\mathbb{P}^{n-1}$. Then in the example above one has $\mathcal{H}^{n-1} = 1$. Indeed, let $\mathcal{H} = \mathcal{O}_{\mathbb{P}^{n-1}}(h)$. Since $L - D \approx \phi^*\mathcal{H}$, we see that $h^0(L-D) = \binom{h+n-1}{n-1}$. From the exact sequence $0 \to L - D \to L \to L_D \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1) \to 0$ we infer that $h^0(L) \geq h^0(L-D) + n$. Since $h^0(L) \leq 2n$ we conclude that $h^0(L-D) \leq n$. Thus, since $n \geq 2$, $\displaystyle\binom{h+n-1}{n-1} \leq n$ implies $h = 1$.

The following example is related to Theorem (7.1) in §7.

Example 5.3 We construct here a smooth hypersurface of degree d in $\mathbb{P}^{2k+1}$ containing a linear $\mathbb{P}^k$, such that the projection from the $\mathbb{P}^k$ associated to $L := \mathcal{O}_X(1)$ has a k-dimensional image.

Consider in $\mathbb{P}^{2k+1}$ the degree d hypersurface defined by the equation

$$\sum_{j=0}^{2k+1} x_j x_{2k+1-j}^{d-1} = 0.$$

Then X is smooth and contains the linear $\mathbb{P}^k$ defined by the equations $x_{2k+1} = \cdots = x_{k+1} = 0$. The projection from this $\mathbb{P}^k$ has image $\mathbb{P}^k$.

Example 5.4 Let $X := \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^k}(1))$, where $\mathcal{E}$ is a rank r vector bundle on $\mathbb{P}^k$ of the form $\mathcal{E} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^k}(a_i)$, $a_i \geq 1$. Then X is of dimension $n = k + r$. Take as $\mathbb{P}^k$ the section of the $\mathbb{P}^r$-bundle $p : X \to \mathbb{P}^k$ corresponding to the quotient

$$\mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^k}(1) \to \mathcal{O}_{\mathbb{P}^k}(1) \to 0.$$

This guarantees that $\xi_{\mathbb{P}^k} \approx \mathcal{O}_{\mathbb{P}^k}(1)$, where $\xi_{\mathbb{P}^k}$ is the restriction to $\mathbb{P}^k$ of the tautological bundle $L := \xi$ of X. Hence in particular $\delta := L^k \cdot \mathbb{P}^k = 1$, i.e., $\mathbb{P}^k$ is linear.

Let $\sigma : \overline{X} \to X$ be the blowing up of X along the $\mathbb{P}^k$. Note that σ induces the blowing up, $\pi : \overline{F} \to \mathbb{P}^r$, at one point, x, of each fiber $F = \mathbb{P}^r$ of p. Consider the morphism $\phi : \overline{X} \to Z$ associated to $L \otimes \mathcal{J}_{\mathbb{P}^k}$. Note that the restriction $\phi_{\overline{F}}$, for each fiber $F = \mathbb{P}^r$, is the morphism given by the line bundle $|\pi^*\mathcal{O}_{\mathbb{P}^r}(1) - \pi^{-1}(x)|$. Therefore $\phi_{\overline{F}}$, being the projection of $\mathbb{P}^r$ from the point x, has lower dimensional image. Since the fibers $F = \mathbb{P}^r$ cover X we thus conclude that ϕ has lower dimensional image.

6 The divisorial case

In this section L always denotes a very ample line bundle on a n-dimensional projective manifold X, such that its global sections, $\Gamma(L)$, embed X in a projective space $\mathbb{P}^{n+r}$. Let $Y = D$ be a smooth connected divisor on X of degree $\delta = L^{n-1} \cdot D$. We assume $n \geq 2$ since the case $n = 1$ is trivial.

Recall that $\delta L - D$ is spanned (see Lemma (2.6)). In the present case we can say considerably more. Let us first show the following fact.

Lemma 6.1 *Let L be a very ample line bundle on a connected projective manifold X of dimension n. Let D be a smooth divisor of degree $\delta = L^{n-1} \cdot D$. Then either $(X, L, \mathcal{O}_X(D)) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(\delta))$, or the restriction $(\delta L - D)_D$ is an ample line bundle on D.*

Proof. By the conductor formula (2.2) and the adjunction formula we have that

$$(\delta - n - 1)L_D - K_D \approx (\delta - n - 1)L_D - (K_X + D)_D \tag{15}$$

is nef. By general adjunction theoretic results (see e.g., [2, (7.2.1)]) we know that $K_X + (n+1)L$ is either ample or $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Therefore we see from (15) that if $(\delta L - D)_D$ is not ample then $K_{X|D} + (n+1)L_D$ is not ample and hence $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. In this case $\mathcal{O}_X(D) \cong \mathcal{O}_{\mathbb{P}^n}(\delta)$. Q.E.D.

Next, we recall the following definition.

Definition 6.2 A line bundle, L, on a projective variety, X, is k-ample for an integer $k \geq 0$, if mL is spanned for some $m > 0$, and the morphism $X \to \mathbb{P}_{\mathbb{C}}$ defined by $\Gamma(mL)$ for such an m has all fibers of dimension $\leq k$.

Theorem 6.3 *Let L be a very ample line bundle on a connected projective manifold X of dimension $n \geq 2$, such that $\Gamma(L)$ embeds X in $\mathbb{P}^{n+r}$. Let D be a smooth divisor on X of degree $\delta = L^{n-1} \cdot D$. Then $\delta L - D$ is 1-ample except in the case when $(X, L, \mathcal{O}_X(D)) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(\delta))$.*

Proof. Let F be a fiber of the morphism associated to $|\delta L - D|$ and assume $\dim F \geq 2$. Then $(\delta L - D)_F \approx \mathcal{O}_F$, so that $D_F \approx \delta L_F$ is ample. This implies that $D \cap F$ contains an effective curve, C, and $D \cdot C > 0$. But $(\delta L - D) \cdot C = 0$ since $\delta L - D$ is trivial on F. If $(X, L, \mathcal{O}_X(D)) \not\cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(\delta))$ this contradicts the ampleness of $(\delta L - D)_D$ (see (6.1)). Q.E.D.

If $\delta > 1$ we can say more.

Theorem 6.4 *Let L be a very ample line bundle on a connected projective manifold X of dimension $n \geq 2$, such that $\Gamma(L)$ embeds X in $\mathbb{P}^{n+r}$. Let D be a smooth divisor on X of degree $\delta = L^{n-1} \cdot D > 1$. Assume that $(X, L, \mathcal{O}_X(D)) \not\cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(\delta))$. Then the morphism associated to $|\delta L - D|$ is birational; moreover, $\delta L - D$ is very ample if $n \geq r+2$.*

Proof. First assume $n \geq r+2$, or, equivalently, $2\dim X - (n+r) \geq 2$. Then by the Barth-Lefschetz theorem (see e.g., [2, (2.3.11)]) we conclude that $\mathrm{Pic}(X) \cong \mathbb{Z}$ with generator the restriction of the hyperplane section bundle on projective space. Since $\delta L - D$ is spanned and not trivial unless $(X, L, \mathcal{O}_X(D)) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(\delta))$ (see (6.1)), we conclude that $\delta L - D$ is a multiple of the restriction of the hyperplane section bundle on projective space. Thus $\delta L - D$ is very ample.

We next assume that $n \leq r+1$. Then

$$\delta - 1 > \frac{n}{r+1}(\delta - 1) - n + 1,$$

and Proposition (3.5) applies to say that $h^0((\delta-1)L - D) > 0$, from which it easily follows that the morphism associated to $|\delta L - D|$ is birational. Q.E.D.

Look at the embedding $X \subset \mathbb{P}^{n+r}$ and let q be the codimension of D in the smallest linear subspace $\mathbb{P}^{n-1+q} \subset \mathbb{P}^{n+r}$ containing it. Let us assume that the Castelnuovo bound conjecture holds true, i.e., $(\delta - q + 1)L - D$ is spanned by its global sections (compare with (4.13)). Clearly we have

$$r \geq q - 1. \tag{16}$$

Recall also the usual relations

$$d \geq r+1 \quad \text{and} \quad \delta \geq q+1. \tag{17}$$

Proposition 6.5 *Let L be a very ample line bundle on a connected projective manifold X of dimension $n \geq 2$, such that $\Gamma(L)$ embeds X in $\mathbb{P}^{n+r}$. Let D be a smooth divisor on X of degree $\delta = L^{n-1} \cdot D > 1$. Let q be the codimension of D in the smallest linear subspace $\mathbb{P}^{n-1+q} \subset \mathbb{P}^{n+r}$ containing it. Assume that $(\delta - q + 1)L - D$ is spanned by its global sections. Then*

1. *If $n \geq r+2$, $(\delta - q+1)L - D$ is very ample unless $X \cong \mathbb{P}^n$ and $\delta L \approx D$;*
2. *If $n \leq r+1$, then the morphism associated to $|(\delta - q + 1)L - D|$ is birational unless $q = r+1$ and either $n = r+1$ or $n < r+1$ and $\delta = r+2$.*

Proof. Assume $n \geq r+2$. Let $d := L^n$. We have the following fact.
Claim. $(\delta - q + 1)L - D$ is not trivial unless $X \cong \mathbb{P}^n$, $\delta L \approx D$.
Proof of Claim. Assume $D \approx (\delta - q + 1)L$. Dotting with L^{n-1} gives $(\delta - q + 1)d = L^{n-1} \cdot D = \delta$, or $(d-1)\delta = d(q-1)$. Using (17) this gives $(d-1)(q+1) \leq d(q-1)$, or

$$2d \leq q+1. \tag{18}$$

Since by (17) and (16), $d \geq r+1 \geq q$, we find $q \leq 1$. Thus (18) yields $d = q = 1$ and hence $r = 0$ by (17). Therefore $X \cong \mathbb{P}^n$, $D \approx \delta L$. $\square$

Since $n \geq r+2$ is equivalent to $2\dim X - (n+r) \geq 2$, by the Barth-Lefschetz theorem (see e.g., [2, (2.3.11)]) we have $\mathrm{Pic}(X) \cong \mathbb{Z}$. Since $(\delta-q+1)L-D$ is spanned and by the Claim we can assume it is not trivial, we conclude that $(\delta-q+1)L-D$ is very ample. This shows 1).

As for 2), assume $n \leq r+1$. If the morphism associated to $|(\delta-q+1)L-D|$ is not birational, then $h^0((\delta-q)L-D) = 0$. Thus, by Proposition (3.5), $\delta - q \leq \frac{n}{r+1}(\delta-1) - n + 1$, or

$$(\delta-1)\left(1-\frac{n}{r+1}\right) \leq q-n, \tag{19}$$

or, by using $r \geq q-1$ from (16),

$$(\delta-1)\left(\frac{r+1-n}{r+1}\right) \leq r+1-n. \tag{20}$$

If $r+1 = n$, then equality holds in (20) and hence in particular $r = q-1$, i.e., D spans $\mathbb{P}^{n+r}$. If $r+1 > n$, inequality (20) yields $\delta - 1 \leq r+1$, or $\delta \leq r+2$. Since $q \leq \delta - 1$ by (17), inequality (19) gives

$$(\delta-1)\left(1-\frac{n}{r+1}\right) \leq \delta-1-n \quad \text{or} \quad n \leq (\delta-1)\frac{n}{r+1}.$$

This implies $r+2 \leq \delta$. Thus $\delta = r+2$. Also, at each step, equalities hold true. Therefore $q = \delta - 1 = r+1$. Q.E.D.

Example 6.6 Notation as in (6.5). We give here an example in the range $n = r+1$ where $|(\delta-q+1)L-D|$ is spanned but the morphism associated to it is not birational, D spans $\mathbb{P}^{n+r}$ and the projection from D has an $(n-1)$-dimensional image.

Consider the Segre embedding $X = \mathbb{P}^1 \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{n+r} = \mathbb{P}^{2n-1}$, $r = n-1$, and let $p_1 : X \to \mathbb{P}^1$, $p_2 : X \to \mathbb{P}^{n-1}$ be the projections on the two factors. Denote $\mathcal{O}(a,b) := p_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^{n-2}}(b)$, for given integers a, b. Let $L := \mathcal{O}(1,1)$, so that $h^0(L) = 2n$. Take a smooth divisor D in the linear system $|\mathcal{O}(2,1)|$. We have $d := L^n = n$ and $\delta := L^{n-1} \cdot D = n+1$. Consider the exact sequence $0 \to L-D \to L \to L_D \to 0$. Note that $L - D = \mathcal{O}(-1,0)$, so $h^0(L-D) = 0$ and, by using Kunnëth's formulas, $h^1(L-D) = 0$. Therefore $h^0(L) = h^0(L_D)$. This means that D spans $\mathbb{P}^{n+r} = \mathbb{P}^{2n-1}$, or $q = r+1 = n$. Then $(\delta-q+1)L-D = 2L-D = \mathcal{O}(0,1)$. Thus $(\delta-q+1)L-D$ is not big, so that the projection from D associated to it is not birational, and has an $(n-1)$-dimensional image.

Example 6.7 Notation as in (6.5). We give here an example in the range $r = n$, where $(\delta-q+1)L-D$ is spanned but not ample, in fact is 1-ample, and the morphism associated to it is birational.

Let $X := \mathbb{P}(\oplus^{n-1}\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Let ξ be the tautological bundle of X and let F be a fiber of the bundle projection $X \to \mathbb{P}^1$. Let $L := \xi + F$ and take a smooth divisor $D \in |\xi + 2F|$. Note that both $\xi + F$ and $\xi + 2F$ are very ample (see e.g., [2, (3.2.4)]).

A standard check shows that $d = L^n = n+1$, $\delta = L^{n-1} \cdot D = n+2$ and $h^0(L) = 2n+1$, $h^0(L-D) = h^0(-F) = 0$, $h^1(L-D) = 0$. Thus $X \subset \mathbb{P}^{2n}$, i.e., $q = r+1 = n+1$. Then

$$(\delta - q + 1)L - D = 2L - D = \xi.$$

The line bundle ξ is spanned but not ample (see e.g., [2, (3.2.4)]) and the morphism associated to $|\xi|$ is the blowing up $X \to \mathbb{P}^n$ of $\mathbb{P}^n$ along $\mathbb{P}^{n-2}$. Hence in particular ξ is 1-ample.

7 The linear case

Let X be a smooth connected projective variety of dimension n, polarized by a very ample line bundle L. In this section we discuss some further results about the structure of projection maps from a k-dimensional subvariety Y of X, in the case when Y is a linear $\mathbb{P}^k$ with respect to L.

In (7.1) we show that if the morphism, ϕ, associated to $L \otimes \mathcal{J}_Y$ as in (4.1) has image dimension $n-k$, then ϕ has $\mathbb{P}^{n-k}$ as image and X is a hypersurface in $\mathbb{P}^{n+1}$. Next we show in (7.2) that assuming "Hartshorne's conjecture" we have a stronger lower bound for the dimension of the image of ϕ. Finally we prove in (7.4) a spannedness result for the adjoint bundle (see also (8.6) for more adjunction theoretic structure type results in the case when Y is a codimension 1 linear $\mathbb{P}^{n-1}$).

Let us explicitly point out the following fact: if Y is a smooth k-dimensional subvariety of (X, L) of degree $\delta = L^k \cdot Y$, then, since $\delta L \otimes \mathcal{J}_Y$ is spanned by global sections by Lemma (2.6), the morphism associated to $|tL \otimes \mathcal{J}_Y|$ is birational for $t \geq \delta + 1$. In particular, if Y is a linear $\mathbb{P}^k$ with respect to L and the projection from Y associated to tL has lower dimensional image, then necessarily $t = 1$.

In the case when Y is a linear $\mathbb{P}^k$ and the projection has image dimension one bigger than the lowest possible value we have the following result. We recall Theorem (4.6) for a general lower bound for the image dimension of ϕ and we refer back to (5.3) which gives in fact an example of the situation discussed below.

Theorem 7.1 *Let L be a very ample line bundle on X, a connected projective manifold of dimension $n \geq 2$. Let Y be a subvariety of X with $(Y, L_Y) \cong (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in* (4.1). *Assume that* $\dim Z = \operatorname{cod}_X Y = n-k$ *and $k \geq 2$. Then X is a hypersurface in $\mathbb{P}^{n+1}$.*

Proof. Set $w := n-k$. Since $\overline{L} - E$ is spanned and gives the projection $\psi : \overline{X} \to \psi(\overline{X})$ and since $\dim Z = w$, we have a surjection of locally free sheaves $\oplus^{w+1}\mathcal{O}_{\overline{X}} \to \overline{L} - E \to 0$. Hence, restricting to E, we have an exact sequence

$$0 \to \mathcal{K} \to \oplus^{w+1}\mathcal{O}_E \to (\overline{L} - E)_E \to 0.$$

Consider the $\mathbb{P}^{w-1}$-bundle map $\pi : E \to Y$. Let $N := N_{X/Y}$ be the normal bundle of Y in X. Notice that $(\overline{L} - E)_E \cong \xi$, the tautological line bundle of $E \cong \mathbb{P}(N^*(L))$. By pushing forward under π, we get an exact sequence on Y

$$0 \to K \to \oplus^{w+1}\mathcal{O}_Y \to N^*(L) \cong \pi_*\xi \to 0. \tag{21}$$

By comparing the ranks, since $N^*(L)$ has rank $\mathrm{cod}_X Y = w$, we conclude that K is a line bundle.

Since $Y \cong \mathbb{P}^k$, $k \geq 2$, the first cohomology of a line bundle is zero, i.e., $h^1(Y, K) = 0$. This means that the sections of $\oplus^{w+1}\mathcal{O}_Y$ surject onto the sections of $N^*(L)$, so $h^0(N^*(L)) \leq w + 1$.

Notice that $\overline{L} - E \approx \phi^*(\mathcal{H})$ for some ample line bundle $\mathcal{H}$ on Z. Since the restriction $\phi_E : E \to Z$ is onto by (4.4), we have $h^0(N^*(L)) = h^0((\overline{L} - E)_E) \geq h^0(\mathcal{H}) = h^0(\overline{L} - E)$. Thus

$$h^0(L \otimes \mathcal{J}_Y) = h^0(\overline{L} - E) \leq w + 1. \tag{22}$$

Now look at the exact sequence

$$0 \to L \otimes \mathcal{J}_Y \to L \to L_Y \to 0.$$

Recall that $L_Y \cong \mathcal{O}_{\mathbb{P}^k}(1)$ since Y is a linear $\mathbb{P}^k$. Therefore, by (22), $h^0(L) \leq h^0(L \otimes \mathcal{J}_Y) + h^0(\mathcal{O}_{\mathbb{P}^k}(1)) \leq w + k + 2 = n + 2$. Thus, either $\Gamma(L)$ embeds X as hypersurface in $\mathbb{P}^{n+1}$, or else $h^0(L) = n + 1$ and $X \cong \mathbb{P}^n$. However, the latter is ruled out by the assumption $\mathrm{dim} Z \geq n - k$. Q.E.D.

A minor modification of the proof of the theorem above gives us the following result, which states that assuming "Hartshorne's conjecture" (see [8]) the image dimension of ϕ has a stronger lower bound unless X is a complete intersection.

Proposition 7.2 *Let X be a smooth connected projective variety of dimension $n \geq 2$. Let L be a very ample line bundle on X. Let Y be a linear $\mathbb{P}^k$ with respect to the embedding given by $\Gamma(L)$. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in (4.1). Assume that Hartshorne's conjecture is true and that X is not a complete intersection. Then* $\mathrm{dim}\phi(\overline{X}) \geq \mathrm{cod}_X Y + \frac{k}{3} - 1$.

Proof. First note that for $k \leq 2$ the bound on $\mathrm{dim}\phi(\overline{X})$ follows from Theorem (4.6) and Corollary (4.7), so we can assume $k \geq 3$.

Set $w := n - k = \mathrm{cod}_X Y$ and $z := \mathrm{dim}\phi(\overline{X})$. Exactly the same argument as in the proof of Theorem (7.1) gives us an exact sequence $0 \to K \to \oplus^{z+1}\mathcal{O}_Y \to N^*(tL) \to 0$ on Y, where K is a vector bundle of rank $z+1-w$ and N is the normal bundle of Y in X.

Assume, by contradiction, that $z < w+\frac{k}{3}-1$, and therefore $\mathrm{rank}(K) = z+1-w < \frac{k}{3}$. Thus from (2.7) we know that K splits as a direct sum of line bundles on $\mathbb{P}^k$ (here we are using our present assumption that $k \geq 3$). Then the first cohomology of K is zero. This means that the sections of $\oplus^{z+1}\mathcal{O}_Y$ surject onto the sections of $N^*(L)$, so $h^0(N^*(L)) \leq z+1$. Again, as in the proof of (7.1), we thus conclude that

$$h^0(L \otimes \mathcal{J}_Y) \leq z+1. \tag{23}$$

Now look at the exact sequence $0 \to L \otimes \mathcal{J}_Y \to L \to L_Y \to 0$. Recall that $L_Y \cong \mathcal{O}_{\mathbb{P}^k}(1)$ since Y is a linear $\mathbb{P}^k$. Therefore, by (23), $h^0(L) \leq h^0(L \otimes \mathcal{J}_Y) + h^0(\mathcal{O}_{\mathbb{P}^k}(1)) \leq z+k+2$. Thus $\Gamma(L)$ embeds X in $\mathbb{P}^{z+k+1}$. A direct numerical check shows that the inequality $z < w + \frac{k}{3} - 1$ implies $n > \frac{2}{3}(z+k+1)$. Since we are assuming that Hartshorne's conjecture is true, we thus conclude that X is a complete intersection. Q.E.D.

We need the following result. The case when $k = 1$ also follows immediately from a result of Ilic [12].

Theorem 7.3 *Let X be a connected projective manifold of dimension $n \geq 2$. Assume that X is a $\mathbb{P}^{n-1}$-bundle $\pi : X \to C$ over a smooth curve C with fibers linear with respect to L, a very ample line bundle on X. Let $Y \subset X$ be a linear $\mathbb{P}^k$ with respect to the embedding given by $\Gamma(L)$. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in (4.1). Then* $\dim Z < n$ *if and only if either*

1. $\dim\pi(Y) = 1$, $\dim Y = 1$ *and Y is a section of π corresponding to a surjection from the vector bundle $\pi_* L$ onto a direct summand $\mathcal{O}_{\mathbb{P}^1}(1)$; or*
2. $\dim\pi(Y) = 0$, $k = n-1$, *and* $(X, L) \cong (\mathbb{P}^{n-1} \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^{n-1}\times\mathbb{P}^1}(1,1))$.

Proof. We leave the reader to check the straightforward assertion that $\dim Z < n$ in cases 1) and 2). Assume now that $\dim Z < n$.

If $\dim\pi(Y) = 1$, then since $\mathbb{P}^k$ cannot map onto a curve if $k \geq 2$, we conclude that $k = 1$ and $C \cong \mathbb{P}^1$. Since Y and fibers of π are linear, we conclude that Y meets any given fiber transversely in exactly one point. Thus Y corresponds to a surjection $\pi_* L \to \mathcal{O}_{\mathbb{P}^1}(L \cdot Y) \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Using the fact that $\pi_* L$ is very ample and a direct sum of line bundles, it is a simple check that $\pi_* L \to \mathcal{O}_{\mathbb{P}^1}(L \cdot Y) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ splits.

Assume now that $\dim\pi(Y) = 0$. If the codimension of Y is one, then we have $0 = (L - Y)^n = L^n - nL^{n-1} \cdot Y = L^n - n$. From this we see that $\pi_* L$ is a very ample rank n vector bundle of degree n. This immediately implies that $(X, L) \cong (\mathbb{P}^{n-1} \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^{n-1}\times\mathbb{P}^1}(1,1))$.

Now we consider the case when the codimension of Y is greater than one. Since $N_{Y/X} \cong \mathcal{O}_{\mathbb{P}^k} \oplus \left(\oplus^{n-1-k}\mathcal{O}_{\mathbb{P}^k}(1)\right)$, it is a straightforward consequence of Lemma (4.2) and the fact that $N^*(L)$ is spanned, that we can choose $n-k-1$ smooth divisors $D_1, \ldots, D_{n-k-1}$ in $|L \otimes \mathcal{J}_Y|$ all meeting transversely in a smooth $(k+1)$-dimensional subvariety $X_{k+1} := D_1 \cap \ldots \cap D_{n-k-1}$ containing Y as a divisor. But since it follows from the last paragraph that $(X_{k+1}, L_{X_{k+1}}) \cong (\mathbb{P}^k \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^k\times\mathbb{P}^1}(1,1))$ we infer that $\pi_{X_{k+1}*}L_{X_{k+1}} \cong \oplus^{k+1}\mathcal{O}_{\mathbb{P}^1}(1)$. Thus we conclude that $\pi_* L \cong (\oplus^{n-k-1}\mathcal{O}_{\mathbb{P}^1}) \oplus (\oplus^{k+1}\mathcal{O}_{\mathbb{P}^1}(1))$. Since $\pi_* L$ is very ample, we conclude that $n = k+1$. Q.E.D.

The case $k = 1$ of the following spannedness result for the adjoint bundle follows from [12].

Theorem 7.4 *Let X be a smooth connected projective variety of dimension $n \geq 2$. Let L be a very ample line bundle on X. Let $Y \subset X$ be a linear $\mathbb{P}^k$ with respect to the embedding given by $\Gamma(L)$. Let $\sigma : \overline{X} \to X$ be the blowing up of X along Y. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in* (4.1). *Assume that* $\dim Z < n$. *Then $K_X + (n-1)L$ is spanned by global sections unless either*

1. $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, $1 \leq k \leq n-1$, *with* $\dim Z = n-k-1$; *or*
2. $(X, L) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, *$\mathcal{Q}$ a quadric in $\mathbb{P}^{n+1}$*, $1 \leq k \leq \left[\frac{n}{2}\right]$, *with* $\dim Z = n-k$; *or*
3. *(X, L) is a scroll, $\pi : X \to C$, over a smooth curve C, i.e., $K_X + nL \approx \pi^* H$ for some ample line bundle H on C, with either*
   (a) $\dim \pi(Y) = 1$, $\dim Y = 1$ *and Y is a section of π corresponding a surjection from the vector bundle $\pi_* L$ onto a direct summand $\mathcal{O}_{\mathbb{P}^1}(1)$; or*
   (b) $\dim \pi(Y) = 0$, $k = n-1$, *and* $(X, L) \cong (\mathbb{P}^{n-1} \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^{n-1}\times\mathbb{P}^1}(1,1))$.

Proof. From general adjunction theory results we know that $K_X + (n-1)L$ is spanned unless either

(i) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$; or

(ii) $(X, L) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, $\mathcal{Q}$ a quadric in $\mathbb{P}^{n+1}$; or

(iii) (X, L) is a scroll, $\pi : X \to C$, over a smooth curve C, i.e., $K_X + nL \approx \pi^* H$ for some ample line bundle H on C.

In case (i), by looking at the projection of $\mathbb{P}^n$ from $\mathbb{P}^k$, we see that $1 \leq k \leq n-1$ with $\dim Z = n-k-1$.

In case (ii) we see that $1 \leq k \leq \left[\frac{n}{2}\right]$ with $\dim Z = n-k$, by looking at the projection of $\mathbb{P}^{n+1}$ from $\mathbb{P}^k$.

In case (iii), use Theorem (7.3). Q.E.D.

Corollary 7.5 *Let X be a smooth connected projective variety of dimension $n \geq 2$. Let L be a very ample line bundle on X. Let $Y \subset X$ be a linear $\mathbb{P}^k$ with respect to the embedding given by $\Gamma(L)$. Let $\phi : \overline{X} \to Z$ be the morphism associated to $L \otimes \mathcal{J}_Y$ as in* (4.1). *Assume that* $\dim Z < n$. *Let $N := N_{Y/X}$ be the normal bundle of Y in X. If (X, L) is not as in one of cases* 1), 2), 3) *of* (7.4), *one has* $c_1(N) \leq n-2-k$.

Proof. By the assumption, $K_X + (n-1)L$ is spanned. On the other hand,

$$(K_X + (n-1)L)_Y \approx K_Y - \det N + (n-1)L_Y \cong \mathcal{O}_{\mathbb{P}^k}(n-2-k) - \det N.$$

Since $(K_X + (n-1)L)_Y \cong \mathcal{O}_{\mathbb{P}^k}(b)$ for some nonnegative integer b, we thus conclude that $\det N \cong \mathcal{O}_{\mathbb{P}^k}(a)$ for some integer $a \leq n-2-k$. Q.E.D.

8 The linear case in codimension 1

Let X be a smooth connected projective variety of dimension $n \geq 2$. Let L be a very ample line bundle on X. Let P be a linear $\mathbb{P}^{n-1} \subset X$ with respect to L, i.e., $\delta = L^{n-1} \cdot P = 1$. Recall that in this case the line bundle $L - P$ is spanned (see the discussion after Lemma (4.3)). We follow the notation of (4.1), with the exception of denoting Y by P to emphasize its special nature. Thus we let $\psi : X \to \psi(X)$ be the morphism associated to $|L - P|$ and $\psi = \mathfrak{s} \circ \phi$ the Remmert-Stein factorization of ψ with $\phi : X \to Z$ having connected fibers and $\mathfrak{s} : Z \to \psi(X)$ finite.

In this section we study the projection from P, a linear $\mathbb{P}^{n-1}$, under the assumption that $n > \dim\phi(X)$. For shortness, it is convenient to refer to the situation above simply saying that (X, L, P) is a $\mathbb{P}^{n-1}$*-degenerate triple.*

First, let us state the following preliminary facts.

Proposition 8.1 *Let X be a connected n-dimensional manifold and let L be very ample line bundle on X. Assume that (X, L, P) is a $\mathbb{P}^{n-1}$-degenerate triple. Let $N := N_{\mathbb{P}^{n-1}/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-s)$ be the normal bundle of $P := \mathbb{P}^{n-1}$ in X. Then we have:*

1. *$s \geq -1$, with equality only if $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;*
2. *if $s \geq 0$, the morphism $\psi : X \to \psi(X)$ associated to $|L - P|$ has an $(n-1)$-dimensional image with all fibers having dimension one; and ψ_P is finite; and*
3. *the degree of (X, L) is given by $d := L^n = \dfrac{(s+1)^n - 1}{s}$ for $s \geq 1$ and by n for $s = 0$.*

Proof. Items 1) and 2) follow immediately from Lemma (6.1) and Theorem (6.3).

As for 3), note that since $L - P$ is not big we have $(L - P)^n = 0$. Then

$$d = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} L^{n-j} \cdot P^j.$$

By noting that $L^{n-j} \cdot P^j = \mathcal{O}_P(1)^{n-j} \cdot \mathcal{O}_P(-s)^{j-1} = (-1)^{j-1} s^{j-1}$, we find $d = \sum_{j=1}^{n} \binom{n}{j} s^{j-1}$. This gives the result. Q.E.D.

In light of the above results we will assume that $(X, L) \ncong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, i.e., $N_{P/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-s)$ with $s \geq 0$.

Let $\mathcal{H}$ be the ample line bundle on Z such that $L - P \approx \phi^*(\mathcal{H})$. Set $\mathfrak{h} = \mathcal{H}^{n-1}$ and $t = L \cdot f$ for a general fiber f of ϕ. We have $L_P - P_P \approx \mathcal{O}_{\mathbb{P}^{n-1}}(s+1) \approx \phi_P^*(\mathcal{H})$. Since $t = \deg \phi_P$, we conclude that

$$t\mathfrak{h} = (s+1)^{n-1}. \qquad (24)$$

Note that the restriction $\phi_P : \mathbb{P}^{n-1} \to Z$ is a t-to-one finite morphism.

Remark 8.2 Note that by (2.4) and (2.5) applied to the finite map ϕ_P we conclude that Z is Cohen-Macaulay, has t-factorial singularities, and $\mathrm{Pic}(Z) \cong \mathbb{Z}$.

Let us give one more class of examples.

With the notation as above, assume that (X, L, P) is a $\mathbb{P}^{n-1}$-degenerate triple with $s \geq 0$ and $t = 1$. Since the restriction ϕ_P is an isomorphism under this assumption we see that, by using also relation (24), $(Z, \mathcal{H}) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s+1))$, and that ϕ is a $\mathbb{P}^1$-bundle (see also [2, (3.2.1)]). We let $V := \phi_* \mathcal{O}_X(P)$ and

$$\mathcal{E} := \phi_* L \cong \phi_*(\mathcal{O}_X(P) \otimes L) \cong \phi_*(\mathcal{O}_X(P) \otimes \phi^* \mathcal{H}) \cong V \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(s+1). \quad (25)$$

Then $X \cong \mathbb{P}(\mathcal{E}) \cong \mathbb{P}(V)$.

Proposition 8.3 *If $s \geq 0$ and $t = 1$ then $(X, L) \cong (\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(s+1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)), \xi)$, where ξ denotes the tautological line bundle on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(s+1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1))$.*

Proof. From the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(P) \to P_P \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-s) \to 0$, by taking the direct image and since the higher direct image functor $R^i \phi_* \mathcal{O}_X$ is zero for $i > 0$, we get the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^{n-1}} \to V \to \mathcal{O}_{\mathbb{P}^{n-1}}(-s) \to 0$. Since $h^1(\mathcal{O}_{\mathbb{P}^{n-1}}(s)) = 0$ we see that this sequence splits. Thus $\mathcal{E} = \phi_* L \cong V \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(s+1) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s+1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. From this the result is clear. Q.E.D.

From relation (24) we see that $s = 0$ implies $t = 1$. This gives the following consequence.

Corollary 8.4 *If $s = 0$ then $(X, L) \cong (\mathbb{P}^{n-1} \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^{n-1} \times \mathbb{P}^1}(1,1))$.*

Remark 8.5 Note that the example of a $\mathbb{P}^{n-1}$-degenerate triple given by $\mathbb{P}^n$ blown up at one point z, $p : X \to \mathbb{P}^n$, with $L = p^* \mathcal{O}_{\mathbb{P}^n}(2) - P$, $P = p^{-1}(z)$, fits in Proposition (8.3) with $s = 1$.

By the above, we can work from now on under the extra assumptions that $s \geq 1$ and $t \geq 2$, where $N_{\mathbb{P}^{n-1}/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-s)$ and $t = \deg \phi_{\mathbb{P}^{n-1}}$.

We can now carry out some more adjunction theoretic analysis, improving, in the case of a codimension 1 linear projective space, the results proved in (7.4). We will also assume $n \geq 3$, since the problem is completely solved when $n = 2$ (see [18], [2, §8.4]). For the structure of the first reduction map occurring in the theorem below we refer to [2, Chap. 7].

Theorem 8.6 *Let X be a smooth connected n-dimensional variety, $n \geq 3$, and let L be very ample line bundle on X. Assume that (X, L, P) is a $\mathbb{P}^{n-1}$-degenerate triple. Let $N := N_{\mathbb{P}^{n-1}/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-s)$. Assume that $s \geq 1$ and $t := \deg \phi_P \geq 2$. Then the first reduction exists, i.e., there exists a map $\pi : X \to X'$ expressing X as the blowup of a projective manifold X' at a finite set B with $K_X + (n-1)L \approx \pi^* H$ for a very ample line bundle H on X'. Moreover it follows that π is an isomorphism unless B is a single point, $s = 1$, and $P := \mathbb{P}^{n-1} = \pi^{-1}(B)$.*

Proof. Set $P := \mathbb{P}^{n-1}$. If $K_X + (n-1)L$ is not spanned, then (X, L) is as in one of cases 1), 3) of (7.4) (notice that case 2) of (7.4) is excluded because we have $\dim P > \left[\frac{n}{2}\right]$). In case 1) we have that $K_X + (n+1)L$ is trivial, which implies that $\mathcal{O}_P \approx (K_X + (n+1)L)_P \approx \mathcal{O}_{\mathbb{P}^{n-1}}(s+1)$. Thus $s = -1$. In case 3), we have $t = 1$. Therefore both cases 1), 3) of (7.4) are excluded in view of our present assumptions that $s \geq 1$ and $t \geq 2$.

Therefore we can assume that $K_X + (n-1)L$ is spanned. It follows [2, §7.3] that either $K_X + (n-1)L$ is nef and big or:

1. $K_X \cong -(n-1)L$; or
2. (X, L) is a quadric fibration, $\pi : X \to C$, over a smooth curve C, i.e., $K_X + (n-1)L \cong \pi^*H$ for some ample line bundle H on C; or
3. (X, L) is a scroll, $\pi : X \to S$, over a smooth surface S, i.e., $K_X + (n-1)L \cong \pi^*H$ for some ample line bundle H on S.

In the first case we have that $\mathcal{O}_f \cong (K_X + (n-1)L)_f$ for a general fiber f of ϕ. Since $(1-n)L \cdot f = (1-n)t = K_X \cdot f = \deg(K_f)$ we conclude that $n = 3$ and $t = L \cdot f = 1$, contradicting our present assumption $t \geq 2$.

Since $P = \mathbb{P}^{n-1}$ can't map to a curve by Lemma (2.3), we conclude in the second case that P is a component of a fiber of π. But since $n \geq 3$ fibers are either irreducible quadrics, or two $\mathbb{P}^{n-1}$'s meeting in a $\mathbb{P}^{n-2}$. Indeed multiple fibers don't happen, since otherwise we could slice down to a surface and have $\mathbb{P}^1$ as a multiple fiber, which is a classical standard impossibility. If we are in the case of two $\mathbb{P}^{n-1}$'s meeting in a $\mathbb{P}^{n-2}$, then we have negative normal bundle for each $\mathbb{P}^{n-1}$ and we can contract one $\mathbb{P}^{n-1}$ to get a map of the other $\mathbb{P}^{n-1}$ to a $(n-1)$-dimensional image but with the intersection $\mathbb{P}^{n-2}$ going to a point, which is not possible again by Lemma (2.3).

In the third case we know from a result of the fourth author [20, Theorem (3.3)] that π is a $\mathbb{P}^{n-2}$-bundle. Thus we conclude that P is a section with $n = 3$. Indeed since fibers of π are one dimensional we conclude that P meets a general fiber f of π in a finite nonempty set. Since $L - P$ is nef and $L \cdot f = 1$ we conclude that $P \cdot f = 1$. Since $(L - P) \cdot f = 0$ it is clear that π is the same as ϕ and $t = 1$.

Thus we see that $K_X + (n-1)L$ is big and the first reduction $\pi : X \to X'$ exists. Assume that π is not an isomorphism. Let F be a positive dimensional fiber of π. We know that F is a linear $\mathbb{P}^{n-1}$ with respect to L and $N_{F/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. If we show that $F = P$ then we see that $s = 1$ and the theorem will be proved. Thus assume that F is not P. Then we see that $F \cap P$ is empty or we would have the absurdity that π maps the positive dimensional subset $F \cap P$ of P to the point $\pi(F)$ without mapping P to the same point. Thus we have $L_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Therefore we see that F is a section of $\phi : X \to Z$. Thus we conclude that ϕ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$. Restricting the bundle to a bundle $\phi_S : S \to R$ on a smooth curve R on Z, we find a $\mathbb{P}^1$-bundle S over R with two disjoint curves, $P \cap S$ and $F \cap S$, each with negative self intersection since both the normal bundles $N_{P/X}$, $N_{F/X}$ are negative. This is absurd. Q.E.D.

We conclude this section by considering the special case of a threefold X.

8.7 The three dimensional case. We use the same notation and assumptions as above. In particular in view of the results above we make the blanket assumption that $s \geq 1$ and $t \geq 2$.

Theorem 8.8 *Let X be a smooth threefold and L a very ample line bundle on X. Assume that (X, L, P) is a $\mathbb{P}^2$-degenerate triple. If $s = 1$ and $\mathfrak{t} := \deg \phi_P \geq 2$, then $t = 4$. In this case X is the blowing up at one point of the complete intersection of three quadrics in $\mathbb{P}^6$.*

Proof. If $s = 1$ then by Proposition (8.1), 3) we see that $L^3 = 7$. Note that we use the classification of degree 7 manifolds given in [13]. By Theorem (8.6) we can assume that $K_X + 2L$ is nef and big. Thus quadric fibrations over curves and scrolls over curves and surfaces are ruled out. By using the degree 7 classification, two possibilities remain.

1. X is the blowing up at one point, $\pi : X \to X'$, of the complete intersection X' of three quadrics in $\mathbb{P}^6$, with π the first reduction map; or

2. there exists a morphism $\rho : X \to C$ of X to a curve C given by the complete linear system $|m(K_X + L)|$ for $m \gg 0$.

In the first case we know from [13] that L embeds X into $\mathbb{P}^5$. This X contains the positive dimensional fiber of π and thus since projection from this linear $\mathbb{P}^2$ must map to $\mathbb{P}^2$ we conclude that this is an example with $s = 1$. Let $f \cong \mathbb{P}^1$ be a fiber of $\phi : X \to Z$. To see what t is, note that $K_X + L$ being nef yields $t = L \cdot f \geq -K_X \cdot f = 2$. By Theorem (8.6) we know that P coincides with the exceptional divisor of π. Moreover, $-K_{X'} \cong \mathcal{O}_{X'}(1) = L'$, the polarization of the first reduction X', which satisfies the condition $L \cong \pi^* L' - P$. Then

$$K_X \cong \pi^* K_{X'} + 2P \cong -L - P + 2P = -L.$$

Hence we have $K_X \cdot f = \deg(K_f) = 0$. Thus we cannot have $t = 2$ since this would imply f was rational. Since we are assuming $t \geq 2$ we conclude by relation (24) that t must equal 4.

In the second case $\rho(P)$ must be a point by Lemma (2.3) and therefore $(K_X + L)_P \cong \mathcal{O}_P$. Since $(K_X + L)_P \cong \mathcal{O}_P(s - 2)$ we get the contradiction $s = 2$. Q.E.D.

Combining Theorem (8.6) and Theorem (8.8) we have the following result.

Corollary 8.9 *Let X be a smooth threefold and L a very ample line bundle on X. Assume that (X, L, P) is a $\mathbb{P}^2$-degenerate triple. If $s \geq 1$ and $\mathfrak{t} := \deg \phi_P \geq 2$, then either X is the blowing up at one point of the complete intersection of three quadrics in $\mathbb{P}^6$, or $K_X + 2L$ is very ample.*

Proof. By (8.6) and (8.8) we know that either $s = 1$ and X is the blowing up at one point of the complete intersection of three quadrics in $\mathbb{P}^6$ or X is isomorphic to its own first reduction. Q.E.D.

Theorem 8.10 *Let X be a smooth threefold and L a very ample line bundle on X. Assume that (X, L, P) is a $\mathbb{P}^2$-degenerate triple. Further assume $s \geq 2$. Then the case $t = 2$ does not occur.*

Proof. By Corollary (8.9) we can assume that (X, L) is its own first reduction. A simple check of the list of pairs with $K_X + L$ not nef (see [2, §7.3]) shows that they cannot occur if $s \geq 2$. Thus we can assume that $K_X + L$ is nef. We know that there is a morphism with connected fibers $\rho : X \to W$ of X onto a normal variety W, given by $|m(K_X + L)|$ for $m \gg 0$, with $K_X + L \cong \rho^* H$ for some ample line bundle H on W. Note that if $t = 2$ then the general fiber of $\phi : X \to Z$ is a conic. Thus $K_X + L$ must be trivial on the general fiber of ϕ. Then there exists a surjective morphism $q : Z \to W$ such that $q \circ \phi = \rho$, whence $\dim W \leq 2$. Note also that $\dim W > 0$. Indeed otherwise $K_X + L$ would be trivial and therefore, since $(K_X + L)_P \cong \mathcal{O}_P(s - 2)$, we would have $s = 2$. But $t = s = 2$ contradicts relation (24).

The divisor P can not be in a fiber of ρ. If it was we would have $(K_X + L)_P \cong \mathcal{O}_P$. This would imply $s = 2$. Then again $t = s = 2$ contradicts relation (24). By using Lemma (2.3) we conclude that $\dim W = 2$ and, since P must map onto W, that all fibers of ρ are one dimensional. By the above, (X, L) is a quadric fibration over the surface W. Then by Besana's results [3] we know that W is smooth and thus by Lazarsfeld's theorem (see e.g., [2, (3.1.7)]) we know that W is $\mathbb{P}^2$. We also see that the maps ρ and ϕ are the same.

Note that by pulling back to P we have

$$m(K_X + L)_P \cong \mathcal{O}_P(m(s - 2)) \cong (L - P)_P \cong \phi_P^* \mathcal{H} \cong \mathcal{O}_{\mathbb{P}^2}(s + 1).$$

This gives $s + 1 = m(s - 2)$ and hence either $s = 5$, $m = 2$, $L - P \cong 2(K_X + L)$, or $s = 3$, $m = 4$, $L - P \cong 4(K_X + L)$. Assume $s = 5$. Then, since $t = 2$, relation (24) gives $\mathfrak{h} = \mathcal{H}^2 = 18$. But since $L - P \cong 2(K_X + L)$ we have the absurdity that $18 = \mathcal{H}^2 = 4H^2$. Assume $s = 3$. Then $\mathfrak{h} = \mathcal{H}^2 = 8$ from relation (24) and $L - P \cong 4(K_X + L)$ gives the absurdity $8 = \mathcal{H}^2 = 16H^2$. Q.E.D.

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Mauro C. Beltrametti
Dipartimento di Matematica
Via Dodecaneso 35
I-16146 Genova, Italy
beltrame@dima.unige.it

Alan Howard
Department of Mathematics
Notre Dame, Indiana, 46556, U.S.A.
howard.1@nd.edu

Andrew J. Sommese
Department of Mathematics
Notre Dame, Indiana, 46556, U.S.A.
sommese@nd.edu
URL: http://www.nd.edu/∼sommese