On the degree and the birationality of the second adjunction mapping

Mauro C. Beltrametti, Andrew J. Sommese

Abstract

Let $\mathcal{L}$ be a very ample line bundle on $\mathcal{M}$, a projective manifold of dimension $n \geq 3$. Under the assumption that $K_{\mathcal{M}} + (n-2)\mathcal{L}$ has Kodaira dimension $n$, we study the degree of the map $\phi$ associated to the complete linear system $|2(K_{\mathcal{M}} + (n-2)\mathcal{L})|$. In particular we show that under a number of conditions, e.g., $n \geq 5$ or $K_{\mathcal{M}} + (n-3)\mathcal{L}$ having nonnegative Kodaira dimension, the degree $\phi$ is one, i.e., $\phi$ is birational. We also show that under a mild condition on the linear system $|K_{\mathcal{M}} + (n-2)\mathcal{L}|$ satisfied for all known examples, $\phi$ is birational unless $(\mathcal{M}, \mathcal{L})$ is a three dimensional variety with very restricted invariants. Moreover there is an example with these invariants such that $\deg \phi = 2$.

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Introduction

Let $\mathcal{L}$ be a very ample line bundle on an $n$-dimensional projective manifold $\mathcal{M}$. We assume throughout this introduction that $n \geq 3$ and also that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$, i.e., that the Kodaira dimension of $K_{\mathcal{M}} + (n-2)\mathcal{L}$ is $n$. This condition on the Kodaira dimension is satisfied except for a short list of special varieties $(\mathcal{M}, \mathcal{L})$ (see e.g., [2]).

The condition $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ implies that $\mathcal{M}$ is the blowing up of a projective manifold $M$, $\pi : \mathcal{M} \to M$, at a finite set such that $K_{\mathcal{M}} + (n-2)\mathcal{L} \cong \pi^*(K_M + (n-2)L)$, where $L := (\pi_*\mathcal{L})^{**}$ is an ample line bundle, $K_M + (n-1)L$ is very ample, and $\mathcal{K}_M := K_M + (n-2)L$ is nef. Thus the a priori meromorphic map associated to $|2(K_{\mathcal{M}} + (n-2)\mathcal{L})|$ factors as $\pi$ composed with the mapping associated to $|2\mathcal{K}_M|$. It is a theorem of the second author (see [2, (13.2.5)]) that in the situation of this paper $2\mathcal{K}_M$ is spanned. Thus we are reduced to describing the structure of the morphism $\phi : M \to \mathbb{P}_c$ associated to $|2\mathcal{K}_M|$. The morphism $\phi$ factors as $r \circ s$, where $r : M \to Y$ is a birational morphism with connected fibers onto a normal projective variety $Y$ and $s$ is finite. The structure of $r$ is completely understood; cf., [2, 12] for a description of this map.
In Theorem (2.1) we show that under the added assumption that there is a smooth divisor \( A \in |\mathcal{K}_M| \) the map \( s \) (and hence \( \phi \)) is birational unless \( n = 3 \), \( \mathcal{K}_M^3 = 1, \mathcal{K}_M^2 \cdot L = 3, \mathcal{K}_M \cdot L^2 = 9 \), and \( A \) is a Del Pezzo surface with \( K^3_A = 1 \). There is moreover a degree 27 threefold \( M \subset \mathbb{P}^{13} \) with these invariants and with \( \deg \phi = 2 \) (see (2.2)). Though we expect such a smooth \( A \) to exist always, this is not known. We show that, without any assumption on the existence of a smooth \( A \in |\mathcal{K}_M| \),

1) the same conclusion holds under the added hypothesis that \( n = \dim M \geq 5 \) (see Corollary (2.4));

2) the morphism \( \phi \) is birational if \( \kappa(K_M + (n - 3)L) \geq 0 \) (see Theorem (2.3)); and

3) \( \deg \phi \leq 7 \) if \( n = 3 \) (see Theorem (3.1)).

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1 Background material

We work over the complex field \( \mathbb{C} \). Throughout the paper we deal with projective varieties \( V \). We denote by \( \mathcal{O}_V \) the structure sheaf of \( V \) and by \( K_V \) the canonical bundle. For any coherent sheaf \( \mathcal{F} \) on \( V \), \( h^i(\mathcal{F}) \) denotes the complex dimension of \( H^i(V, \mathcal{F}) \).

Let \( L \) be a line bundle on \( V \). The line bundle \( L \) is said to be numerically effective (nef, for short) if \( L \cdot C \geq 0 \) for all effective curves \( C \) on \( V \). \( L \) is said to be big if \( \kappa(L) = \dim V \), where \( \kappa(L) \) denotes the Kodaira dimension of \( L \). If \( L \) is nef then this is equivalent to \( c_1(L)^n > 0 \), where \( c_1(L) \) is the first Chern class of \( L \) and \( n = \dim V \).

1.1 Notation. The notation used in this paper are standard from algebraic geometry. Let us only fix the following.

\( \approx \), the linear equivalence of line bundles;

\( \chi(L) = \sum_i (-1)^i h^i(L) \), the Euler characteristic of a line bundle \( L \);

\( |L| \), the complete linear system associated to a line bundle \( L \);

\( \Gamma(L) \), the space of the global sections of a line bundle, \( L \), on a variety \( V \); we say that \( L \) is spanned if it is spanned at all points of \( V \) by \( \Gamma(L) \);

\( q(V) = h^1(\mathcal{O}_V) \), the irregularity, for \( V \) smooth;

\( \kappa(D) \), the Kodaira dimension of the line bundle associated to a Cartier divisor \( D \) on \( V \);
\[ \kappa(V) := \kappa(K_V), \] the Kodaira dimension of \( V \), for \( V \) smooth.

Line bundles and divisors are used with little (or no) distinction. We almost always use the additive notation.

1.2 Genus formula. For a line bundle \( L \) on an irreducible normal variety \( V \) of dimension \( n \) the sectional genus, \( g(L) = g(V, L) \), of \( (V, L) \) is defined by \( 2g(L) - 2 = (K_V + (n - 1)L) \cdot L^{n-1} \). Note that if \( |L| \) contains \( n - 1 \) elements \( H_1, \ldots, H_{n-1} \) meeting in a reduced irreducible curve \( C \) contained in the smooth points of \( V \), then \( g(L) = g(C) = 1 - \chi(\mathcal{O}_C) \), the arithmetic genus of \( C \).

1.3 Castelnuovo’s bound. Let \( C \) be a reduced irreducible projective curve. Assume that \( \psi : C \to \mathbb{P}^N \) is a generically one-to-one morphism, and that \( \psi(C) \) does not lie in any hyperplane. Let \( d \) denote the degree of \( \psi(C) \) in \( \mathbb{P}^N \). Let \( g(C) \) be the arithmetic genus of \( C \). Then Castelnuovo’s bound (see, e.g., [6, Theorem 3.7]) reads

\[
g(C) \leq \text{Castel}(d, N) := \left[ \frac{d - 2}{N - 1} \right] \left( d - N - \left( \left[ \frac{d - 2}{N - 1} \right] - 1 \right) \frac{N - 1}{2} \right),
\]

where \( \left[ x \right] \) means the greatest integer \( \leq x \).

1.4 Reductions. (See e.g., [2, Chapters 7, 12]). Let \( (\mathcal{M}, \mathcal{L}) \) be a smooth variety of dimension \( n \geq 2 \) polarized with a very ample line bundle \( \mathcal{L} \). A smooth polarized variety \( (\mathcal{M}, \mathcal{L}) \) is called a reduction of \( (\mathcal{M}, \mathcal{L}) \) if there is a morphism \( \pi : \mathcal{M} \to M \) expressing \( \mathcal{M} \) as the blowing up of \( X \) at a finite set of points, \( B \), such that \( L := (\pi_* \mathcal{L})^* \) is ample and \( \mathcal{L} \approx \pi^* L - [\pi^{-1}(B)] \) or, equivalently, \( K_M + (n - 1) \mathcal{L} \approx \pi^* (K_M + (n - 1) L) \).

Note that there is a one-to-one correspondence between smooth divisors of \( |L| \) which contain the set \( B \) and smooth divisors of \( |\mathcal{L}| \).

Except for an explicit list of well understood pairs \( (\mathcal{M}, \mathcal{L}) \) (see in particular [2, §§7.2, 7.3, 7.4]) we can assume:

a) \( K_M + (n - 1) \mathcal{L} \) is spanned and big, and \( K_M + (n - 1) L \) is very ample. Note that this reduction, \( (M, L) \), is unique up to isomorphism. We will refer to it as the first reduction of \( (\mathcal{M}, \mathcal{L}) \);

b) \( K_M + (n - 2) L \) is nef and big, for \( n \geq 3 \).

Since by the above we can assume that \( K_M + (n - 2) L \) is nef and big, from the Kawamata-Shokurov base point free theorem (see [7, §3]) we know that \( m(K_M + (n - 2) L) \), for \( m \gg 0 \), gives rise to a morphism \( \varphi : M \to X \) with connected fibers and normal image. Thus there is an ample line bundle \( \mathcal{K} \) on \( X \) such that
\( K_M + (n-2)L \approx \varphi^*K \). The morphism \( \varphi \) is very well behaved (see e.g., [2, §§7.5, 7.6, 7.7 and Chapter 12]). Furthermore \( X \) has terminal, 2-Gorenstein (i.e., \( 2K_X \) is a line bundle) isolated singularities and \( K \approx K_X + (n-2)D \), where \( D := (\varphi_*L)'' \) is a 2-Cartier divisor such that \( 2L \approx \varphi^*(2D) - \Delta \) for some effective divisor \( \Delta \) on \( M \) which is \( \varphi \)-exceptional and \( \dim \varphi(\Delta) \leq 1 \) (see [2, (7.5.7)]). The pair \((X, D)\) is known as the second reduction of \((\mathcal{M}, \mathcal{L})\). For definition and properties of terminal singularities we also refer to [7].

We say that \((\mathcal{M}, \mathcal{L})\) is of log-general type if \( \kappa(K_M + (n-2)\mathcal{L}) = n \). Notice that this is equivalent to saying that \( K_M + (n-2)L \) is nef and big (see [2, (7.6.9)]). Let \( \hat{S} \) be the smooth surface obtained as transversal intersection of \( n-2 \) general members of \( \mathcal{L} \) and let \( S := \pi(\hat{S}) \) be the corresponding smooth surface in \( M \). Since \( K_M + (n-2)L \) is nef and big the canonical bundle \( K_S \) of \( S \) is nef and big, so that \( S \) is a minimal surface of general type (see also [2, (7.6.10)]). We have

\[
K_S \cdot K_S < 9\chi(O_S). \tag{2}
\]

Indeed, Miyaoka’s inequality yields \( K_S \cdot K_S \leq 9\chi(O_S) \). Note that the equality cannot happen. Otherwise \( S \) is ball quotient and hence a \( K(\pi, 1) \), which contradicts [2, (1.3)].

For further properties of log-general type polarized pairs see, e.g., [2, §13.2] and [3, (0.10)].

1.5 Pluridegrees. Let \((\mathcal{M}, \mathcal{L})\), \((M, L)\) be as in \((1.4)\). Define the pluridegrees, for \( j = 0, \ldots, n = \dim \mathcal{M} \), by

\[
\hat{d}_j := (K_M + (n-2)\mathcal{L})^j \cdot \mathcal{L}^{n-j} \quad \text{and} \quad d_j := (K_M + (n-2)L)^j \cdot L^{n-j}. \]

If \( \gamma \) denotes the number of points blown up under \( \pi : \mathcal{M} \to M \), then because \( K_M + (n-2)\mathcal{L} \approx \pi^*(K_M + (n-2)L) + \sum_i E_i \), \( E_i \) the exceptional divisors, the invariants \( \hat{d}_j, d_j \) are related by

\[
\hat{d}_j = d_j - (-1)^j \gamma.
\]

We put \( \hat{d} := \hat{d}_0 \), \( d := d_0 \). If \( K_M + (n-2)L \) is nef, by the generalized Hodge index theorem (see e.g., [2, (2.5.1), (13.1)]) one has

\[
d_j^2 \geq d_{j-1}d_{j+1}, \quad j = 1, \ldots, n-1, \tag{3}
\]

and the parity Lemma (13.1.1) of [2] says that

\[
d_j \equiv d_{j+1} \mod(2) \quad \text{for} \quad j = 0, \ldots, n-1 \quad \text{and} \quad j \text{ even}. \tag{4}
\]

Moreover if \( K_M + (n-2)L \) is nef and big, i.e., if \((\mathcal{M}, \mathcal{L})\) is of log-general type, the \( d_j \)'s are positive.
If the second reduction, $(X, \mathcal{D}), \varphi : M \to X$, with $\mathcal{D} = (\varphi_*L)^\times, \mathcal{K} \approx K_X + (n - 2)\mathcal{D}$, of $(M, \mathcal{L})$ exists, then we define

$$d^j_0 := \mathcal{K}^j \cdot \mathcal{D}^{n-j}, \quad j = 0, \ldots, n, \quad d^j := d^j_0.$$ Note that $d_j = d^j_0$ for $j \geq 2$. To see this recall that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ for some effective Cartier divisor $\Delta$ which is $\varphi$-exceptional (see (1.4)) and compute, for $j \geq 2$,

$$2^{n-j}d_j = (K_M + (n - 2)L)^j \cdot (2L)^{n-j} = (\varphi^*\mathcal{K})^j \cdot (\varphi^*(2\mathcal{D}) - \Delta)^{n-j} = 2^{n-j}\mathcal{K}^{n-j} \cdot \mathcal{D}^j = 2^{n-j}d^j_0,$$

where the last but one equality follows from the fact that $\dim \varphi(\Delta) \leq 1$.

The following is a consequence of the Tsuji inequality (see [13, (5.2)]), the log version of the usual Yau inequality (see also [2, (13.1.7), (13.1.8)]).

**Proposition 1.6 (Tsuji inequality)** Let $\mathcal{M}$ be a smooth 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that the first reduction, $(M, L)$, of $(M, \mathcal{L})$ exists. Let $d_j, 0 \leq j \leq 3$, be the pluridegrees of $(M, L)$. Assume that $K_M + L$ is nef. Let $S$ be a smooth element in $[L]$. Then we have

$$(K_M + L)^3 + \frac{8}{3}K_S \cdot L_S \leq 32(2h^0(K_M + L) - \chi(\mathcal{O}_S)), \quad \text{or}$$

$$h^0(K_M + L) \geq \frac{\chi(\mathcal{O}_S)}{2} + \frac{d_1}{24} + \frac{d_3}{64}.$$

The following lower bound for the degree is not optimal, but it is sufficient for our purposes.

**Lemma 1.7** Let $\mathcal{M}$ be a smooth $n$-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $(M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Then either $[\mathcal{L}]$ embeds $\mathcal{M}$ in $\mathbb{P}^{n+1}$ or $d \geq \hat{d} := \mathcal{L}^n \geq 8$.

**Proof.** We can assume that $\Gamma(\mathcal{L})$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq n + 2$. Let $\hat{S}$ be the smooth surface obtained as transversal intersection of $n - 2$ general members of $[\mathcal{L}]$. Since $\hat{S}$ is of general type we have by a result of Castelnuovo (see [2, (8.1)], [9, (0.6)]) that $\hat{d} > 2(N - n) + 2 \geq 6$. Thus $d \geq \hat{d} \geq 7$. Assume $\hat{d} = 7$. Then $N = n + 2$ since otherwise $\hat{d} > 2(N - n) + 2 \geq 8$. Thus by Castelnuovo’s bound we conclude that $g(C) \leq 6$ for any smooth curve section $C$ of $\mathcal{M} \subset \mathbb{P}^{n+2}$. Therefore, by the genus formula, $7 + d_1 \leq 7 + \hat{d}_1 = 2g(C) - 2 \leq 10$, or $d_1 \leq 3$. But then $d_2 \leq 1$ by the Hodge index relation $d_1^2 \geq dd_2$. Since $(\mathcal{M}, \mathcal{L})$ is of log-general type we know that $d_j \geq 1$ for each $j = 1, \ldots, n$. Therefore we conclude that $d_2 = 1$ and $d_2^2 \geq d_1d_3$ implies $d_1 = 1$. This gives the absurdity that $d = 1$. Q.E.D.
2 Two birationality results for the second adjunction mapping

Let $\mathcal{M}$ be a smooth connected $n$-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, D)$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reductions of $(\mathcal{M}, \mathcal{L})$ as in (1.4).

Set $\mathcal{K}_M := K_M + (n-2)L$. In this section we study the birationality of the map given by the complete linear system $|2\mathcal{K}_M|$ under the assumption that $|K_M + (n-2)L|$ contains a smooth element (see (2.1)). We consider this result to be one of the most important results in this paper since it is likely optimal. We also prove the birationality of $|2\mathcal{K}_M|$ in the case when $\kappa(K_M + (n - 3)\mathcal{L}) \geq 0$ (see (2.3)).

Note that, since $\mathcal{K}_M \approx \varphi^*\mathcal{K}$, the map associated to $|2\mathcal{K}|$ is a birational morphism if and only if the map associated to $|2\mathcal{K}_M|$ is a birational morphism.

Let us recall the following basic result of the second author (see e.g., [2, (13.2.5)])

- With the notations as above, assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Then $2\mathcal{K}_M$ is spanned by its global sections.

The first result we have is the following.

**Theorem 2.1** Let $\mathcal{M}$ be a smooth connected $n$-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. Assume that there exists a smooth element $A$ in the complete linear system $|\mathcal{K}_M|$. Then the map associated to $|2\mathcal{K}_M|$ is a birational morphism unless $n = 3$, $d_3 = 1$, $d_2 = 3$, $d_1 = 9$ and $A$ is a Del Pezzo surface with $K_A^2 = 1$.

**Proof.** Let $L_A$ be the restriction of $L$ to $A$. Consider on $M$ the exact sequence

$$0 \to K_M + (n - 2)L \to 2\mathcal{K}_M \to K_A + (\dim A - 1)L_A \to 0.$$ 

Since $2\mathcal{K}_M$ is spanned by $\bullet$ and $h^1(K_M + (n - 2)L) = 0$ we see that $K_A + (\dim A - 1)L_A$ is spanned and defines a morphism $\psi_A$ which coincides with the restriction, $\phi_A$, of the morphism $\phi$ associated to $|2\mathcal{K}_M|$. Note that the connected part of the morphism $\psi_A$ is the first reduction map, $\pi_A : A \to A'$, of the pair $(A, L_A)$ and that the positive dimensional fibers of $\pi_A$ are contained in the exceptional set of $\phi$. We have $\psi_A = s_A \circ \pi_A$, where $s_A : A' \to \mathbb{P}_C$ is a finite-to-one morphism.

Assume first $n \geq 4$. Then by a well known result due to the second author and Van de Ven (see e.g., [2, (11.3.1)]), $s_A$ is an embedding. Let $t$ be the sheet number of $\phi$, i.e., $\phi$ is a generically $t$-to-one morphism. Choose a general point $x \in M$. Then $\phi^{-1}(\phi(x))$ consists of $t$ distinct points. Note that, since $2A$ belongs to the linear system defining $\phi$, a smooth element $A \in |\mathcal{K}_M|$ containing $x$ contains all the $t$ points. Therefore $\psi_A^{-1}(\psi_A(x))$ consists of $t$ distinct points and hence, since $s_A$ is an embedding, $\pi_A^{-1}(\pi_A(x))$ consists of $t$ distinct points. This contradicts the fact that $\pi_A$ has connected fibers. Thus we conclude that $t = 1$. 

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It remains to consider the case when $n = 3$. In this case exactly the same argument works unless the polarized surface $(A, L_A)$ is one of the four exceptional pairs described e.g., in [2, (10.3.1)].

The first possibility is that $A$ is a degree 9 Del Pezzo surface, $K_A^2 = 1$, $L_A \approx -3K_A$. Then $L_A \approx -3(2K_M + L)_A$, or $(6K_M + 4L)_A \approx O_A$. It thus follows that $(3K_M - L)_A \approx O_A$ and hence, since $A \in |K_M|$, we have $(3K_M - L) \cdot K_M \cdot D = 0$ for any effective divisor $D$ on $M$. Therefore, by taking $D = A$ and $D = L$, we find $3d_3 = d_2$ and $3d_2 = d_1$ respectively. Thus $d_1 = 9d_3$. If $d_3 = 1$, we have $d_1 = 9$, $d_2 = 3$ and we find the 3-dimensional exceptional case as in the statement. Note that the case $d_3 \geq 2$ cannot occur, since in this case $d_1 \geq 18$ and we get the numerical contradiction $9 = L_A \cdot L_A = L \cdot L \cdot K_M = d_1 \geq 18$.

The second possibility is that $A$ is a degree 8 Del Pezzo surface, $K_A^2 = 2$, $L_A \approx -2K_A$. The same argument as in the case considered above gives now $(4K_M - L)_A \approx O_A$ and hence $d_1 = 4d_2$, $d_2 = 4d_3$. Thus we find the numerical contradiction $8 = L_A \cdot L_A = L \cdot L \cdot K_M = d_1 = 16d_3$.

The third possibility is when $A$ is the blowing up at one point, $r : A \to A'$, of a degree 8 Del Pezzo surface $(A', L')$ with $L' \approx -2K_{A'}$, $K_{A'}^2 = 2$. In this case $2K_A + L_A \approx r^*(2K_{A'} + L') + \ell \approx \ell$, where $\ell$ is the exceptional line of $r$. Therefore $2K_A + L_A$ has a section. Since $A \in |K_M|$ it follows from $K_A \approx (2K_M - L)_A$ that $(4K_M + 3L)_A \approx (4K_M - L)_A$ has a section. Then from

$$(4K_M - L) \cdot K_M = (4K_M - L)_A \cdot K_M|_A = (2K_A + L_A) \cdot K_M|_A = \ell \cdot K_M|_A = 0$$

and

$$(4K_M - L) \cdot K_M \cdot L = (2K_A + L_A) \cdot L = \ell \cdot L_A = 1$$

we get $4d_3 = d_2$ and $4d_2 = d_1 + 1$ respectively. But $d_1 = K_M \cdot L \cdot L = L_A \cdot L_A = 7$, so that we find the numerical contradiction $7 = d_1 = 16d_3 - 1 \geq 15$.

The fourth possibility is when $A$ is a $\mathbb{P}^1$-bundle over a smooth elliptic curve of invariant $e = -1$, $L_A \approx 3E$, $E$ a section of minimal self-intersection. Let $f \cong \mathbb{P}^1$ be a fiber of the $\mathbb{P}^1$-bundle. Then $K_A \approx -2E + f$ and $(K_A + L_A) \cdot L_A = (E + f) \cdot 3E = 6$. Thus from $2L_A \cdot K_M|_A = (K_A + L_A) \cdot L_A = 6$ we get $L_A \cdot K_M|_A = L \cdot K_M = K_M|_A = d_2 = 3$. Moreover $d_1 = L_A \cdot L_A = 9$, so that the Hodge inequality $d_1 \cdot d_3 \leq d_2^2$ yields $d_3 = 1$. This gives the contradiction $g(M) = 0 ([4, (3.2)])$. Q.E.D.

**Example 2.2** Notation as in (2.1). Note that the exceptional case of a polarized 3-fold $(M, L)$ with the morphism associated to $|2(K_M + L)|$ not birational and with invariants $d_3 = 1$, $d_2 = 3$, $d_1 = 9$ as in (2.1) really occurs. The example in [8] has $\mathcal{L} = 3H$ very ample with $K_M \cong -2H$. Thus all reduction maps are isomorphisms with $K_M = K_M + L \approx H$ having $h^0(K_M) = 3$, $d_3 = K_M^3 = 1$, $d_2 = K_M^2 \cdot \mathcal{L} = 3H^3 = 3$, $d_1 = K_M^2 \cdot \mathcal{L} = 3H^3 = 9$. The morphism $\phi$ associated to $|2K_M|$ has degree 2 (see also [11, (5.3)])]. Indeed, from the exact sequence

$$0 \to K_M \cong H \to 2H \to 2H_H \to 0,$$

we obtain $h^0(2K_M) = 7$ and $\phi : M \to \mathbb{P}^6$ maps $M$ surjectively onto the cone $\mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(2))$ over $(\mathbb{P}^2, O_{\mathbb{P}^2}(2))$. Since $(2K_M)^3 = 8$ and the tautological bundle of $\mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(2))$ satisfies $\xi^3 = 4$ we conclude that $\phi$ has degree 2.
A second result we have is the following.

**Theorem 2.3** Let $\mathcal{M}$ be a smooth connected $n$-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $\kappa(K_{\mathcal{M}} + (n - 3)\mathcal{L}) \geq 0$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $K_{\mathcal{M}} := K_M + (n - 2)L$. Then the map associated to $|2K_{\mathcal{M}}|$ is a birational morphism.

**Proof.** We claim that $d_2 := K_S^2 \geq 10$, where $S = \pi(\hat{S})$ and $\hat{S}$ is the transversal intersection of $n - 2$ general members of $|\mathcal{L}|$. To see this consider the 3-fold section $\mathcal{V} = \mathcal{V}_3$ of $\mathcal{M}$ obtained as transversal intersection of $n - 3$ general members of $|\mathcal{L}|$. Let $\mathcal{L}_\mathcal{V}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Let $V := \pi(\mathcal{V})$ and let $L_V$ be the restriction of $L$ to $V$. Then the reduction is compatible with the restriction, i.e., $(\mathcal{V}, L_V)$ is the first reduction of $(V, L_V)$. Let $d = d_0, d_1, d_2, d_3$ be the pluridegrees of $(V, L_V)$ as in (1.5), so that $d_2 = K_S^2$. Note that the assumption $\kappa(K_{\mathcal{M}} + (n - 3)\mathcal{L}) \geq 0$ implies $\kappa(V) \geq 0$. Then from [12, (1.5), (3.1)] we know that

$$d_3 \geq d_2 \geq d_1 \geq d.$$  

(5)

From Lemma (1.7) we have $d \geq 8$. First consider the case when $|\mathcal{L}_\mathcal{V}|$ embeds $\mathcal{V}$ in $\mathbb{P}^N, N \geq 6$. If $d = 8, 9$, Castelnuovo’s bound (1.3) gives $g := g(\mathcal{L}_\mathcal{V}) \leq 5, 7$ respectively. Since $d_1 \geq d$ by (5) the genus formula $d + d_1 = 2g - 2$ gives a numerical contradiction.

Thus we can assume that $|\mathcal{L}_\mathcal{V}|$ embeds $\mathcal{V}$ in $\mathbb{P}^5$. By looking over the list of small degree 3-folds in $\mathbb{P}^5$ (see e.g., [5, Chapter 6]) we see that the only possible cases with $d = 8, 9$ are when $\mathcal{V}$ is the complete intersection of either a quadric and a quartic or two cubics in $\mathbb{P}^5$ and $\mathcal{L}_\mathcal{V} = \mathcal{O}_V(1)$. Accordingly, $\mathcal{M}$ is the complete intersection either of type $(2, 4)$ or of type $(3, 3)$ in $\mathbb{P}^{n+2}$ with $\mathcal{L} \approx \mathcal{O}_M(1)$. In both cases $K_{\mathcal{M}} + (n - 2)\mathcal{L}$ is very ample. Thus from now on we can assume that $d_2 \geq 10$.

First assume $n = 3$, so that $\kappa(M) \geq 0$. Recalling $\bullet$, consider the morphism $\phi : M \to Y$ associated to $|2K_{\mathcal{M}}|$. Assume that $\phi$ is not birational, i.e., is not generically one-to-one. Thus there exists a dense open set $U \subset Y$ such that for any point $y \in Y$, the fiber $\phi^{-1}(y)$ contains at least two points $x_1 := x_{1,y}, x_2 := x_{2,y}$ (depending on $y$) such that $\phi(x_1) = \phi(x_2) = y$. From a Bertini’s type theorem (see e.g., [2, (1.7.9)]) we know that there is a smooth surface $\hat{S} \in |\mathcal{L}|$ passing through $\pi^{-1}(x_1), \pi^{-1}(x_2)$. Then the image $S = \pi(\hat{S})$ is a smooth surface in $|L|$ passing through $x_1, x_2$.

The exact sequence

$$0 \to K_M + K_N \to 2K_M \to 2K_S \to 0$$

gives a surjection $\Gamma(2K_M) \to \Gamma(2K_S) \to 0$. Thus we see that $\phi$ restricts to the bicanonical map $\phi_S$ associated to $|2K_S|$. Hence in particular $\phi_S$ is not an embedding. Recall that $S$ is a minimal surface of general type under our assumptions (see (1.4)). Then we know by Reider’s theorem (see e.g., [2, (8.5.1)]) that there exists an effective curve $C_y \subset S$, $C_y$ depending on $y$ and containing $x_1, x_2$, such that

$$K_S \cdot C_y - 2 \leq C_y \cdot C_y < \frac{K_S \cdot C_y}{2} < 2.$$  

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Therefore, since $K_S$ is nef, we have $0 \leq K_S \cdot C_y \leq 3$. The case $K_S \cdot C_y = 3$ is excluded. Indeed in this case $C_y^2 = 1$ and the Hodge index relation $d_2 \leq K_S^2 C_y^2 \leq (K_S \cdot C_y)^2$ contradicts the assumption $d_2 \geq 10$.

Note that when $y$ varies in $U$, the curves $\{C_y\}_{y \in U}$ cover a dense open set $\phi^{-1}(U)$ of $M$. This is clear since the points $x_{1,y}, x_{2,y}$ cover $\phi^{-1}(U)$.

Assume $K_S \cdot C_y = 0$. Then, by the genus formula, $C_y^2 = -2$ and $C_y$ is the union of rational curves. Thus by the above, there is a dense open set $U \subset M$ covered by rational curves. This contradicts the assumption $\kappa(M) \geq 0$ (see e.g., [10, Part I, (5.8)]). Since $\kappa(M) \geq 0$, a multiple $tK_M$ of the canonical bundle $K_M$ is effective for some $t \gg 0$. Since the curve $C_y$ moves it thus follows that $K_M \cdot C_y \geq 0$.

If $K_S \cdot C_y = 1, 2$, from $K_S \cdot C_y = (K_M + L) \cdot C_y \geq L \cdot C_y$ we obtain $L \cdot C_y \leq 2$. Since $L$ is very ample outside of a finite set of points, this implies that $C_y$ is a rational curve. Thus we conclude as in the previous case.

It remains to consider the general case when $n \geq 4$. Let $\mathcal{V}_t$ be the $t$-fold section of $\mathcal{M}$ obtained as transversal intersection of $n-t$ general members of $|\mathcal{L}|$, $t = 2, \ldots, n$, $\mathcal{V}_n := \mathcal{M}$, $\mathcal{V}_2 := \hat{S}$. Let $\mathcal{L}_t$ be the restriction of $\mathcal{L}$ to $\mathcal{V}_t$. Let $\nu_t := \pi(\mathcal{V}_t)$ and let $\mathcal{L}_t$ be the restriction of $L$ to $\nu_t$. Reductions are compatible with restrictions, i.e., $(\nu_t, \mathcal{L}_t)$ is the first reduction of $(\mathcal{V}_t, \mathcal{L}_t)$. Set $\mathcal{K}_t := \mathcal{K}_{\nu_t} + (t-2)L_t$. Consider the exact sequence, for $t = 3, \ldots, n-1$,

$$0 \to \mathcal{K}_{\nu_t} + \mathcal{K}_t + (n-3)L_t \to 2\mathcal{K}_t \to 2\mathcal{K}_{t-1} \to 0,$$

which gives a surjective map $\Gamma(2\mathcal{K}_t) \to \Gamma(2\mathcal{K}_{t-1})$. Assuming $\phi$ is not birational and using induction on $t$, we conclude as above that the bicanonical map of the surface section $V_2 := S$ of $M$ with respect to $L$ is not an embedding. Recalling that the assumption $\kappa(K_M + (n-3)\mathcal{L}) \geq 0$ implies $\kappa(\mathcal{V}_3) = \kappa(V_3) \geq 0$, the above argument shows that we can reduce to the 3-fold case. Q.E.D.

By using results from [1] we obtain from Theorem (2.3) the following rather strong consequence.

**Corollary 2.4** Let $\mathcal{M}$ be a smooth connected $n$-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $K \approx K_X + (n-2)\mathcal{D}$, be the first and the second reductions of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. If $n \geq 5$, then the map associated to $|2\mathcal{K}_M|$ is a birational morphism.

**Proof.** First assume $n \geq 7$. Then by [1, (3.1)] we know that $K_X + (n-3)\mathcal{K}$ is nef. Thus $t(K_X + (n-3)\mathcal{K})$ is effective for some integer $t \in \mathbb{Z}$, i.e., $K_X + (n-3)\mathcal{K}$ is $\mathbb{Q}$-effective. Therefore $\kappa(K_X + (n-3)\mathcal{K}) \geq 0$. Hence from [2, (7.6.1), (7.6.2)] we have

$$\kappa(K_X + (n-3)\mathcal{D}) = \kappa(K_M + (n-3)\mathcal{L}) = \kappa(K_M + (n-3)L) \geq 0.$$

Thus Theorem (2.3) applies to give the result.
Assume \( n = 6 \). Then from \([1, (4.1)]\) we know that \( K_X + 3\mathcal{K} = K_X + (n - 3)\mathcal{K} \) is nef unless \((X, \mathcal{K}) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))\). In this case \( \mathcal{K} \) is very ample, so \( |2\mathcal{K}_M| = |\sigma^*\mathcal{O}_{\mathbb{P}^6}(2)| \) defines a birational morphism. Thus the same argument as above lets us conclude that \( \kappa(K_M + 3L) \geq 0 \) and hence that \( |2\mathcal{K}_M| \) defines a birational morphism by (2.3).

Assume \( n = 5 \). Then from \([1, (4.1)]\) we know that \( K_X + 3\mathcal{K} \) is nef unless either \((X, \mathcal{K}) \cong (\mathbb{Q}, \mathcal{O}_{\mathbb{Q}}(1))\), \( \mathcal{Q} \) hyperquadric in \( \mathbb{P}^6 \), or \( L^5 = 121(\neq \frac{3^n - 1}{2}) \) and \( X \) is a singular 2-Gorenstein Fano 5-fold with \( 2K_X \approx -7\mathcal{K} \).

In the first case, \( \mathcal{K} \) is very ample, so that \( |2\mathcal{K}_M| = |\sigma^*\mathcal{O}_{\mathbb{Q}}(2)| \) gives a birational morphism again by (2.3).

In the second case, note that \( X \) is not Gorenstein since otherwise \( K_X \approx -7A \) for some ample line bundle \( A \) on \( X \). This contradicts the well known fact that \( X \) has index \( \leq \dim X + 1 = 6 \) (see also \([2, (3.3.2)]\)). Thus, since \( X \) is not Gorenstein, \([2, (0.3.3)]\) applies to say that \( h^0(K_M + 2\mathcal{L}) = h^0(K_M + 2L) > 0 \).

Write \( 2\mathcal{K}_M = \mathcal{K}_M + (K_M + 2L) + L \). Since \( \mathcal{K}_M \approx \varphi^*(\mathcal{K}) \), \( |L| \) gives a birational map (\( L \) is very ample off a finite set of points) and \( K_M + 2L \) is effective, we see that \( 2\mathcal{K}_M \) is the sum of a nef and big and an effective line bundle. Thus we conclude that \( |2\mathcal{K}_M| \) gives a birational morphism.

Q.E.D.

The following two lemmas are not essentially used in the paper, but have some interest in themselves.

**Lemma 2.5** Let \( \mathcal{M} \) be a smooth connected \( n \)-fold polarized by a very ample line bundle \( \mathcal{L} \), \( n \geq 3 \). Assume that \((\mathcal{M}, \mathcal{L})\) is of log-general type. Let \( \pi : (\mathcal{M}, \mathcal{L}) \to (M, L) \) be the first reduction of \((\mathcal{M}, \mathcal{L})\). Let \( \mathcal{K}_M := K_M + (n - 2)L \). If \( h^0(K_M + \mathcal{K}_M + tL) \neq 0 \) for \( t \leq n - 3 \), then the morphism given by \( |2\mathcal{K}_M| \) is birational.

**Proof.** Write \( 2\mathcal{K}_M \) as \( 2\mathcal{K}_M = K_M + \mathcal{K}_M + tL + (n - 2 - t)L \). Since \( K_M + \mathcal{K}_M + tL \) is effective, \( |L| \) gives a birational map and \( n - 2 - t > 0 \) we are done. Q.E.D.

**Lemma 2.6** Let \( \mathcal{M} \) be a smooth connected \( n \)-fold polarized by a very ample line bundle \( \mathcal{L} \), \( n \geq 3 \). Assume that \((\mathcal{M}, \mathcal{L})\) is of log-general type. Let \( \pi : (\mathcal{M}, \mathcal{L}) \to (M, L) \) be the first reduction of \((\mathcal{M}, \mathcal{L})\). Let \( \mathcal{K}_M := K_M + (n - 2)L \). Let \( V \) be the \( t \)-fold section of \((M, L)\) obtained as transversal intersection of \( n - t \) general members of \(|L|\). Let \( \mathcal{L}_V \) be the restriction of \( L \) to \( V \) and let \( \mathcal{K}_V := K_V + (t - 2)\mathcal{L}_V \). If \( t \geq 3 \) and the map given by \( |2\mathcal{K}_M| \) is not birational, then

\[
h^0(K_V + \mathcal{K}_V + j\mathcal{L}_V) = 0 \quad \text{for} \quad 0 \leq j \leq t - 3.
\]

**Proof.** Consider on \( M \) the Koszul complex

\[
0 \to -(n - t)L \to \cdots \to \oplus^{n-t-1}(-2L) \to \oplus^{n-t}(-L) \to \mathcal{O}_M \to \mathcal{O}_V \to 0.
\]

By tensoring with \( \mathcal{K}_M + K_M + (n - t + j)L \) we get the exact sequence

\[
0 \to K_M + \mathcal{K}_M + jL \to \cdots \to K_M + \mathcal{K}_M + (n - t + j)L \to K_V + \mathcal{K}_V + j\mathcal{L}_V \to 0.
\]
Note that $h^1(K_M + \mathcal{K}_M + jL) = 0$ by Kodaira vanishing and $h^0(K_M + \mathcal{K}_M + (n-t+j)L) = 0$ by Lemma (2.5) since $n-t+j \leq n-3$. Thus $h^0(K_V + \mathcal{K}_V + jL_V) = 0$. Q.E.D.

As a consequence of (2.5) and (2.6) we have the following general fact.

**Remark 2.7** Let $\mathcal{M}$ be a smooth connected $n$-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \rightarrow (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + (n-2)L$. If the morphism associated to $|2\mathcal{K}_M|$ is not birational, then $h^i(\mathcal{O}_M) = 0$, $i \geq 3$.

To see this, assume first $n = 3$. Then $h^3(\mathcal{O}_M) = h^0(K_M)$. If $h^0(K_M) > 0$, we would have an inclusion $0 \rightarrow L \rightarrow K_M + L$, so that the morphism associated to $|\mathcal{K}_M| = |K_M + L|$ would be birational.

To show the statement in the general case, assume e.g., $n = 4$. Let $V$ be the general smooth element of $|L|$ and consider the exact sequence

$$0 \rightarrow -L \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_V \rightarrow 0.$$  

Look at the cohomology associated to this sequence. Note that $h^i(-L) = 0$, $i = 0, 1, 2, 3$. Moreover $h^i(\mathcal{O}_M) = h^0(K_M) = 0$ since otherwise we would have an inclusion

$$0 \rightarrow (n-2)L = 2L \rightarrow K_M + 2L = \mathcal{K}_M,$$

and hence the same contradiction as above. Let $L_V$ be the restriction of $L$ to $V$. Set $\mathcal{K}_V := K_V + 2L_V$. By the assumption, it follows that also the morphism associated to $|2\mathcal{K}_M|$ is not birational. Therefore Lemma (2.6) yields $h^0(K_V + \mathcal{K}_V) = 0$ and thus $h^0(K_V) = h^3(\mathcal{O}_V) = 0$. Then $h^3(\mathcal{O}_M) = 0$.

In the general case the same argument as above gives the result, starting from the Koszul complex

$$0 \rightarrow -(n-3)L \cdots \rightarrow \oplus^{n-4}(-2L) \rightarrow \oplus^{n-3}(-L) \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_V \rightarrow 0.$$

## 3 A bound for the degree of the second adjunction mapping in the case $n = 3$

We keep the notation and the assumptions as at the beginning of §2. This section is devoted to the proof of the following result.

**Theorem 3.1** Let $\mathcal{M}$ be a smooth connected threefold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \rightarrow (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let $\mathcal{K}_M := K_M + L$. Let $t$ be the degree of the morphism, $\phi$, associated to $|2\mathcal{K}_M|$. Then $t \leq 7$.

**Proof.** Let $d_j$, $j = 0, 1, 2, 3$, $d_0 = d$, be the pluridegrees of $(M, L)$. Let $S$ be a smooth general member of $|L|$.
Note that we can assume $h^0(K_M + K_M) = h^0(2K_M - L) = 0$. Indeed otherwise $\Gamma(2K_M - L)$ gives an embedding $\Gamma(L) \hookrightarrow \Gamma(2K_M)$ and hence $\Gamma(2K_M)$ gives a birational morphism, given on a Zariski open set by sections of $\Gamma(L)$. Therefore a Riemann-Roch computation (see [4, §6]) gives the general relation

$$4\chi(O_S) + d_2 = d_3 + h,$$

where $h := h^0(K_M + L)$. Note also that $K_M$ is nef and big. Then the exact sequence

$$0 \to K_M + K_M \to 2K_M \to 2K_S \to 0$$

yields $h^0(2K_M) = d_2 + \chi(O_S)$. Thus we obtain

$$(2K_M)^3 = 8d_3 = t\deg \phi(M) \geq t(d_2 + \chi(O_S) - 3).$$

Reasoning by contradiction, assume $t \geq 8$. Then (7) yields $d_3 \geq d_2 + \chi(O_S) - 3$, which, combined with (6), reads $\chi(O_S) \geq 2h - 1$. From this and Tsuji’s inequality (1.6) we infer that

$$\frac{d_1}{12} + \frac{d_3}{32} \leq 1,$$

i.e.,

$$\frac{d_1}{12} + \frac{d_3}{32} = 1.$$  

(8)

Hence in particular $d_3 \geq 2$ since $d_1$ is integer.

Recall that $h \geq 2$ from [3, (1.2)]. Thus (6) and the above inequality $\chi(O_S) \geq 2h - 1$ give

$$\frac{d_3 - d_2}{2} = 2\chi(O_S) - 3h \geq h - 2 \geq 0,$$

so that $d_3 \geq d - 2$.

Recall that $d \geq 8$ by Lemma (1.7) and $d_3 \geq 2$ by the above. Then simply starting from $d \geq 8$, $d_1 \geq 1$, $d_2 \geq 1$ and $d_3 \geq 2$ and by using repeatedly the Hodge index inequalities (3) we obtain $d \geq 8$, $d_1 \geq 8$, $d_2 \geq 8$, $d_3 \geq 8$.

Assume $\tilde{d} := \mathcal{L}^3 = 8$. Then $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^5$. Indeed otherwise Castelnuovo’s bound (1) gives $g(\mathcal{L}) \leq \text{Castel}(8,4) \leq 5$ and hence we would have the numerical contradiction $16 \leq \tilde{d} + \tilde{d}_1 = d + d_1 = 2g(\mathcal{L}) - 2 \leq 8$. By looking over the list of 3-folds in $\mathbb{P}^5$ of degree $\leq 12$ (see [5, Chapter 6]) we see that the only degree 8 log-general type case is the complete intersection of a quadric and a quartic. In this case $K_M \approx O_M(1)$ and therefore $t = 1$, contradicting the present assumption $t \geq 8$.

Thus we can assume $\tilde{d} \geq 9$, so that $d \geq 9$. Then starting from $d \geq 9$, $d_1 \geq 1$, $d_2 \geq 1$, $d_3 \geq 2$ and using repeatedly the Hodge index inequalities (3) we obtain $d \geq 9$, $d_1 \geq 9$, $d_2 \geq 9$, $d_3 \geq 9$. Now use relation (8) to get the numerical contradiction

$$1 = \frac{d_1}{12} + \frac{d_3}{32} \geq \frac{9}{12} + \frac{9}{32} > \frac{9}{12} + \frac{8}{32} = 1.$$

Thus we conclude that $t \leq 7$. Q.E.D.
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Mauro C. Beltrametti
Dipartimento di Matematica
Via Dodecaneso 35
I-16146 Genova, Italy
beltrame@dima.unige.it

Andrew J. Sommese
Department of Mathematics
University of Notre Dame
Notre Dame, Indiana, 46556, U.S.A.
sommese@nd.edu
http://www.nd.edu/~sommese