DISCRIMINANT LOCI OF VARIETIES WITH SMOOTH NORMALIZATION

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Abstract. Let X be a smooth complex projective n-fold endowed with an ample and spanned line bundle L. Under the assumption that $\Gamma(L)$ defines a generically one-to-one map we describe the singular set of the general element in the main component of the discriminant locus of $|L|$. This description is used to show that (X, L) is covered by linear $\mathbb{P}^k$'s, where $k+1$ stands for the codimension of the main component. We also give some applications relating k to the spectral value of (X, L) and discuss some examples.

Introduction

Let $X^\#$ be an irreducible and reduced n-dimensional variety in the complex projective space $\mathbb{P}^N$. Assume that $X^\#$ is nondegenerate, i. e., that no proper linear subspace of $\mathbb{P}^N$ contains $X^\#$. If $X^\#$ is smooth the discriminant variety of $X^\#$, i. e. , the subvariety $\mathcal{D}$ of $\mathbb{P}^{N*}$ consisting of all singular hyperplane sections of $X^\#$ is irreducible and there are many remarkable relations connecting the geometry of $X^\#$ and $\mathcal{D}$ [Kl, Section I-7]. However, even for mildly singular $X^\#$, $\mathcal{D}$ does not have to be irreducible and the refined theory for smooth $X^\#$ breaks down. Nevertheless one can identify the main component of $\mathcal{D}$ with the dual variety of $X^\#$, for which some general results are known also when $X^\#$ is singular. E. g. , a classical fact going back to Bertini is that the general contact locus is a linear space.

The aim of this article is twofold. First, continuing the theory started in [LPS2], we recover a part of the classical perspective in the framework of polarized manifolds (Section 2). Secondly we extend the refined theory, holding in the smooth case, to varieties admitting mild singularities (Section 3). Actually we study the situation when the normalization X of $X^\#$ is smooth. Note that, restrictive as it is, this assumption is rather natural when we consider that every irreducible curve has a smooth normalization. Let $\nu : X \to X^\#$ denote the normalization map and

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set $L := \nu^*\mathcal{O}_{\mathbb{P}^N}(1)$ and $V := \nu^* H^0(\mathcal{O}_{\mathbb{P}^N}(1))$. Let $|V|$ denote the linear system associated to this vector space of sections of L. Note that the finite birational morphism ν is the map associated to $|V|$. Though the set of singular $D \in |V|$ may have many components, and the classical proofs break down, many of the important results still hold in this setting, while they fail in a more general context (Section 4).

More precisely, let $\mathcal{D}_0 \subset |V|$ denote the closure of those $D \in |V|$ singular at a general point of X. Then $\mathcal{D}_0$ is irreducible and Theorem (2.3) states that if the codimension of $\mathcal{D}_0$ is $k+1$ then for a general element $A \in \mathcal{D}_0$ it follows that

$$(\mathrm{Sing}(A), L_{\mathrm{Sing}(A)}) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1)).$$

Moreover $\mathrm{Sing}(A)$ is a locus of nondegenerate quadratic singularities. As a consequence, if $k > 0$, it turns out that X is covered by rational varieties (Corollary (2.5)) and is of negative Kodaira dimension. Further if $k > 0$, then a basic result of Ein [E, Theorem 2.3] from the smooth case on the normal bundle of the general contact locus still holds: it relies on Theorem (1.2), which shows that there is a symmetric isomorphism

$$N_{\mathrm{Sing}(A)/X} \cong N^*_{\mathrm{Sing}(A)/X} \otimes L.$$

¿From this we also get an extension of Landman's parity result to our setting (Proposition (3.2)). Moreover we derive Theorem (3.3), which is a generalization of a result of Beltrametti, Sommese and Wiśniewski [BSW], i. e. if $k > 0$, then $K_X + \left(\dfrac{n+k}{2} + 1\right) L$ is nef but not ample and there is a contraction of an extremal ray $\rho : X \to Y$ onto a normal projective variety Y, with $K_X + \left(\dfrac{n+k}{2} + 1\right) L = \rho^* H$ for an ample line bundle H on Y. Using this we get a classification result of varieties with positive k, $k \geq n-6$, Theorem (3.5). In particular the classification of projective n-folds of positive defect with $n \leq 7$ due to Beltrametti, Fania, and Sommese [BFS] extends to our setting. Finally examples showing some of the many possibilities for the decomposition of $\mathcal{D}$ into irreducible components are given (Section 4).

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0. Background and general facts

We work over the complex number field $\mathbb{C}$. We use standard symbols in algebraic geometry. Following a current abuse we use the additive notation for the tensor product of line bundles. Let X be a smooth connected projective manifold of dimension n; we denote by K_X the canonical bundle of X and we denote by $\mathcal{F}_Y$ the pull back of a vector bundle $\mathcal{F}$ on X via an embedding $Y \hookrightarrow X$.

(0.1) *Jet bundles.* Let L be a line bundle on X. For any non-negative integer r we denote by $J_rL = J_r(X, L)$ the r-th jet bundle of L, which is the vector bundle of rank $\binom{n+r}{n}$ associated to the sheaf $\pi^*L/(\pi^*L \otimes \mathcal{I}_\Delta^{r+1})$, where $\pi : X \times X \to X$ is the projection onto the first factor, $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal $\Delta \subset X \times X$, and the tensor product is with respect to $\mathcal{O}_{X\times X}$. We recall that the fibre of J_rL over each point $x \in X$ is $(J_rL)_x = L_x/\mathfrak{m}_x^{r+1}$, where $\mathfrak{m}_x$ stands for the maximal ideal of L_x. In particular $J_0L = L$. Interpreting germs of J_rL as Taylor expansions of holomorphic sections of L truncated after the r-th term, the map sending such a germ to the Taylor expansion truncated after the $(r-1)$-th term defines a bundle homomorphism $J_rL \to J_{r-1}L$ giving rise to the following exact sequence

$$0 \to T_X^{*\,(r)} \otimes L \to J_rL \to J_{r-1}L \to 0,$$

where $T_X^{*\,(r)}$ denotes the r-th symmetric power of the cotangent bundle of X. Moreover for every subspace $V \subseteq H^0(L)$ there is a natural homomorphism $j_r : V \to H^0(J_rL)$ sending every section $s \in V$ to its r-th jet.

¿From now on in this Section we assume that L is an ample line bundle on X, spanned (i. e. globally generated at every point $x \in X$) by a vector subspace $V \subseteq H^0(X, L)$.

(0.2) *Discriminant locus.* We shall freely use the notation in [LPS2, Section 0] apart from some minor changes. In particular if $x \in X$, we denote by $V - mx$ the subspace of V consisting of the sections $s \in V$ such that $j_{m-1}(x) = 0$. We set $|V|$ ($|V - mx|$) to denote the linear space of divisors corresponding to the elements of V ($V - mx$). In particular $|V - 2x|$ consists of the elements of $|V|$ which are singular at x. We denote by $\mathcal{D} = \mathcal{D}(X, V)$ the discriminant locus of V, i. e. the subset of $|V|$ parametrizing its singular elements. Define by

$$\mathcal{S} = \mathcal{S}(X, V) = \{(x, D) \in X \times |V| : D \in |V - 2x|\}$$

the global singular set and let p, q be the projections of $X \times |V|$ onto the factors X and $|V|$ respectively. Then $\mathcal{D} = q(\mathcal{S})$. Throughout the paper we use the expression "classical case" to mean that L is a very ample line bundle and $V = H^0(L)$. Note that in this case $\mathcal{D}$ is the dual variety of X embedded in a projective space by the complete linear system $|L|$.

(0.3) *Jumping sets.* For L and V as above with $\dim V = N + 1$, we denote by $\phi_V : X \to \mathbb{P}^N$ the finite morphism defined by $|V|$. We write ϕ_L for $\phi_{H^0(L)}$. According to [LPS2, (1.1)] the i-th jumping set of V ($i = 1, \dots, n$) is defined as

$$\mathcal{J}_i = \mathcal{J}_i(V) = \{x \in X : \mathrm{rk}(d\phi_V)(x) \le n - i\}.$$

Recall that [LPS2, Proposition 1.3]

$$\mathcal{J}_i = \{x \in X : \mathrm{codim}_{|V-x|}|V - 2x| \le n - i\}.$$

We use X_i to denote $\mathcal{J}_i - \mathcal{J}_{i+1}$, where we make the conventions that $\mathcal{J}_0 = X$ and $\mathcal{J}_{n+1} = \emptyset$. Note that X_0 is the open set where $d\phi_V$ has maximal rank. Recall that $\dim \mathcal{J}_i \leq n - i$ [LPS2, Theorem 1.2], hence

$$\dim X_i \leq n - i. \tag{0.3.1}$$

Note that if $\dim X_i = n - i$ then $\dim \mathcal{S}_i = N - 1$, where $\mathcal{S}_i = \overline{p_{|\mathcal{S}}^{-1}(X_i)}$ and $\mathcal{S}$ is the global singular set appearing in (0.2). Actually if $x \in X_i$ then $|V - 2x|$ has codimension $n - i + 1$ in $|V|$. So we get

$$\dim \mathcal{S}_i = \dim X_i + \dim |V| - (n - i + 1) = N - 1.$$

Let $\mathcal{D}_i$ denote the closure in $|V|$ of the set of $H \in |V|$ such that H is singular on X at a point of X_i. In other words, $\mathcal{D}_i = q(\mathcal{S}_i)$. Note that $\mathcal{S}_0$ is the closure of $\mathbb{P}(K)$, where K is the dual of the kernel of the evaluation morphism $X_0 \times V \to (J_1 L)_{X_0}$ defined by sending (x, s) to the germ of $j_1(s)$ at x. Since K is a vector bundle of rank $N - n$ on X_0, we see that $\mathcal{S}_0$ is irreducible of dimension $N - 1$; hence $\mathcal{D}_0 = q(\mathcal{S}_0)$ is an irreducible subvariety of dimension $\leq N - 1$. We call $\mathcal{D}_0 = \mathcal{D}_0(X, V)$ the main component of the discriminant locus. Note that $\mathcal{D}_0$ is denoted by $\mathcal{D}'$ in [LPS2] and $\mathcal{D} = \cup_{i=0}^{n} \mathcal{D}_i$.

(0.4) *Defects.* Let X, L and V be as above. According to [LPS2, Section 2] we define the defect of (X, V) to be the integer

$$\operatorname{def}(X, V) = \operatorname{codim}_{|V|} \mathcal{D}(X, V) - 1.$$

Recall that this definition is independent of the subspace V spanning L and so we shall also write $\operatorname{def}(X, L) = \operatorname{def}(X, V)$. Recalling the properties of the main component $\mathcal{D}_0(X, V)$ we can also define the main defect as

$$\operatorname{def}_0(X, V) = \operatorname{codim}_{|V|} \mathcal{D}_0(X, V) - 1.$$

We write def_0 (def) instead of $\operatorname{def}_0(X, V)$ ($\operatorname{def}(X, V)$) when there is no danger of confusion. Note that since $\mathcal{D}_0 \subseteq \mathcal{D}$, we have $\operatorname{def}_0 \geq \operatorname{def}$.

We insert two general results which we will need in Section 2.

(0.5) Lemma. *Let Λ be a general linear space in $|V|$ and let A_a, A_b be the divisors corresponding to two points a, b belonging to two distinct connected components of $\mathcal{D} \cap \Lambda$. Then*

$$\operatorname{Sing}(A_a) \cap \operatorname{Sing}(A_b) = \emptyset.$$

Proof. Let s_a, s_b be sections in V corresponding to a and b. Assume, by contradiction, that there exists a point $x \in \operatorname{Sing}(A_a) \cap \operatorname{Sing}(A_b)$. Then x is a singular point for all elements in the pencil $\lambda s_a + \mu s_b$. Hence the line $< a, b >$ is contained in $\mathcal{D} \cap \Lambda$. But then a and b would lie in the same connected component of $\mathcal{D} \cap \Lambda$, a contradiction. $\square$

(0.6) Lemma. *Let $[s_0]$ be a general point on a codimension $k+1$ irreducible component $\mathcal{C}$ of $\mathcal{D}$ and let A be the corresponding divisor. Then* $\mathrm{Sing}(A)$ *is Cohen-Macaulay of pure dimension k.*

Proof. Let $s_1, \ldots, s_{k+1}$ be general sections in V and let $W = < s_0, \ldots, s_{k+1} >$. Let G_i, $i = 0, \ldots, s$ be the connected components of $\mathbb{P}(W) \cap \mathcal{D}$ and let $[s_0] \in G_0$. Note that some components G_i may have positive dimension; however

(0.6.1) G_0 consists of the single point $[s_0]$.

Otherwise, $[s_0]$ would lie on the intersection of $\mathcal{C}$ with another component of $\mathcal{D}$ of higher dimension; but this cannot happen, $[s_0]$ being general. Note that $\mathrm{Sing}(A)$ is contained in $j_1(W)^{-1}(0)$, the set of the singular loci of the singular elements in $\mathbb{P}(W)$. Then by (0.6.1) and Lemma (0.5) we see that all the connected components of $\mathrm{Sing}(A)$ are connected components of $j_1(W)^{-1}(0)$. On the other hand $j_1(W)^{-1}(0)$ is nonempty and then it is Cohen-Macaulay of pure dimension k by [LPS2, Theorem 2.4]. Therefore so are its connected components, which proves the assertion. □

The proof of Lemma (0.6) shows that for a general $\mathbb{P}(W)$, every 0-dimensional connected component of $\mathcal{D} \cap \mathbb{P}(W)$ gives rise to some connected component of $j_1(W)^{-1}(0)$. So for every irreducible codimension $k+1$ component of $\mathcal{D}$ there is at least one connected component of $\mathcal{D} \cap \mathbb{P}(W)$ consisting of a single point like $[s_0]$ in the proof above. So we have the following

(0.7) *Remark.* Let L be an ample line bundle on a projective manifold X, spanned by a subspace $V \subseteq H^0(L)$. The number of the codimension $k+1$ connected components of $\mathcal{D}$ is bounded by the number of the connected components of the general representative of $c_k(J_1 L)$ (which in turn is bounded by $c_k(J_1 L) \cdot L^{n-k}$).

1. Nondegenerate quadratic singularities

Let L be a line bundle on a smooth complex projective n-fold, $n \geq 2$ and assume that $|L|$ contains a reduced divisor A. Let s be a section of L defining A. We say that a point $x \in A$ is an *isolated nondegenerate quadratic singularity* if, with local coordinates $x_1, \ldots, x_n$ around x, s can be written in the form

$$s = \sum_{i,j} a_{ij} x_i x_j + \text{h.o.t.} \qquad (a_{ij} = a_{ji}),$$

with

$$\det(a_{ij}) \neq 0,$$

where h.o.t. means higher order terms. Now let P be a smooth subvariety of A of dimension $k > 0$. We say that P is a *locus of nondegenerate quadratic singularities* of A if for every $x \in P$ there exist k smooth hypersurfaces $H_1, \ldots, H_k$ meeting transversally along a submanifold Γ such that $x = P \cap \Gamma$ is an isolated nondegenerate quadratic singularity of $A \cap \Gamma$.

Isolated nondegenerate quadratic singularities can be characterized by means of the jet formalism in the following way.

(1.1) Proposition. *Let L be a line bundle on X with $h^0(L) \geq 2$, let $A \in |L|$ be a reduced divisor having a singularity at x. Assume that there exists a section $t \in \Gamma(L)$ not vanishing at x. Then x is an isolated nondegenerate quadratic singularity of A if and only if x is a smooth point of $(j_1(t) \wedge j_1(s))^{-1}(0)$, where s is a section in $\Gamma(L)$ defining A.*

Proof. Choose local coordinates $x_1, \ldots, x_n$ on X around x. We can assume that any section t as above is 1 around x; on the other hand, since A is singular at x, we have

$$s = \sum_{i,j} a_{ij} x_i x_j + \text{h.o.t} \qquad (a_{ij} = a_{ji}),$$

where h.o.t. means higher order terms. Then $j_1(t) = (1, 0, \ldots, 0)$ around x, while

$$j_1(s) = (s, \sum_{j=1}^{n} a_{1,j} x_j + \text{h.o.t.}, \ldots, \sum_{j=1}^{n} a_{n,j} x_j + \text{h.o.t.}).$$

Thus

$$j_1(t) \wedge j_1(s) = (\sum_{j=1}^{n} a_{1,j} x_j + \text{h.o.t.}, \ldots, \sum_{j=1}^{n} a_{n,j} x_j + \text{h.o.t.}).$$

For shortness set $\psi = j_1(t) \wedge j_1(s)$. Now it is clear that x is a smooth point of $\psi^{-1}(0)$ if and only if $d\psi$ has rank n at x. But this is equivalent to x being an isolated nondegenerate quadratic singularity of A since $d\psi(x) = (a_{ij})$. $\square$

The property of the normal bundle of the general contact locus, discovered by Ein in the classical case, has an analogue for the normal bundle of a smooth locus of nondegenerate quadratic singularities.

(1.2) Theorem (Generalized Ein Lemma [E, Theorem 2.2])**.** *Let L be a line bundle on X having a section defining an irreducible hypersurface A. Assume that A has a smooth locus P of nondegenerate quadratic singularities with* $\dim P > 0$. *Then we have a symmetric isomorphism*

$$N_{P/X} \cong N^*_{P/X} \otimes L.$$

Proof. The proof runs as in [BS, Thm. 14.4.1, p. 346]. Let $s \in \Gamma(L)$ be the section defining A; then $j_1(s)$ is zero on P. Consider the exact sequence

$$0 \to T_X^{*(2)} \otimes L \to J_2 L \xrightarrow{\alpha} J_1 L \to 0,$$

where $J_m L$ is the m-th jet bundle of L. Around every point $x \in P$ choose local coordinates $z_1, \ldots, z_n$ on X such that P is defined by $z_{k+1} = \cdots = z_n = 0$. Now look at the second jet $j_2(s)$. Since $\alpha(j_2(s)) = j_1(s)$ is trivial on P we have

$(j_2(s))_P \in H^0((T_X^{*(2)} \otimes L)_P)$. Since $s(P) = 0$ all partial derivatives in the z_i directions ($i = 1, \ldots, k$) are zero. This shows that in fact

$$(j_2(s))_P \in H^0(N_{P/X}^{*(2)} \otimes L_P) \subseteq H^0((T_X^{*(2)} \otimes L)_P).$$

Recalling that

$$N_{P/X}^{*(2)} \otimes L_P \subseteq N_{P/X}^* \otimes N_{P/X}^* \otimes L_P \cong \mathcal{H}om(N_{P/X}, N_{P/X}^* \otimes L_P)$$

we thus se that $j_2(s)$ defines a homomorphism $h(s) : N_{P/X} \to N_{P/X}^* \otimes L_P$. Note that $h(s)$ is represented at every point $x \in P$ by the Hessian matrix of s with respect to the coordinates $z_{k+1}, \ldots, z_n$. But this matrix has maximal rank since x is a nondegenerate quadratic singularity. Moreover it is symmetric, hence $h(s)$ is the required symmetric isomorphism. □

2. The main theorem

Let us start by relating the title of the paper to the set-up we consider in this Section.

Let $X^\# \subset \mathbb{P}^N$ be an irreducible variety, let $\nu : X \to X^\#$ be its normalization and assume that X is smooth. Set $L := \nu^*(\mathcal{O}_{X^\#}(1))$ and call V the subspace of $H^0(L)$ obtained by pulling back via ν the image of the restriction homomorphism

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(X^\#, \mathcal{O}_{X^\#}(1)).$$

Then ϕ_V, the morphism associated with V, coincides with ν. Since ν is a finite birational morphism we conclude that L is ample, V spans L and ϕ_V is generically one-to-one.

Conversely, consider a triple (X, L, V) consisting of a smooth projective n-fold, an ample line bundle L on X and a vector subspace $V \subseteq H^0(L)$ spanning L and giving a generically one-to-one morphism ϕ_V. Set $X^\# = \phi_V(X)$; then $X^\#$ has a smooth normalization isomorphic to X. To see this, let $\nu : \tilde{X} \to X^\#$ be the normalization morphism. Then $\phi_V : X \to X^\#$ factors through ν and a morphism $f : X \to \tilde{X}$, which is birational so being both ϕ_V and ν. Then all fibers of f are connected by the Zariski main theorem. But all fibers of f must have dimension 0. Otherwise X would contain a curve Γ with $\Gamma L = 0$, contradicting the ampleness of L. Therefore f is an isomorphism, hence $\tilde{X} \cong X$ is smooth.

In view of this equivalence, we consider the following set-up.

(2.0.1) L is an ample line bundle on a smooth complex projective n-fold X, $n \geq 2$, which is spanned by some subspace $V \subseteq H^0(L)$ and the associated morphism $\phi_V : X \to \mathbb{P}^N$ is generically one-to-one.

We set also $\phi = \phi_V$ where there is no danger of confusion. Our first goal is to analyze $\mathcal{D}_0$, the main component of the discriminant locus $\mathcal{D}(X, V)$ defined in (0.3). Recall that on X_0 we have the exact sequence

(2.0.2) $$0 \to K^* \to X_0 \times V \to J_1(X_0, L) \to 0.$$

and that $\mathcal{D}_0$ is the image of the closure of the projective bundle $\mathbb{P}(K)$ over X_0 via the second projection $q : X \times |V| \to |V|$, i. e.

$$\mathcal{D}_0 = q(\overline{\mathbb{P}(K)}). \tag{2.0.3}$$

Let $U \subseteq X_0$ be the open subset where ϕ is one-to-one. Since $\phi_{|U}$ is an embedding, we have

(2.1) *Remark.* Let $x \in U$; an element $D \in |V|$ is singular at x if and only if $D = \phi^* h$, h being a tangent hyperplane to $\phi(X)$ at $\phi(x)$.

(2.2) Lemma. *Assume that $\mathcal{D}_0$ has codimension one in $|V|$ and let $A \in \mathcal{D}_0$ be a general element. Then*

$$\mathrm{Sing}(A) \cap (X \setminus U) = \emptyset.$$

Proof. By contradiction, assume that

(2.2.1) the general $A \in \mathcal{D}_0$ has some singular points in the closed subset $X \setminus U$.

Recalling (2.0.3), assumption (2.1.1) means that the map q from $\overline{\mathbb{P}(K)}_{X \setminus U}$ to $\mathcal{D}_0$ is dominant, hence there exists an irreducible component Z of $X \setminus U$ such that $q_{|Z} : \overline{\mathbb{P}(K)}_Z \to |V|$ maps onto a dense subset of $\mathcal{D}_0$ and so

$$\dim \overline{\mathbb{P}(K)}_Z \geq \dim \mathcal{D}_0. \tag{2.2.2}$$

But this is impossible. Actually, by assumption

$$\dim \mathcal{D}_0 = \dim |V| - 1 = \dim V - 2, \tag{2.2.3}$$

On the other hand, since Z is a proper analytic subset of X, we conclude that $\overline{\mathbb{P}(K)}_Z$ is a proper analytic subset of $\overline{\mathbb{P}(K)}$. Thus, recalling (2.0.2), we get

$$\begin{aligned} \dim \overline{\mathbb{P}(K)}_Z &\leq \dim \overline{\mathbb{P}(K)} - 1 \\ &= \dim X_0 + \dim V - \mathrm{rank} J_1(X_0, L) - 1 - 1 \\ &= n + \dim V - n - 1 - 1 - 1 \\ &= \dim V - 3. \end{aligned}$$

So by (2.2.3)

$$\dim \overline{\mathbb{P}(K)}_Z < \dim \mathcal{D}_0,$$

which contradicts (2.2.2). $\square$

When L is a very ample line bundle it is a classical fact that for the general $A \in \mathcal{D}(X, L)$ the singular locus of A is a linear $\mathbb{P}^k$, where $k = \mathrm{def}(X, L)$. On the other hand, when $k = 0$ it is also well known that $\mathrm{Sing}(A)$ is an isolated nondegenerate quadratic singularity. Here we restore these facts in our set-up and at the same time we prove that if $k \geq 1$ then $\mathrm{Sing}(A)$, for a general $A \in \mathcal{D}_0$, is a locus of nondegenerate quadratic singularities.

(2.3) Theorem. *Let X, L and V be as in* (2.0.1)*. Assume that $\mathcal{D}_0$ has codimension $k+1$ in $|V|$ and let $A \in \mathcal{D}_0$ be a general element. Then* $(\mathrm{Sing}(A), L_{\mathrm{Sing}(A)}) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$*. Moreover* $\mathrm{Sing}(A)$ *is a locus of nondegenerate quadratic singularities.*

Proof. The proof consists of two parts dealing respectively with cases $k = 0$ and $k > 0$.

Part I. Assume $k = 0$. We have to prove that the general element $H \in \mathcal{D}_0$ has only one singularity, which is an isolated nondegenerate quadratic singularity.

In view of Lemma (2.2) all the singularities of A are contained in the open subset U defined above. Then, since $\phi_{|U}$ is one-to-one, A corresponds to a tangent hyperplane h to $\phi(X)$, by Remark (2.1). Note also that by Remark (2.1), combined with our assumption, the dual variety $(\phi(X))^*$ of $\phi(X)$ has dimension

$$\dim(\phi(X))^* = \dim \mathcal{D}_0 = N - 1. \tag{2.3.1}$$

In $\phi(X) \times \mathbb{P}^{N\,*}$ consider the tangency correspondence

$$\mathcal{P} := \overline{\{(y,h) \in (\phi(X))_{\mathrm{sm}} \times \mathbb{P}^{N\,*} \ : \ h \text{ is tangent to } \phi(X) \text{ at } y\}}$$

(cf. [Z, p. 20]) and let $\gamma : \mathcal{P} \to (\phi(X))^*$ be the Gauss map induced by the projection to the second factor. Note that $\dim \mathcal{P} = N - 1 = \dim(\phi(X))^*$ by (2.3.1), hence by [Z, Theorem 2.3, c), p. 21] γ is birational. This means that h, the general tangent hyperplane, is tangent at a single point $y \in \phi(X)$. It thus follows that A is singular only at $\phi^{-1}(y)$, $\phi_{|U}$ being one-to-one.

To show that $\phi^{-1}(y)$ is a nondegenerate quadratic singularity of A we proceed as follows. Let $\pi : \mathbb{P}^N \to \mathbb{P}^{n+1}$ be a projection from a linear subspace not meeting $\phi(X)$. Then the composed map $\psi := \pi \circ \phi : X \to \mathbb{P}^{n+1}$ is generically one-to-one onto its image. By Remark (2.1) we know that $A = \phi^* h$, where h is a hyperplane of $\mathbb{P}^N$ tangent to $\phi(X)$ at y. Assume that $h \cap \phi(X)$ has a worse singularity than a nondegenerate quadratic one at y. Then since it is an isolated singularity, the same should happen for the corresponding hyperplane of $\mathbb{P}^{n+1}$ tangent to $\psi(X)$ at $\pi(y)$. So it is enough to prove that the section with a hyperplane tangent to $\psi(X)$ at a general point u has a nondegenerate quadratic singularity at u. Let $g : (\psi(X))_{\mathrm{sm}} \to (\psi(X))^*$ be the map sending the smooth point u to the hyperplane of $\mathbb{P}^{n+1}$ tangent to $\psi(X)$ at u, which is uniquely defined since $\psi(X)$ is a hypersurface.

Now consider the tangency correspondence in $\psi(X) \times \mathbb{P}^{n+1\,*}$. Since there is no danger of confusion let us use the same letter $\mathcal{P}$ as before to denote it and let $p : \mathcal{P} \to \psi(X)$ be the restriction of the first projection. Note that p is birational $\psi(X)$ being a hypersurface and that p^{-1} is defined on $(\psi(X))_{\mathrm{sm}}$. Then

$$g = \gamma \circ p^{-1}, \tag{2.3.2}$$

where $\gamma : \mathcal{P} \to (\psi(X))^*$ is the Gauss map. Note that since $\psi(X)$ is a projection of $\phi(X)$, by duality, the dual variety $(\psi(X))^* \subset \mathbb{P}^{n+1\,*}$ is a linear slice of $(\phi(X))^* \subset \mathbb{P}^{N\,*}$. By (2.3.1) we thus get

$$\dim(\psi(X))^* = \dim(\phi(X))^* - (N - (n+1)) = n.$$

But in the hypersurface case we also have $\dim \mathcal{P} = n$. So by [Z, Theorem 2.3, c), p. 21] again we conclude that γ is birational; as a consequence by (2.3.2) g is birational, hence at a general point $u \in (\psi(X))_{\mathrm{sm}}$ the differential dg has maximal rank.

To make this condition explicit we use a local argument, which is inspired by [K, pp. 221–225]. Let $\psi(X)$ be defined by

$$F(Z_0, \dots, Z_{n+1}) = 0,$$

where F is a homogeneous polynomial. Then for any $v \in (\psi(X))_{\mathrm{sm}}$

$$g(v) = (\frac{\partial F}{\partial Z_0} : \frac{\partial F}{\partial Z_1} : \cdots : \frac{\partial F}{\partial Z_{n+1}})(v).$$

Let $u = (1 : 0 : \cdots : 0) \in (\psi(X))_{\mathrm{sm}}$. Then using affine coordinates $z_i = Z_i/Z_0$, $i = 1, \dots, n$, the corresponding affine part of $\psi(X)$ is defined by

$$f(z_1, \dots, z_{n+1}) = F(1, z_1, \dots, z_{n+1}) = 0.$$

Up to a linear change of coordinates we can assume that the tangent hyperplane at u has equation $z_{n+1} = 0$. Thus by the implicit function theorem there exists a holomorphic function $h = h(z_1, \dots, z_n)$ in a neighborhood of $0 \in \mathbb{C}^n$, vanishing to the second order at 0, such that in a suitable neighborhood of u in $\mathbb{C}^{n+1}$, f can be expressed in the form

$$z_{n+1} - h(z_1, \dots, z_n). \tag{2.3.3}$$

Now, by the Euler theorem

$$\sum_{i=0}^{n+1} \frac{\partial F}{\partial Z_i} Z_i = dF,$$

where d is the degree of F. So at points of $\psi(X)$ in our affine chart we have

$$\frac{\partial F}{\partial Z_0} = -\sum_{i=1}^{n+1} \frac{\partial F}{\partial Z_i} z_i = -\sum_{i=1}^{n+1} \frac{\partial f}{\partial z_i} z_i.$$

Moreover, from (2.3.3) we get

$$\frac{\partial F}{\partial Z_i} = \frac{\partial f}{\partial z_i} = -\frac{\partial h}{\partial z_i}, \quad \text{for} \quad i = 1, \dots, n,$$

and

$$\frac{\partial F}{\partial Z_{n+1}} = \frac{\partial f}{\partial z_{n+1}} = 1.$$

We thus see that in a neighbourhood of u on $(\psi(X))_{\mathrm{sm}}$

$$g(v) = (-\sum_{i=1}^{n+1} \frac{\partial f}{\partial z_i} z_i : \frac{\partial f}{\partial z_1} : \cdots : \frac{\partial f}{\partial z_{n+1}})(v)$$
$$= (\sum_{i=1}^{n} \frac{\partial h}{\partial z_i} z_i - h : -\frac{\partial h}{\partial z_1} : \cdots : -\frac{\partial h}{\partial z_n} : 1)(v).$$

Let $W_0 : \cdots : W_{n+1}$ be the homogeneous coordinates in $\mathbb{P}^{n+1\,*}$. Since the last coordinate of $g(v)$ is not zero, using affine coordinates $w_i = W_{i-1}/W_{n+1}$, $i = 1, \ldots, n+1$, we get the following local expression for g:

$$g(v) = (\sum_{i=1}^{n} \frac{\partial h}{\partial z_i} z_i - h, -\frac{\partial h}{\partial z_1}, \ldots, -\frac{\partial h}{\partial z_n})(v).$$

By evaluating dg at u we thus see that the condition that $\mathrm{rank}(dg(u)) = n$ is equivalent to the fact that the Hessian matrix $[\frac{\partial^2 h}{\partial z_i \partial z_j}](u)$ is nondegenerate. But this exactly means that u is a nondegenerate quadratic singularity for the section of $\psi(X)$ with the hyperplane defined by $z_{n+1} = 0$. $\square$

Part II. Now assume that $k > 0$. Let $x \in \mathrm{Sing}(A)$; choose k general elements $H_1, \ldots, H_k \in |V|$ and set $\Gamma = H_1 \cap \cdots \cap H_k$. Note that L_Γ is ample and spanned by V_Γ. Moreover, due to the genericity of the H_i's, it is clear that the morphism ϕ_{V_Γ} is generically one-to-one. Since $\phi_{V_\Gamma}(\Gamma)$ is a general linear slice of $\phi_V(X)$ by k hyperplanes, the dual variety $(\phi_{V_\Gamma}(\Gamma))^*$ is a general projection of $(\phi_V(X))^* \subset \mathbb{P}^{N*}$ into a $\mathbb{P}^{N-k}$ inside $\mathbb{P}^{N*}$. So the main component $\mathcal{D}_0(\Gamma, V_\Gamma)$ has codimension 1, namely $\mathrm{def}_0(\Gamma, V_\Gamma) = 0$. Then Part I applies and we have that $\mathrm{Sing}(A_\Gamma)$ is a single point representing an isolated nondegenerate quadratic singularity for A_Γ. Now, due to Lemma (0.6) we know that $\dim \mathrm{Sing}(A) = k$ and then by the ampleness of L we have that

$$\mathrm{Sing}(A) \cap \Gamma \neq \emptyset.$$

On the other hand

$$\mathrm{Sing}(A) \cap \Gamma \subseteq \mathrm{Sing}(A_\Gamma),$$

the inclusion being scheme-theoretic, due to the generality of Γ. As we said, the right hand term is a point. Thus $\mathrm{Sing}(A) \cap \Gamma$ is a single point. Moreover $\mathrm{Sing}(A)$ is Cohen-Macaulay of pure codimension k by Lemma (0.6) again. Therefore $\mathrm{Sing}(A)$ is irreducible and generically reduced, hence reduced. Now look at the polarized variety $(\mathrm{Sing}(A), L_{\mathrm{Sing}(A)})$. We know that $\mathrm{Sing}(A)$ is Cohen-Macaulay,

$$L^k_{\mathrm{Sing}(A)} = 1,$$

and $L_{\mathrm{Sing}(A)}$ is ample and satisfies the condition $h^0(L_{\mathrm{Sing}(A)}) \geq k+1$, due to the spannedness of L. Hence [F, Theorem 1.1, p. 22] applies and we get

$$(\mathrm{Sing}(A), L_{\mathrm{Sing}(A)}) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1)).$$

Now fix any $x \in \mathrm{Sing}(A)$ and recall that $x \in U$, by Lemma (2.2). Choose $H_1, \dots, H_k$ as above but in $|V - x|$. Then the same argument as above shows that $x = \mathrm{Sing}(A) \cap \Gamma = \mathrm{Sing}(A_\Gamma)$ is an isolated nondegenerate quadratic singularity. This concludes the proof. □

As a consequence of Theorem (2.3) we obtain that the conjecture stated in [LPS2, Conjecture 2.11] is true in the case when ϕ is generically one-to-one. Recall that if (X, L) is a polarized variety, a *linear* $\mathbb{P}^k$ of (X, L) (a *line* if $k = 1$) is a subvariety $P \subset X$ such that $(P, L_P) \cong (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$. We need the following

(2.4) Lemma. *Let L be an ample and spanned line bundle on an irreducible reduced projective variety X. If there is a linear $\mathbb{P}^k$ through the general point of X, then there is a linear $\mathbb{P}^k$ through every point.*

Proof. Choose a component Z of the Hilbert scheme of the linear $\mathbb{P}^k$'s of (X, L) containing a point corresponding to the linear $\mathbb{P}^k$ through a general point of X. Let $p : \mathcal{H}_Z \to Z$ be the universal projection, where $\mathcal{H}_Z$ is the total space of the Hilbert scheme over Z. Let $q : \mathcal{H}_Z \to X$ be the natural map. Let $x \in X$ be an arbitrary point and let $z \in Z$ be such that $x \in q(p^{-1}(z))$. Call G the object $q(p^{-1}(z))$ with all its possible embedded components removed. If we show that G is a linear $\mathbb{P}^k$ we are done. Since p is flat, its fibers F have constant dimension $= k$, and the intersection index $(q^*L)^k F$ is independent of the fiber F and so equal to 1. Note that $q_{|F}$ is an isomorphism, since $\mathcal{H}_Z$ is a family of algebraic subsets of X. Then $L^k q(p^{-1}(z)) = 1$. Since L is ample we thus get $L^k G = 1$. This shows that G is irreducible and reduced. Let $\nu : \widetilde{G} \to G$ be the normalization. Then $\nu^* L$ is ample and spanned and $\nu^* H^0(L)$ gives a morphism $\widetilde{G} \to \mathbb{P}^k$ which is finite to one and generically one-to-one. Thus by the Zariski main theorem this map is an isomorphism, say f. On the other hand f factors through $\widetilde{G} \xrightarrow{\nu} G \to \mathbb{P}^k$. This shows that $G \cong \mathbb{P}^k$. □

(2.5) Corollary. *Let X, L, V be as in* (2.0.1) *and assume that* $k = \mathrm{def}_0(X, V) > 0$. *Then the $\mathbb{P}^k$'s arising as singular loci of the general elements of $\mathcal{D}_0(X, V)$ cover X. In particular X is covered by lines.*

Proof. By Theorem (2.3) there is an open dense subset of X covered by linear $\mathbb{P}^k$'s. Then the assertion follows from Lemma (2.4). □

3. Applications

In this section we discuss some applications of the main theorem recovering several known results on contact loci in our more general setting. Throughout all this Section we assume that X, L and V are as (2.0.1). Since in general ϕ_V may not be an embedding of X in $\mathbb{P}^N$ we cannot follow Ein's approach [E, Theorem 2.3]. To overcome this point we need the following technical

(3.1) Lemma. *Let X be a complex projective manifold and let $\mathcal{X} = \{P_t\} \xrightarrow{p} \mathcal{T}$ be a deformation family of a submanifold $P = P_0$ of X covering a dense open subset of X. Then for a general $\tau \in \mathcal{T}$ the normal bundle $N_{P_\tau/X}$ is spanned by global sections at a general point.*

Proof. Let $\varphi : \mathcal{X} \to X$ be the obvious morphism sending isomorphically every P_t onto its image in X. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & T_{P_t} & \to & (T_{\mathcal{X}})_{P_t} & \to & N_{P_t/\mathcal{X}} = p^*T_{\mathcal{T},t} & \to & 0 \\ & & \| & & \downarrow{\scriptstyle d\varphi} & & \downarrow{\scriptstyle f} & & \\ 0 & \to & T_{P_t} & \to & (T_X)_{P_t} & \xrightarrow{g} & N_{P_t/X} & \to & 0 \end{array},$$

where f is induced by $d\varphi$. Due to the assumption $d\varphi$ is a surjection at a general point $x \in P_t$. Since g is surjective we conclude that f is a surjection. On the other hand at a smooth point $t = \tau \in \mathcal{T}$ we have $p^*T_{\mathcal{T},\tau} = \mathbb{C}^{\dim \mathcal{T}}$, hence the trivial bundle surjects onto $N_{P_\tau/X}$ at $x \in P_\tau$, which proves the spannedness at x. $\square$

Now assume that $k = \text{def}_0(X, V) > 0$. Then, by Corollary (2.5), Lemma (3.1) applies to the $\mathbb{P}^k$'s arising as the singular loci of the general elements of $\mathcal{D}_0(X, V)$. Therefore there exists such a $\mathbb{P}^k$ such that $N_{\mathbb{P}^k/X}$ is spanned at the general point. For a line $l \subset \mathbb{P}^k$ passing through that point, hence for a general line, we have

$$(N_{\mathbb{P}^k/X})_l \cong \oplus_{i=1}^{n-k} \mathcal{O}_{\mathbb{P}^1}(a_i) \qquad \text{with} \qquad a_i \geq 0,$$

due to the spannedness. Now, by Theorem (1.2) we get

$$\oplus_{i=1}^{n-k} \mathcal{O}_{\mathbb{P}^1}(a_i) = \oplus_{j=1}^{n-k} \mathcal{O}_{\mathbb{P}^1}(1 - a_j).$$

Hence the set of the a_i's coincides with the set of the integers $1 - a_j$. This implies that

$$(3.2.0) \qquad (N_{\mathbb{P}^k/X})_l \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus(\frac{n-k}{2})} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(\frac{n-k}{2})}.$$

In particular we get the following extension of Landman's parity result.

(3.2) Proposition. *Let X, L and V be as in* (2.0.1). *If* $\text{def}_0 > 0$ *then*

$$n \equiv \text{def}_0 \mod 2.$$

Moreover since $N_{l/\mathbb{P}^k} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(k-1)}$, in view of (3.2.0) the exact sequence

$$0 \to N_{l/\mathbb{P}^k} \to N_{l/X} \to (N_{\mathbb{P}^k/X})_l \to 0$$

splits and gives

$$(3.2.1) \qquad N_{l/X} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus(\frac{n-k}{2})} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(\frac{n+k}{2}-1)}.$$

Note that by adjunction we thus get

$$-2 = \deg K_l = K_X l + \deg N_{l/X} = K_X l + \frac{n+k}{2} - 1. \tag{3.2.2}$$

This gives

$$(-K_X)_{\mathbb{P}^k} \cong \mathcal{O}_{\mathbb{P}^k}(\frac{n+k+2}{2}). \tag{3.2.3}$$

In particular this shows that K_X is not nef. We recall that the *nef value* of a polarized manifold (X, L) with K_X not nef is the lowest positive real number $\tau = \tau(X, L)$ such that $K_X + \tau L$ is nef but not ample [BS, Lemma 1.5.5, p. 26]. By a well known result of Kawamata τ is a rational number and there exists a morphism $\Phi : X \to Y$ with connected fibers onto a normal projective variety Y such that $N(K_X + \tau L) = \Phi^* h$, where h is an ample line bundle on Y and N is a positive number such that $N\tau$ is an integer. We refer to Φ as to the *nef value morphism.*

As a consequence of the above discussion we get the following

(3.3) Theorem. *Let X, L, V be as in* (2.0.1)*. If $k = \mathrm{def}_0(X, L) > 0$ then the nef value of (X, L) is $\tau = \frac{n+k}{2} + 1$ and the nef value morphism $\Phi : X \to Y$ is a Mori contraction. Moreover* $\dim Y \leq \frac{n-k}{2}$.

Proof. By Corollary (2.5) X is covered by lines. Let $l \subset X$ be such a line. By (3.2.2) we get

$$\nu := -2 - K_X l = \frac{n+k}{2} - 1 \geq \frac{n-2}{2}.$$

Then [BSW, Theorem 2.3] applies. Noting that $(X, L) = (\mathbb{P}^{n/2} \times \mathbb{P}^{n/2}, \mathcal{O}(1, 1))$ has defect zero, this gives the assertion on Φ. The last assertion follows from [BSW, Lemma 1.4.4]. □

In [LPS2, Theorem 2.8] we proved that if $V \subseteq H^0(L)$ spans an ample line bundle L on a projective manifold X of dimension n, then def $\leq n$ with equality if and only if $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. If ϕ_V is not generically one-to-one this is no longer true with def_0 instead of def. For instance, as Example (4.1) will show, it can happen that $\mathrm{def}_0 = n >$ def. However, in the setting (2.0.1) we can restore the above result with def_0 instead of def. On the other hand, Theorem (3.3) allows us to generalize a circle of ideas developed in the classical case in [BFS] to our more general set-up. Actually, the positive defect classification in [BFS] carries over in our case. To prove our result we need the following

(3.4) Lemma. *Let X, L and V be as in* (2.0.1) *with* $\dim X = n$*. If* $k := \mathrm{def}_0 > 0$ *then the nef value morphism Φ cannot be a quadric fibration.*

Proof. By contradiction, let $\Phi : X \to Y$ be a quadric fibration over a normal projective variety Y. Then $\tau = n - y$, where $y := \dim Y$. Since $k > 0$, by Theorem (3.3) we get

$$k = n - 2y - 2. \tag{3.4.1}$$

By Theorem (2.3) a general element $H \in |L-2x|$ with x general, is singular along a positive dimensional linear space $P = \mathbb{P}^k$. Since P cannot fibre over Y we have that $P \subset F$, where where $F = \Phi^{-1}(\Phi(x))$. Moreover $F \cong \mathbb{Q}^{n-y}$ is a smooth quadric, since x is a general point of X. Now either $H \supset F$, or $H \cap F$ is an element of $|L_F|$ with a singular set containing P, hence at least one-dimensional. But, since F is a smooth quadric, it is self dual under the map sending a point to the unique tangent hyperplane at that point. Hence the elements of $|L_F|$ that are singular have only a single isolated singularity. This shows that $F \subset H$. Now set $F_t = \Phi^{-1}(t)$ where $t \in Y$ and consider

$$\mathcal{G} := \overline{\{H \in |L| \ : \ H \supset F_t \text{ for a general } t \in Y\}}.$$

We have shown that

$$\mathcal{D}_0 \subseteq \mathcal{G}. \tag{3.4.2}$$

We have

$$\dim \mathcal{G} \leq \dim Y + \dim |L - F_t|, \tag{3.4.3}$$

for $F_t \cong \mathbb{Q}^{n-y}$ a general fiber of Φ. Now note that for the general F_t the restriction homomorphism $H^0(L) \to H^0(L_{F_t})$ is surjective since the morphism ϕ_L (and hence its restriction to a general fiber F_t) is finite and generically one-to-one. Therefore $h^0(L-F_t) = h^0(L) - h^0(L_{F_t})$. Since $L_{F_t} = \mathcal{O}_{\mathbb{Q}^{n-y}}(1)$, this gives $h^0(L_{F_t}) = n-y+2$. In view of (3.4.3) we thus get $\dim \mathcal{G} \leq \dim |L| - n + 2y - 2$. But then, recalling (3.4.1), the inclusion (3.4.2) gives

$$\dim |L| - n + 2y + 1 = \dim |L| - (k+1) = \dim \mathcal{D}_0 \leq \dim \mathcal{G} \leq \dim |L| - n + 2y - 2,$$

a contradiction. □

(3.5) Theorem. *Let X, L and V be as in* (2.0.1) *with* $\dim X = n$ *and let* $k := \operatorname{def}_0(X, V) > 0$. *Then $k \leq n$ with equality if and only if $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Moreover, if $n - 8 < k < n$, then one of the following cases holds:*

a) $k = n - 2$ and (X, L) is a scroll over a smooth curve;

b) $k = n - 4$ and (X, L) is either

- b1) *the grassmannian $G(1, 4)$ of lines of $\mathbb{P}^4$, with L the line bundle giving the Plücker embedding and its section with a general hyperplane, or*
- b2) *a scroll over a smooth surface;*

c) $k = n - 6$ and (X, L) is either

- c1) *a Mukai manifold with* $\operatorname{Pic}(X) \cong \mathbb{Z}$ *and $-K_X = (n-2)L$,*
- c2) *a Del Pezzo fibration over a smooth curve, or*
- c3) *a scroll over a normal threefold.*

Proof. As already observed, K_X is not nef, hence the results on the adjunction theoretic structure of (X, L) apply (see [BS, Ch. 7] and alternately [LPS1, Theorem 1.2 and Theorem 2.3] for the first part of the proof). Assume that $k \geq n$; then $\tau \geq n+1$ by Theorem (3.3). Hence by [BS, Proposition 7.2.2, p. 160] we see that $\tau = n+1$, i. e. , $k = n$ and $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Now assume that $n - 8 < k < n$. Then by Proposition (3.2) it can only be $k = n-2$, $n-4$, $n-6$.

Let $k = n-2$. Then $\tau = n$ by Theorem (3.3). By [BS, Proposition 7.2.2, p. 160] (X, L) is either as in *a)*, or $K_X + nL$ is nef and big, in which case however we would get $\dim Y = n$, contradicting Theorem (3.3). Note that the pair $(X, L) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ cannot occur since in this case $\mathrm{def} = \mathrm{def}_0 = 0$.

Let $k = n-4$; then $\tau = n-1$ by Theorem (3.3). By [BS, Theorem 7.3.2, p. 169], taking into account Lemma (3.4), we conclude that (X, L) is either: i) a Del Pezzo manifold, ii) a scroll over a normal surface, or Φ is birational, which however contradicts Theorem (3.3). In case i), due to the assumption in (2.0.1), Φ_L is generically one-to-one, hence L is very ample, by the classification [F, Ch. I, §8]. So the only Del Pezzo manifolds occurring in this setting are those listed in b1). Case ii) leads to b2) since, X being smooth, the surface Y itself has to be smooth.

Note that when $k < n-4$ we have $\tau < n-1$ by Theorem (3.3). So by [BS, Theorem 7.3.2, p. 169] $K_X + (n-1)L$ is ample, which shows that (X, L) coincides with its first reduction (e. g. , see [BS, (7.3.3), p. 171]). Now let $k = n-6$; then $\tau = n-2$ and so by [BS, Theorem 7.5.3, p. 176], taking into account Lemma (3.4) again, we conclude that (X, L) is either: a Mukai manifold, a pair as in c2), c3), or $K_X + (n-2)L$ is nef and big, in which case however, $\dim Y = n$, contradicting Theorem (3.3). Note that in the first case since $n \geq 7$ we have $n - 2 = \mathrm{index}(X) \geq \frac{n}{2} + 1$, hence $\mathrm{Pic}(X) \cong \mathbb{Z}$, due to [W]. □

Note that Theorem (3.5) gives a complete classification of pairs as in (2.0.1) with $\mathrm{def}_0 > 0$ and $n \leq 7$. The result is formally the same as in the case when L is very ample [BFS, §2].

We conclude this Section with another application of Corollary (2.5).

(3.6) Proposition. *Let X, L and V be as in* (2.0.1). *If* $\mathrm{def}_0 > 0$, *then* $\mathcal{J}_m(V) = \emptyset$ *for* $2m > n - \mathrm{def}_0$.

Proof. Let $k := \mathrm{def}_0$. Assume that $x \in \mathcal{J}_m(V)$ and let $Z := \cap_{D \in |V-2x|} D$. Since $\mathrm{codim}_{|V-x|}|V - 2x| \leq n - m$, we can choose $n - m' \leq n - m$ general elements $s_1, \ldots, s_{n-m'} \in V - x$ such that their classes in $|V - x|$ together with $|V - 2x|$ generate the linear space $|V - x|$. Since

$$\bigcap_{D \in |V-x|} D = \Big(\bigcap_{D \in |V-2x|} D\Big) \cap \Big(\bigcap_{i=1}^{n-m'} s_i^{-1}(0)\Big)$$

is finite, we conclude that

$$\dim Z \le n - m'. \tag{3.6.1}$$

Note that every element $D \in |V - 2x|$ contains any line l of (X, L) passing through x, otherwise $1 = Dl \ge \mathrm{mult}_x(D) \ge 2$. Hence Z contains the locus of the points of X lying on the lines through x. In particular

$$Z \supseteq G, \tag{3.6.2}$$

where G is the locus of the points of X lying on the lines through x contained in the $\mathbb{P}^k$'s of singularities of the general elements of $\mathcal{D}_0$. Note that

$$\dim G \ge \frac{n+k}{2}. \tag{3.6.3}$$

Actually for a line l contained in the $\mathbb{P}^k$ of singularities of a general $D \in \mathcal{D}_0$ we have $-K_X l = \frac{n+k}{2} + 1$ by (3.2.2). On the other hand such lines cover X by Corollary (2.5), hence the dimension of the family of lines through x is $\ge -K_X l - 2$ by [BS, Remark 6.3.4, p. 129]. Thus, recalling (3.6.1), (3.6.2), (3.6.3), we have $n - m \ge n - m' \ge \dim Z \ge \dim G \ge \frac{n+k}{2}$, i.e., $n \ge 2m + k$. Thus if $2m > n - k$ we conclude that $\mathcal{J}_m(V)$ is empty. □

4. Examples

In this section we give some examples of the nonclassical behavior of the discriminant variety and the degeneracy loci $\mathcal{J}_i(V)$ in the set-up of (2.0.1) studied in this paper. In (4.1) we show how the conclusion of Theorem (2.3) can fail for an arbitrary ample and spanned line bundle L on X when the map associated to $|L|$ is not birational. In (4.2.2) we produce an example of a plane curve whose discriminant variety has two components. Example (4.2.4) shows that taking a product of the curve in (4.2.2) with $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ we get only one irreducible component. In this and the remaining examples we show some of the possible configurations of the $\mathcal{J}_i$'s.

(4.1) Let Y be a projective n-fold and let $\mathcal{L} \in \mathrm{Pic}(Y)$ be a very ample line bundle. Let $\pi : X \to Y$ be a finite morphism of degree $m \ge 2$ branched along an irreducible reduced divisor $B \subset Y$. On X consider the line bundle $L := \pi^*\mathcal{L}$, which is ample and spanned, and assume that

$$H^0(L) = \pi^* H^0(\mathcal{L}). \tag{4.1.1}$$

Due to (4.1.1) ϕ_L factors through π and the embedding $\phi_{\mathcal{L}} : Y \to \mathbb{P}^N$, and we can identify $\mathbb{P}(H^0(L))$ with the dual of the projective space $\mathbb{P}^N$.

Still (4.1.1) says that every $D \in |L|$ has the form $D = \pi^*H$, with $H \in |\mathcal{L}|$. Hence D is singular if and only if H is either singular or tangent (or in an even more special position with respect) to B, as an immediate local check shows. Therefore via the identification $\mathbb{P}(H^0(L)) = \mathbb{P}^{N*}$, we see that $\mathcal{D}(X,L)$ contains at least the following two components: $\mathcal{D}_0(X,L) = (\phi_{\mathcal{L}}(Y))^*$ and $\mathcal{D}_1(X,L) = (\phi_{\mathcal{L}}(B))^*$. Of course some more components may occur, corresponding to a stratification of the singular locus of B.

Note that ϕ_L is not generically one-to-one since $m \geq 2$, and that the general element of $\mathcal{D}_0(X,L)$ has exactly m isolated nondegenerate quadratic singularities. Moreover $\mathrm{def}_0(X,L) = \mathrm{def}(Y,\mathcal{L})$; in particular if $(Y,\mathcal{L})$ has degenerate dual variety we get examples where $\mathrm{def}_0(X,L) > 0$. In the limit case $(Y,\mathcal{L}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ the first component $\mathcal{D}_0(X,L)$ disappears while the second one $\mathcal{D}_1(X,L)$ is a hypersurface, since B itself is a hypersurface of $\mathbb{P}^n$. In other words we get $\mathrm{def}_0(X,L) = n$ while $\mathrm{def}(X,L) = 0$. Moreover, by applying [LPS2, (2.8)] we get the following

(4.1.2) Proposition. *Let* $\pi : X \to Y$*,* $\mathcal{L}$ *and* L *be as in* (4.1) *and assume that* (4.1.1) *holds. Then* $\mathrm{def}_0(X,L) \leq n-2$ *unless* $(Y,\mathcal{L}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(4.1.3)*Example.* To give a concrete example, let Y be a projective n-fold polarized by a very ample line bundle $\mathcal{L}$. Let B be a smooth element of $|mt\mathcal{L}|$ with $t \geq 2$, $m \geq 3$. Then let $\pi : X \to Y$ be the cyclic m-sheeted cover branched along B and set $L := \pi^*\mathcal{L}$ as before.

Now let $B' \subset X$ be the ramification divisor of π. Since π maps B' isomorphically to B we have an isomorphism $H^0(L_{B'}) \cong H^0(\mathcal{L}_B)$. On the other hand from the exact sequence

$$0 \to \mathcal{L} - B = (1-tm)\mathcal{L} \to \mathcal{L} \to \mathcal{L}_B \to 0,$$

noting that $h^i((1-tm)\mathcal{L}) = 0$ for $i = 0,1$, we get the isomorphism $H^0(\mathcal{L}) \cong H^0(\mathcal{L}_B)$. Recalling that $B' \in |\pi^* t\mathcal{L}|$, π induces the following exact sequence on X

$$0 \to L - B' = (1-t)\pi^*\mathcal{L} \to L \to L_{B'} \to 0.$$

Since $t \geq 2$ we also have $h^i((1-t)\pi^*\mathcal{L}) = 0$, for $i = 0,1$, hence there is a further isomorphism $H^0(L) \cong H^0(L_{B'})$. By putting together these isomorphisms we finally see that (4.1.1) holds.

So the discussion in (4.1) applies to this situation and since B is smooth $\mathcal{D}(X,L)$ consists exactly of two components. As before, the general element of $\mathcal{D}_0(X,L)$ has exactly m isolated nondegenerate quadratic singularities, but now, since π is cyclic of degree $m \geq 3$, a local check shows that any element of $\mathcal{D}_1(X,L)$ (coming from a divisor H tangent to B) has a singular point which is worse than a nondegenerate quadratic singularity.

(4.2) We are giving some examples of ample line bundles L on a projective n-fold X with a vector subspace $V \subseteq H^0(L)$ spanning L and giving rise to a generically one-to-one map ϕ_V.

(4.2.1) Let L be a very ample line bundle on a projective n-fold X and let $\phi_L : X \to \mathbb{P}^N$ be the corresponding embedding. Assume that $N \geq n+2$ and let P be a general $\mathbb{P}^{N-n-2} \subset \mathbb{P}^N$. As is known, projecting $\phi_L(X)$ from P gives a generically one-to-one map onto the image. So let V be the vector subspace of $H^0(L)$ such that $\mathbb{P}(V) = \phi_L^*|\mathcal{O}_{\mathbb{P}^N}(1) - P|$ (the linear system of hyperplanes containing P). Then V spans L and the map ϕ_V is generically one-to-one.

In the next examples L will be not very ample, but $V = H^0(L)$.

(4.2.2) *Example.* Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 4$ having a single cusp and no more singular points. Let $\nu : \tilde{C} \to C$ be the normalization and let $\mathcal{L} = \nu^*\mathcal{O}_{\mathbb{P}^2}(1)$. $\mathcal{L}$ is ample and spanned and ν is generically one-to-one. So it is enough to show that $\phi_{\mathcal{L}} = \nu$, or equivalently that $h^0(\mathcal{L}) = 3$. By contradiction, assume that $h^0(\mathcal{L}) \geq 4$. Then since $\phi_{\mathcal{L}}(\tilde{C})$ is not a plane curve, the genus $g(\tilde{C})$ is bounded from above by the Castelnuovo bound. On the other hand $g(\tilde{C}) = g(C) - 1 = \frac{1}{2}d(d-3)$, since C is a plane curve. But this contradicts the Castelnuovo bound, since $d \geq 4$.

Note that the same argument works for a plane curve C having a single node, or more generally for plane curves having a small number of singular points with respect to the degree; e. g. for C a plane quintic having a single node, in which case $\mathcal{L} = K_{\tilde{C}} - p$, where p is a point of $\tilde{C}$.

The following remark is obvious.

(4.2.3) *Remark.* Let $\mathcal{L}_i$ be an ample and spanned line bundle on a projective manifold Y_i and assume that $\Gamma(\mathcal{L}_i)$ gives a generically one-to-one map, for $i = 1, 2$. Set $X := Y_1 \times Y_2$ and let p_i denote the projection of X onto the i-th factor. Then the line bundle

$$\mathcal{L}_1 \boxtimes \mathcal{L}_2 := p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$$

is ample and spanned and the map defined by its sections is generically one-to-one.

Now let $\tilde{C}$ and $\mathcal{L}$ be as in (4.2.2) and let $z \in \tilde{C}$ be the point mapped by ν to the cusp of C. Then $(\tilde{C})_0 = \tilde{C} - \{z\}$ and $(\tilde{C})_1 = \{z\}$. Now, for any $n \geq 2$ set $X = \tilde{C}^n$ and let $L := \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L}$; then $\dim X_i = n - i$ for every $i = 0, 1, \ldots, n$. So we have equality in (0.3.1).

(4.2.4) *Example.* Let $\tilde{C}$, $\mathcal{L}$ and z be as in (4.2.3) and consider the pair

$$(S, L) := (\tilde{C} \times \mathbb{P}^1, \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)).$$

By Remark (4.2.3) L is ample and spanned and its sections define a generically one-to-one map. In fact ϕ_L factors through $(\phi_{\mathcal{L}}, \mathrm{id}_{\mathbb{P}^1})$ and the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$. Note that (S, L) is a scroll: let $\pi : S \to \tilde{C}$ be its projection and let $Z = \pi^{-1}(z)$ be the special fibre. We have $(S)_0 = S \setminus Z$, while $(S)_1 = Z$. Since $h^0(L) = 6$ we see that the global singular sets

$$\mathcal{S}_0 = \overline{\{(x, H) \in (S)_0 \times |L| \ : \ x \in \mathrm{Sing}(H)\}}$$

and

$$\mathcal{S}_1 = \{(x, H) \in Z \times |L| \ : \ x \in \mathrm{Sing}(H)\}$$

have both dimension 4. Now look at their images $\mathcal{D}_0$ and $\mathcal{D}_1$ under the second projection of $S \times |L|$. Note that (S, L) being a scroll its nef value is 2, hence $\mathrm{def}_0(S, L) = 0$ by Theorem (3.3). So $\mathcal{D}_0$ is a hypersurface of $|L| = \mathbb{P}^{5*}$. Our aim is to show that $\mathcal{D}_1$ is strictly contained in $\mathcal{D}_0$.

Let $x \in S$. Since (S, L) is a scroll we have that all elements $H \in |L - 2x|$ have the form

$$(+) \qquad H = F + \Gamma,$$

where F is the fibre of π through x and Γ is an effective divisor inducing $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibres. Moreover if $x \in (S)_0$ and H is a general element of $\mathcal{D}_0$ then Γ is irreducible, hence smooth. Otherwise H would have more than a single nondegenerate quadratic singularity, which would contradict Theorem (2.3). Note however that if $x \in Z$, then

$$(++) \qquad H = 2Z + C,$$

where C also is an effective divisor inducing $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibres. Actually $H - Z \in |(\mathcal{L} - [z]) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)|$. On the other hand since $|\mathcal{L} - z| = |\mathcal{L} - 2z|$ we have that z is a base point of the linear system $|(\mathcal{L} - [z])|$, hence Z is a fixed component of $|(\mathcal{L} - [z]) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)|$.

The strict inequality $\mathcal{D}_1 \subset \mathcal{D}_0$ will follow once we have shown that $\mathcal{D}_1 \subseteq \mathcal{D}_0$. Actually the general $H \in \mathcal{D}_0$ has a single nondegenerate quadratic singularity while we have seen that all elements $H \in \mathcal{D}_1$ are singular along Z. So in view of (+) and (++) it only remains to prove that $Z + C$ occurs as a limit of Γ's for $y := \pi(x)$ tending to z. Since $h^0(\mathcal{L} - [y]) = 2$ for all $y \in \tilde{C}$, it is enough to show that the line bundle $\mathcal{M}_z := (\mathcal{L} - [z]) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ occurs as limit of $\mathcal{M}_y := (\mathcal{L} - [y]) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ for y tending to z. Consider the threefold $\mathcal{X} = S \times \tilde{C} = \tilde{C} \times \mathbb{P}^1 \times \tilde{C}$ and let p_i ($i = 1, 2, 3$) be the projections onto the three factors. Denote by Δ the inverse image of the diagonal of $\tilde{C} \times \tilde{C}$ via the map (p_1, p_3). The line bundle on $\mathcal{X}$ given by $\mathcal{M} = (p_1^*\mathcal{L} - \Delta) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ restricts to $p_3^{-1}(y)$ as $\mathcal{M}_y$ and to $p_3^{-1}(z)$ as $\mathcal{M}_z$ and so we are done.

(4.2.5) *Example.* Let Σ be the Del Pezzo surface with $K_\Sigma^2 = 2$ and let $L = -2K_\Sigma$. As is well known the map given by $\Gamma(K_\Sigma + L) = \Gamma(-K_\Sigma)$ is a double cover $\varphi : \Sigma \to \mathbb{P}^2$ branched along a smooth quartic B. Let $B' = \varphi^{-1}(B)$. Let $C \in |L|$ be a general element and let x be a point of $C \cap B'$. Let γ be a smooth conic passing through $\varphi(x)$ but containing no other point of the finite set $\varphi(C \cap B')$. Then $C_0 = \varphi^*\gamma$ is a smooth hyperelliptic curve in the linear system $\varphi^*|\mathcal{O}_{\mathbb{P}^2}(2)| \subset |L|$ As is known (e.g. see [BS, Example 10.2.4, p. 260]) the hyperelliptic locus of $|L|$ is exactly the sublinear system $\varphi^*|\mathcal{O}_{\mathbb{P}^2}(2)|$, which is a hyperplane of $|L|$. So, since C is general, C_0 is the unique hyperelliptic curve in the pencil $\mathcal{P}$ generated by C_0 and C. By blowing-up Σ at the base points of $\mathcal{P}$ we get a surface S and a morphism $p : S \to \mathbb{P}^1$. Let F be its general fibre; then F is a smooth non-hyperelliptic curve of genus 3 as C; we still denote by C_0 the special hyperelliptic fibre of p. Finally let $\sigma : S \to \Sigma$ be the blowing-up and let E_i $(i = 1, \ldots, 8)$ denote the exceptional curve corresponding to the i-th base point of $\mathcal{P}$. Since $p^*\mathcal{O}_{\mathbb{P}^1}(1) = F = \sigma^* L - \sum_i E_i$, we get

$$K_S + p^*\mathcal{O}_{\mathbb{P}^1}(1) = \sigma^*(K_\Sigma + L) = \sigma^*(-K_\Sigma).$$

Hence $K_S + p^*\mathcal{O}_{\mathbb{P}^1}(1)$ is spanned and the E_i's are the only curves on S having intersection zero with it. Now let $\mathcal{L} := K_S + p^*\mathcal{O}_{\mathbb{P}^1}(\delta)$, with $\delta >> 0$. Writing

$$(*) \qquad \mathcal{L} = (K_S + p^*\mathcal{O}_{\mathbb{P}^1}(1)) + p^*\mathcal{O}_{\mathbb{P}^1}(\delta - 1)$$

we see that $\mathcal{L}$ is spanned; moreover it is ample, since $E_i F = 1$. In addition the morphism defined by $\Gamma(\mathcal{L})$ is generically one-to-one. Actually two points lying on distinct fibres of S can be separated by using a section of $\mathcal{O}_{\mathbb{P}^1}(\delta - 1)$. Now take two distinct points lying on the same general fibre F. Recalling (*), by adjunction we see that

$$(**) \qquad \mathcal{L}_F = (K_S + p^*\mathcal{O}_{\mathbb{P}^1}(1))_F = K_F.$$

On the other hand the restriction homomorphism $H^0(K_S + F) \to H^0(K_F)$ is an isomorphism, S being rational. Therefore there exists a section of $\mathcal{L}$ separating our points. Note however that $\mathcal{L}$ cannot be very ample, since $\mathcal{L}_{C_0} = K_{C_0}$, C_0 is hyperelliptic and $H^0(K_S + C_0) \to H^0(K_{C_0})$ is still an isomorphism. Now let y be a hyperelliptic branch point of C_0 and choose local coordinates (z, w) around y in such a way that C_0 is locally defined by $w = 0$. Then applying (**) to C_0 instead of F shows that any section $s \in \Gamma(\mathcal{L})$ has a local expression of the form

$$s = a + bw + cz^2 + \ldots,$$

(i.e. it does non contain the linear term in z). This shows that $|L - 2y|$ has codimension 1 in $|L - y|$, i.e. $y \in \mathcal{J}_1$. So $\mathcal{J}_1$ contains the set Z consisting of the 8 hyperelliptic branch points of C_0. Since $\phi_{\mathcal{L}}$ is an embedding on $X \setminus C_0$ and unramified on $C_0 \setminus Z$, we conclude that $\mathcal{J}_1 = Z$. Thus

$$S_0 = S \setminus Z, \quad S_1 = Z, \quad S_2 = \emptyset.$$

(4.2.6) *Example.* Let $(S, \mathcal{L})$ be as in (4.2.5) and for any $m \geq 2$ set $X = S^{2m}$ and let $L = \mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L}$. Then L is ample and spanned and the map it defines is generically one-to-one as follows by applying (4.2.3) inductively. Moreover $\dim X_i = n - 2i = 2(m - i)$ for $i = 0, 1, \ldots, m$ and $X_{n-2i-1} = \emptyset$. So for $i \geq 1$ we have strict inequality in (0.3.1).

Finally let $(S, \mathcal{L}_1 = \mathcal{L})$ be as in (4.2.5), let $(T = \tilde{C} \times \tilde{C}, \mathcal{L}_2 = \mathcal{L} \boxtimes \mathcal{L})$, where $(\tilde{C}, \mathcal{L})$ is as in (4.2.2) and set $X = S \times T$, $L = \mathcal{L}_1 \boxtimes \mathcal{L}_2$. Then for (X, L) we have $\dim X_i = n - i$ for $i \leq 2$, while $\dim X_3 = 4$ and $X_4 = \emptyset$.

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