

# INVARIANT CURVES FOR BIRATIONAL SURFACE MAPS

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ABSTRACT. We classify invariant curves for birational surface maps that are expanding on cohomology. When the expansion is exponential, the arithmetic genus of an invariant curve is at most one. This implies severe constraints on both the type and number of irreducible components of the curve. In the case of an invariant curve with genus equal to one, we show that there is an associated invariant meromorphic two form.

## 1. INTRODUCTION

One of the first things (see e.g. [Bea], p65) one learns about dynamics on the Riemann sphere is that no non-trivial rational function  $f : \mathbf{P}^1 \dashrightarrow \mathbf{P}^1$  leaves more than two points totally invariant. This fact, an elementary consequence of the Riemann-Hurwitz Theorem, has been generalized [FS], [SSU], [BCS] to holomorphic maps  $f : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$  in any dimension. For example, if  $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$  is holomorphic, then the largest totally  $f$ -invariant curve  $C = f^{-1}(C)$  is a union of at most three lines. In this paper, we classify curves invariant under a non-trivial birational map of two complex variables.

There are two important differences between the birational and holomorphic cases. First of all, it is much too restrictive to consider only those curves which are totally invariant in the strictest sense. If  $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$  is a birational map that is not linear, then no curve  $C$  can be  $f$ -invariant if one allows components of the critical set of  $f$  in the preimage. So instead, we define the preimage  $f^{-1}(C)$  of a curve  $C$  to exclude the critical set, and we say that  $C$  is *invariant* if  $C = f^{-1}(C)$ . That is,  $C$  is equal to its proper transform under  $f$ .

The second way in which the birational case is different is that one cannot hope to obtain a general theorem concerning birational maps of  $\mathbf{P}^2$  without also considering birational maps of other complex surfaces. The problem is as follows. If one begins with, say, a linear map  $L : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$  that preserves a line  $\ell$  and then conjugates with a cremona transformation  $g : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$  of very high algebraic degree  $d(g)$ , then the birational map  $f := g \circ L \circ g^{-1} : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$  will leave invariant the curve  $g(\ell)$ , which has degree  $d(g)$ . Hence there is no limit on the degree of a birationally invariant curve.

We observe, however, that the map  $f$  in such an example always degenerates when one starts to iterate it. In particular, the algebraic degree  $d(f^n)$  of  $f^n$  is not  $d(f)^n$  as one would hope. Rather, it is bounded above by (and in typical cases equal to)  $d(g)^2$ . In a sense, both the map  $f$  and the invariant curve  $g(\ell)$  are relatively simple objects disguised as something

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more complicated by a poor birational choice of coordinate. In particular, it was shown in [DF] that by blowing up points in  $\mathbf{P}^2$  and lifting everything to the new complex surface, one can always arrive at a birational map that is *algebraically stable*. That is,  $(f^n)^* = (f^*)^n$  where  $f^*$  is the induced linear action of  $f$  on  $\text{Pic}(X)$ . In this case, the degree of  $f$  is replaced by the spectral radius  $\lambda(f)$  of  $f^*$ , i.e. the so-called (*first*) *dynamical degree* of  $f$ . Our first main result (see section 3) is

**Theorem 1.1.** *Let  $f : X \dashrightarrow X$  be an algebraically stable birational map of a complex projective surface with  $\lambda(f) > 1$ . Then the genus of any connected  $f$ -invariant curve  $C$  is zero or one.*

By genus here, we mean what is commonly called the *arithmetic genus* of  $C$ . In particular, if  $C$  is irreducible, then the Riemann surface obtained by desingularizing  $C$  is either the Riemann sphere or a torus. And if  $X = \mathbf{P}^2$ , then the theorem amounts to saying that the degree of an  $f$ -invariant curve is at most three. In fact, we will prove (see Section 3) somewhat more than the assertion in Theorem 1.1, classifying invariant curves for any bimeromorphic map  $f : X \dashrightarrow X$  of any compact Kähler surface  $X$  for which the sequence  $\|f^{n*}\|$  is unbounded.

Beyond the intrinsic interest of Theorem 1.1, we note that invariant curves play a decisive role in many dynamically interesting examples of birational maps. For instance, Example 4 in [Fav] concerns a birational map that restricts to a rotation on a particular line, and this example has turned out to be important for testing the limits of what is known about ergodic theory of birational maps. In another direction, the papers [BD3] and [BD1] give detailed descriptions of the real dynamics of some families of birational maps, and the analysis in these papers depends heavily on the fact that indeterminacy orbits of the maps are constrained to lie in invariant curves. More generally, it is natural to consider the class of birational maps on a surface that preserve a given meromorphic two form. The support of the divisor of such a two form will necessarily be invariant in some sense. We show in section 4 that

**Theorem 1.2.** *Let  $f : X \dashrightarrow X$  be an algebraically stable birational map of a complex projective surface, and  $C$  a connected  $f$ -invariant curve of genus one. By contracting curves in  $X$ , one can arrange additionally that  $-C$  is the divisor of a meromorphic two form  $\eta$  satisfying  $f^*\eta = c\eta$ . The constant  $c \in \mathbf{C}$  is determined solely by the curve  $C$  and the induced automorphism  $f : C \dashrightarrow C$ .*

Among other things, this theorem allows us to be quite precise about the possibilities for a genus one invariant curve (see Corollary 4.3).

The plan of the paper is as follows. Section 2 contains (mostly well-known) definitions and results concerning geometry of surfaces and birational maps on surfaces. It also reviews a classification of birational self-maps from [DF] that we will rely on heavily. Section 3 presents, among other things, the proof of Theorem 1.1. The central ingredient is Corollary 3.3 which says that if  $C$  is an invariant curve and  $K_X$  the canonical class of  $X$ , then  $(C + K_X) \cdot \theta \leq 0$  for some nef class  $\theta \in H_{\mathbf{R}}^{1,1}(X)$ . Section 4 concerns the case of genus one invariant curves and contains the proof of Theorem 1.2. Section 5 discusses the number of irreducible components of an invariant curve. In the genus one case, we give an upper bound that depends only on the surface and not the map. In the genus zero case, we give an upper bound that is more

complicated, but aside from one exceptional situation, the bound again depends only on the surface.

## 2. BACKGROUND

**2.1. Complex surfaces.** Throughout this paper,  $X$  will denote a complex surface, by which we mean a connected compact complex manifold of complex dimension two. Usually  $X$  will be rational. The book [BHPV] is a good general reference for complex surfaces. Here we recount only needed facts.

Given divisors,  $D, D'$  on  $X$ , we will write  $D \sim D'$  to denote linear equivalence,  $D \leq D'$  if  $D' = D + E$  where  $E$  is an effective divisor and  $D \lesssim D'$  if  $D' - D$  is linearly equivalent to an effective divisor. By a *curve* in  $X$ , we will mean a reduced effective divisor. We let  $\text{Pic}(X)$  denote the Picard group on  $X$ , i.e. divisors modulo linear equivalence. We let  $K_X \in \text{Pic}(X)$  denote the (class of) a canonical divisor on  $X$ , which is to say, the divisor of a meromorphic two form on  $X$ .

Taking chern classes associates each element of  $\text{Pic}(X)$  with a cohomology class in  $H^{1,1}(X) \cap H^2(X, \mathbf{Z})$ . We will have need of the larger group  $H_{\mathbf{R}}^{1,1}(X) := H^{1,1}(X) \cap H^2(X, \mathbf{R})$ . We call a class  $\theta \in H_{\mathbf{R}}^{1,1}(X)$  *nef* if  $\theta^2 \geq 0$  and  $\theta \cdot C \geq 0$  for any complex curve. We will repeatedly rely on the following consequence of the Hodge index theorem.

**Theorem 2.1.** *If  $\theta \in H_{\mathbf{R}}^{1,1}(X)$  is a non-trivial nef class, and  $C$  is a curve, then  $\theta \cdot C = 0$  implies that either*

- *the intersection form is negative when restricted to divisors supported on  $C$ ; or*
- *$\theta^2 = 0$  and there exists an effective divisor  $D$  supported on  $C$  such that  $D \sim t\theta$  for some  $t > 0$ .*

*In particular if  $\theta$  has positive self-intersection, then the intersection form is negative definite on  $C$ .*

*Proof.* The hypotheses imply that  $\theta \cdot D = 0$  for every divisor  $D$  supported on  $C$ . Suppose that the intersection form restricted to  $C$  is not negative definite. That is, there is a non-trivial divisor  $D$  with  $\text{supp } D \subset C$  and  $D^2 \geq 0$ . Then we may write  $D = D_+ - D_-$  as a difference of effective divisors supported on  $C$  with no irreducible components in common. Since  $D_+ \cdot D_- \geq 0$ , we have

$$0 \leq D^2 \leq D_+^2 + D_-^2,$$

so replacing  $D$  with  $D_+$  or  $D_-$  allows us to assume that  $D$  is effective. In particular,  $D$  represents a non-trivial class in  $H_{\mathbf{R}}^{1,1}(X)$ . Since  $D \cdot \theta = 0$  and  $\theta^2, D^2 \geq 0$ , we see that the intersection form is non-negative on the subspace of  $H_{\mathbf{R}}^{1,1}(X)$  generated by  $D$  and  $\theta$ . By Corollary 2.15 in [BHPV, page 143], such a subspace must be one-dimensional. Thus  $D = t\theta$  for some  $t > 0$ .  $\square$

By the *genus*  $g(C)$  of a curve  $C \subset X$ , we will mean the quantity  $1 - \chi(\mathcal{O}_C)$ , or equivalently,  $1 + h^0(K_C)$  minus the number of connected components of  $C$ . If  $C$  is smooth and irreducible, then  $g(C)$  is just the usual genus of  $C$  as a Riemann surface. If  $C$  is merely irreducible, then  $g(C)$  is usually called the *arithmetic genus* of  $C$ , and in this case it dominates the genus of the Riemann surface obtained by desingularizing  $C$ . If  $C$  is connected then  $g(C) \geq 0$ , but

our notion of genus is a bit non-standard in that we do not generally require connectedness of  $C$  in what follows. For any curve  $C$ , connected or not, we have the following *genus formula*

$$(1) \quad g(C) = \frac{C \cdot (C + K_X)}{2} + 1.$$

**2.2. Birational maps.** Now suppose that  $Y$  is a second complex surface and  $f : X \rightarrow Y$  is a birational map of  $X$  onto  $Y$ . That is,  $f$  maps some Zariski open subset of  $X$  biholomorphically onto its image in  $Y$ . In general the complement of this subset will consist of a finite union of rational curves collapsed by  $f$  to points, and a finite set  $I(f)$  of points on which  $f$  cannot be defined as a continuous map. We call the contracted curves *exceptional* and the points in  $I(f)$  *indeterminate* for  $f$ . The birational inverse  $f^{-1} : Y \rightarrow X$  of  $f$  is obtained by inverting  $f$  on the Zariski open set where  $f$  acts biholomorphically. Note that what we call a birational map is perhaps more commonly called a birational correspondence, the former term often being understood to mean that  $I(f) = \emptyset$ .

We adopt the following conventions concerning images of proper subvarieties of  $X$ . If  $C \subset X$  is an irreducible curve, then  $f(C)$  is defined to be  $\overline{f(C - I(f))}$ , which is a point if  $C$  is exceptional for  $f$  and a curve otherwise. If  $p \in X$  is a point of indeterminacy, then  $f(p)$  will denote the union of  $f^{-1}$ -exceptional curves that  $f^{-1}$  maps to  $p$ . We apply the same conventions to images under  $f^{-1}$ .

Our convention for the inverse image of an irreducible curve extends by linearity to a *proper transform* action  $f^\#D$  of  $f$  on divisors  $D$ , provided we identify points with zero. We also have the *total transform* action  $f^*D$  of  $f$  on divisors obtained by pulling back local defining functions for  $D$  by  $f$ . Total transform has the advantage that it preserves linear equivalence and therefore descends to a linear map  $f^* : \text{Pic}(Y) \rightarrow \text{Pic}(X)$ . We denote the proper and total transform under  $f^{-1}$  by  $f_*$  and  $f_\#$ , respectively.

In general,  $f^*D - f^\#D$  is an effective divisor with support equal to a union of exceptional curves mapped by  $f$  to points in  $\text{supp } D$ . It will be important for us to be more precise about this point. To do so, we use the ‘graph’  $\Gamma(f)$  of  $f$  obtained by minimally desingularizing the variety

$$\overline{\{(x, f(x)) \in X \times Y : x \notin I(f)\}}.$$

We let  $\pi_1 : \Gamma(f) \rightarrow X$ ,  $\pi_2 : \Gamma(f) \rightarrow Y$  denote projections onto first and second coordinates. Thus  $\Gamma(f)$  is an irreducible complex surface and  $\pi_1, \pi_2$  are *proper modifications* of their respective targets, each holomorphic and birational and therefore each equal to a finite composition of point blowups. One sees readily that  $f = \pi_2 \circ \pi_1^{-1}$ , and that the exceptional and indeterminacy sets of  $f$  are the images under  $\pi_1$  of the exceptional sets of  $\pi_2$  and  $\pi_1$ , respectively. Given a decomposition  $\sigma_n \circ \cdots \circ \sigma_1$  of  $\pi_2$  into point blowups, we let  $E(\sigma_j)$  denote the center of the blowup  $\sigma_j$  and

$$\hat{E}_j(f) = \sigma_1^* \cdots \sigma_{j-1}^* E(\sigma_j), \quad E_j(f) = \pi_{1*} \hat{E}_j(f).$$

In particular,  $\bigcup \text{supp } E_j(f)$  is the exceptional set. We call the individual divisors  $E_j(f)$  the *exceptional components* of  $f$  and call their sum  $E(f) := \sum E_j(f)$  the *exceptional divisor* of  $f$ . It should be noted that, as we have defined them, the exceptional components of  $f$  are connected, but in general they are neither reduced nor irreducible.

The following proposition assembles some further information about the exceptional components. These can be readily deduced from well-known facts about point blowups. We recall that the *multiplicity* of a curve  $C$  at a point  $p$  is just the minimal multiplicity of the intersection of  $C$  with an analytic disk meeting  $C$  only at  $p$ .

**Proposition 2.2.** *Let  $\sigma_j$ ,  $E_j(f)$ , and  $E(f)$  be as above, and  $C \subset X$  be a curve.*

- $E(f) = (f^*\eta) - f^*(\eta)$  for any meromorphic two form  $\eta$  on  $X$  (here  $(\eta)$  denotes the divisor of  $\eta$ ). Less precisely,  $E(f) \sim K_X - f^*K_X$ .
- $E_j(f)$  and  $E_i(f)$  have irreducible components in common if and only if  $f(E_j(f)) = f(E_i(f))$ . If this is the case, then  $i \leq j$  implies that  $E_i(f) \leq E_j(f)$ .
- The multiplicity with which an irreducible curve  $E$  occurs in  $E(f)$  is bounded above by a constant that depends only on the number of exceptional components  $E_j(f)$  that include  $E$ .
- $f^*C - f^\sharp C = \sum c_j E_j(f)$  where  $c_j$  is the multiplicity of  $(\sigma_n \circ \dots \circ \sigma_{j+1})^\sharp(C)$  at the point  $\sigma_j(E(\sigma_j))$ .
- In particular,  $c_j$  vanishes if  $p_j := f(E_j(f)) \notin C$ ,  $c_j \leq 1$  if  $p_j$  is a smooth point of  $C$ , and  $c_j > 0$  if  $p_j \in C$  and  $E_j(f)$  is not dominated by any other exceptional component of  $f$ .
- Hence (in light of the 2nd and 5th items),  $\text{supp } f^*C - f^\sharp C = f^{-1}(C \cap I(f^{-1}))$ .

We will also need the following elementary fact.

**Lemma 2.3.** *Let  $C \subset X$  be a curve such that no component of  $C$  is exceptional for  $f$ . If  $p \in C - I(f)$ , then multiplicity of  $f(C)$  at  $f(p)$  is no smaller than that of  $C$  at  $p$ . In particular,  $f(p)$  is singular for  $f(C)$  if  $p$  is singular for  $C$ .*

**2.3. Classification of birational self-maps.** Supposing that  $f : X \dashrightarrow X$  is a birational self-map, we now recall some additional information from [DF]. First of all, there are pullback and pushforward actions  $f^*, f_* : H_{\mathbf{R}}^{1,1}(X) \dashrightarrow H_{\mathbf{R}}^{1,1}(X)$  compatible with the total transforms  $f^*, f_* : \text{Pic}(X) \dashrightarrow \text{Pic}(X)$ . The actions are adjoint with respect to intersections, which is to say that

$$(2) \quad f^* \alpha \cdot \beta = \alpha \cdot f_* \beta,$$

for all  $\alpha, \beta \in H_{\mathbf{R}}^{1,1}(X)$ . Less obviously,  $f^{n*}$  is ‘intersection increasing’, meaning

$$(f^{n*} \alpha)^2 \geq \alpha^2$$

The *first dynamical degree* of  $f$  is the quantity

$$\lambda(f) := \lim_{n \rightarrow \infty} \|f^{n*}\|^{1/n} \geq 1.$$

It is less clear than it might seem that  $\lambda(f)$  is well-defined, as it can happen that  $(f^n)^* \neq (f^*)^n$  for  $n$  large enough. However,  $\lambda(f)$  can be shown to be invariant under birational change of coordinate and one can take advantage of this to choose a good surface on which to work.

**Theorem 2.4.** *The following are equivalent for a birational map  $f : X \dashrightarrow X$  on a complex surface.*

- $(f^n)^* = (f^*)^n$  for all  $n \in \mathbf{Z}$ .
- $I(f^n) \cap I(f^{-n}) = \emptyset$  for all  $n \in \mathbf{N}$ .

- $f^n(C) \notin I(f)$  for any  $f$ -exceptional curve  $C$ .
- $f^{-n}(C) \notin I(f^{-1})$  for any  $f^{-1}$  exceptional curve  $C$ .

By blowing up finitely many points in  $X$ , one can always arrange that these conditions are satisfied.

We will call maps satisfying the equivalent conditions of this theorem AS (for *algebraically* or *analytically stable*). If  $f$  is AS, then  $\lambda = \lambda(f)$  is just the spectral radius of  $f^*$ . If  $X$  is Kähler, then there is a nef class  $\theta^+$  satisfying

$$f^*\theta^+ = \lambda\theta^+.$$

From (2) we have that  $\lambda(f^{-1}) = \lambda(f)$ , so we let  $\theta^-$  denote the corresponding class for  $f^{-1}$ . The following theorem summarizes many of the main results of [DF], and we will rely heavily on it here.

**Theorem 2.5.** *If  $f : X \dashrightarrow X$  is an AS birational map of a complex Kähler surface  $X$  with  $\lambda(f) = 1$ , then exactly one of the following is true (after contracting curves in  $\text{supp } E(f^n)$ , if necessary).*

- $\|f^{n*}\|$  is bounded independent of  $n$ , and  $f$  is an automorphism some iterate of which is isotopic to the identity.
- $\|f^{n*}\| \sim n$  and  $f$  preserves a rational fibration. In this case  $\theta^+ = \theta^-$  is the class of a generic fiber.
- $\|f^{n*}\| \sim n^2$  and  $f$  is an automorphism preserving an elliptic fibration. Again  $\theta^+ = \theta^-$  is the class of a generic fiber.

If, on the other hand,  $\lambda(f) > 1$ , then  $\theta^+ \cdot \theta^- > 0$  and either  $X$  is rational or  $f$  is (up to contracting exceptional curves) an automorphism of a torus, an Enriques surface, or a K3 surface.

We remark that the classes  $\theta^\pm$  are unique up to positive multiples whenever  $\|f^{n*}\|$  is unbounded, and indeed under the unboundedness assumption, we have

$$\lim_{n \rightarrow \infty} \frac{f^{n*}\theta}{\|f^{n*}\|} = c\theta^+$$

for any Kähler class  $\theta$  and some constant  $c = c(\theta) > 0$ . In what follows, we will largely ignore the case in which  $\|f^{n*}\|$  is bounded. After all, if some iterate of  $f$  is the identity map, then every curve in  $X$  will be  $f$ -invariant.

To close this section, we recall a result from [BD2], which we will use in section 4.

**Theorem 2.6.** *If  $f : X \dashrightarrow X$  is an AS birational map of a complex Kähler surface  $X$  with  $\lambda(f) > 1$ , then after contracting curves in  $\text{supp } E(f^n)$ , we can arrange additionally that  $\theta^+ \cdot f(p) > 0$  for every  $p \in I(f)$  and  $\theta^- \cdot f^{-1}(p) > 0$  for every  $p \in I(f^{-1})$ .*

### 3. INVARIANT CURVES

Unless otherwise noted in what follows,  $f : X \dashrightarrow X$  will always be an AS birational map on a complex Kähler surface  $X$ . We will call a curve  $C \subset X$  *invariant* for  $f$  if  $f(C) = C$ . If  $C$  is  $f$ -invariant, then  $f$  lifts to a biholomorphism of the desingularization of  $C$ . In particular

$f$  permutes the irreducible components of  $C$  and no such component is exceptional. Clearly  $C$  is  $f$ -invariant if and only if  $C$  is  $f^{-1}$ -invariant.

**Theorem 3.1.** *If  $C$  is an  $f$ -invariant curve, then  $C \cap I(f)$  consists of smooth points of  $C$ .*

The proof of this result boils down, essentially, to the fact that if  $p = f(p)$  is a fixed singular point of  $C$  that also lies in an exceptional curve, then  $f^n(p)$  should eventually be much more singular for  $C$  than  $p$  is, contradicting  $f$ -invariance of  $C$ .

*Proof.* In order to keep the notation simpler, we will prove the equivalent statement that  $C \cap I(f^{-1})$  consists of smooth points of  $C$ . We suppose in order to reach a contradiction that some point  $p \in I(f^{-1})$  is also a singular point of  $C$ . Since  $f$  is AS,  $f^k(p)$  is well-defined for all  $k \geq 0$ . By Lemma 2.3 and invariance of  $C$ , all points in the forward orbit of  $p$  are singular for  $C$ . But the singular set of  $C$  is finite, so replacing  $f$  by  $f^k$  if necessary, we may assume that  $p$  is fixed by  $f$  and that  $C$  is the connected curve obtained by keeping only those irreducible components containing  $p$ .

Because  $p \in I(f^{-1})$ , we have that  $f^{-1}(p)$  is a connected curve in  $\text{supp } E(f)$ , and because  $f(p) = p$  we see that  $p \in E \subset f^{-1}(p)$  for some irreducible exceptional curve  $E$ . In particular,  $f^*E \geq E$  by the fifth conclusion in Proposition 2.2. More generally, for  $k \geq 1$ , the first item in Proposition 2.2 and the fact that  $f$  is AS imply that

$$(3) \quad E(f^k) = (f^{k*}\eta) - f^{k*}(\eta) = \sum_{j=0}^{k-1} f^{j*}((f^{(k-j)*}\eta) - f^*(f^{(k-j-1)*}\eta)) = \sum_{j=0}^{k-1} f^{j*}E(f) \geq kE.$$

We will complete the proof by suitably interpreting (3) from another point of view.

Let  $\Gamma$  be a minimal desingularization of the graph of  $f^k$  and  $\pi_1, \pi_2 : \Gamma \rightarrow X$  be projections onto source and target. Because  $p \in I(f^{-1})$  and  $f$  is AS, it follows that  $p \notin I(f^k)$ . Hence there is a neighborhood  $U \ni p$  such that  $\pi_1$  maps  $\pi_1^{-1}(U)$  biholomorphically onto  $U$ . We let  $C_0 = \pi_1^\sharp(C)$ ,  $E_0 = \pi_1^\sharp E$ ,  $p_0 = \pi_1^{-1}(p)$  and observe that  $C_0$  is a connected curve meeting  $E_0$  at  $p_0$ . After decomposing  $\pi_2 = \sigma_n \circ \cdots \circ \sigma_1$  into a sequence of point blowups, we further define

$$C_j = \sigma_j \circ \cdots \circ \sigma_1(C_0), \quad E_j = \sigma_j \circ \cdots \circ \sigma_1(E_0), \quad p_j = \sigma_j \circ \cdots \circ \sigma_1(p_0).$$

In particular,  $C_j$  is connected and meets  $E_j$  (which will be a point for  $j$  large enough) at  $p_j$ . By  $f$  invariance of  $p$  and  $C$ , we see that  $C_n = C$  and  $p_n = p$ .

Let  $m \in \mathbf{N}$  be larger than  $g(C)$ . By the third item in Proposition 2.2, we can choose  $k$  large enough in (3) to deduce that there are at least  $m$ -indices  $j$  for which  $p_j = E_j = \sigma_j(E(\sigma_j))$  is the point blown up by  $\sigma_j$ . It is well-known (see e.g. [GH], p506) that blowing up a singular point of a curve strictly decreases its genus. Thus,  $g(C_{j-1}) \leq g(C_j) - 1$  for  $m$  different values of  $j$ . The cumulative effect of this is that

$$g(C_0) \leq g(C) - m < 0,$$

contradicting the fact that the genus of a connected curve is always non-negative.  $\square$

**Corollary 3.2.** *Let  $C \subset X$  be an  $f$ -invariant curve. Then up to linear equivalence, we have*

$$f^*(C + K_X) \lesssim C + K_X$$

*Proof.* In light of Theorem 3.1 and the fifth and first items in Proposition 2.2, we have

$$f^*C \leq C + E(f) \sim C + K_X - f^*K_X,$$

which rearranges to give the inequality we seek.  $\square$

**Corollary 3.3.** *Suppose that  $\|f^{n*}\|$  is unbounded. Then for any  $f$ -invariant curve  $C$ , we have*

$$\theta^\pm \cdot (C + K) \leq 0$$

*In particular, when  $\lambda(f) > 1$ ,  $(C + K) \cdot \theta \leq 0$  for some nef class  $\theta$  with  $\theta^2 > 0$ .*

*Proof.* Let  $\theta$  be a Kähler class on  $X$ . Then by Corollary 3.2, we have

$$0 \geq \frac{1}{\|f^{n*}\|} \theta \cdot ((C + K) - f_*^n(C + K)) = \frac{\theta - f^{n*}\theta}{\|f^{n*}\|} \cdot (C + K) \rightarrow c\theta^+ \cdot (C + K),$$

which verifies the first assertion. If  $\lambda(f) > 1$ , then Theorem 2.5 tells us that  $(\theta^+ + \theta^-)^2 \geq 2\theta^+ \cdot \theta^- > 0$ . Hence the second assertion holds for the particular class  $\theta^- + \theta^+$ .  $\square$

**Theorem 3.4.** *Suppose that  $\|f^{n*}\| \sim n^2$ . Then any connected  $f$  invariant curve is contained in a fiber of the elliptic fibration preserved by  $f$ .*

*Proof.* Let  $S$  be a Riemann surface and  $\pi : X \rightarrow S$  be the elliptic fibration preserved by  $f$ . Then  $\theta^+$  is the class of a fiber of  $\pi$ . Since the self-intersection of a fiber is zero and generic fibers are smooth elliptic curves, the genus formula tells us that  $0 = \theta^+(\theta^+ + K_X) = \theta^+ \cdot K_X$ . Therefore, from Corollary 3.3 we see that any  $f$  invariant curve  $C$  satisfies  $C \cdot \theta^+ \leq 0$ . Since  $\theta^+$  is nef, it follows that  $C \cdot \theta^+ = 0$ . This can only happen if  $C$  is contained in some fiber of  $\pi$ .  $\square$

**Theorem 3.5.** *Suppose that  $\|f^{n*}\| \sim n$  and let  $\pi : X \rightarrow S$  denote the rational fibration preserved by  $f$ . Let  $C$  denote the union of all irreducible  $f$ -invariant curves not contained in fibers of  $\pi$ . If  $C$  is non-empty, then exactly one of the following is true.*

- $C$  consists of one or two irreducible components, each mapped biholomorphically by  $\pi$  onto  $S$ .
- $C$  consists of one irreducible component, and  $\pi : C \rightarrow S$  is a two to one branched cover. In this case, some power of  $f$  induces the identity map on  $S$ .

In particular if the surface  $X$  is rational, then the base of the rational fibration is  $\mathbf{P}^1$ . Therefore,  $f$ -invariant curves must be rational or, in the case where some iterate of the induced map is trivial, hyperelliptic. We observe that hyperelliptic curves really do arise in this fashion. If, for example,  $A$  and  $B$  are meromorphic functions on  $\mathbf{P}^1$  and  $f : \mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow \mathbf{P}^1 \times \mathbf{P}^1$  is given by

$$f(x, y) = \left( x, -\frac{A(x)y + B(x)}{y} \right),$$

then  $f$  restricts to the identity map on the curve  $y^2 + A(x)y + B(x) = 0$ .



*Proof.* As in the previous proof,  $\theta^+$  is the class of a generic fiber of  $\pi$ , and the genus formula implies that  $\theta^+ \cdot K_X = -2$  (this time fibers are rational rather than elliptic curves). Therefore if  $C$  is an  $f$ -invariant curve, we see from Corollary 3.3 that  $C \cdot \theta^+ \leq 2$ . After removing all components of  $C$  contained in fibers of  $\pi$ , we obtain that  $\pi|_C$  is an at most 2 to 1 branched cover of  $S$  semiconjugating  $f|_C$  to an automorphism  $\tilde{f} : \mathbf{P}^1 \dashrightarrow \mathbf{P}^1$ . In particular,  $C$  has at most two irreducible components, and if there are two, they are necessarily isomorphic to  $S$ . If  $\tilde{f}$  is not periodic, then  $S$  is either a torus and  $\tilde{f}$  an ergodic translation or  $S$  is  $\mathbf{P}^1$  and  $\tilde{f}$  a linear fractional transformation. In either case  $f|_C$  is also aperiodic. Hence  $C$  is also a torus or an elliptic curve. In the case where  $\tilde{f}$  is a linear fractional transformation,  $f^n|_C$  must have a finite, non-zero number of fixed points for all  $n \in \mathbf{N}$  which means that  $C$  is rational.  $\square$

**Theorem 3.6.** *Let  $f : X \dashrightarrow X$  be an AS birational map with  $\lambda(f) > 1$ . Then any  $f$ -invariant curve  $C$  satisfies  $g(C) \leq 1$ . In particular, if  $C$  is connected, then  $g(C)$  is 0 or 1.*

*Proof.* Let us suppose first that  $X$  is irrational. Then by Theorem 2.5, we may blow down curves in  $E(f^n)$  (in particular not components of  $C$ ) and assume that  $X$  is an Enriques surface, a K3 surface or a torus, and  $f$  is an automorphism. In the last two cases  $K_X = 0$ , so Corollary 3.3 tells us that  $C \cdot \theta = 0$  for some nef class  $\theta$  with positive self-intersection. It follows that  $C \cdot (C + K_X) = C^2 < 0$ , so by the genus formula  $C$  has genus zero and is in fact a smooth rational curve with self-intersection  $-2$ . If  $X$  is an Enriques surface, it is double-covered by a K3-surface ([BHPV], page 339), and the theorem follows from lifting everything to the K3 surface.

Now we turn to the rational case. With  $V$  as in the theorem and  $K_X, K_C$  the canonical bundles of  $X$  and  $C$ , respectively, we have the standard short exact sequence of line bundles

$$0 \rightarrow K_X \rightarrow K_X + C \rightarrow K_C \rightarrow 0,$$

which gives rise to the long exact sequence

$$\cdots \rightarrow H^0(X, K_X + C) \rightarrow H^0(C, K_C) \rightarrow H^1(X, K_X) \rightarrow \cdots$$

However, since  $X$  is a rational surface,  $H^1(X, K_X)$  vanishes, so the sections of  $K_X + C$  surject onto those of  $K_C$ . But Corollary 3.3 tells us that  $K_X + C$  has non-positive intersection with a nef class with positive self-intersection. It follows then from Theorem 2.1 that

$$0 \leq h^0(K_C) \leq \dim h^0(K_X + C) \leq 1.$$

We conclude

$$g(C) = 1 - h^0(\mathcal{O}_C) + h^0(K_C) \leq h^0(K_C) \leq 1.$$

$\square$

#### 4. GENUS 1

For the remainder of this paper, we will assume  $f : X \dashrightarrow X$  is a birational map on a complex Kähler surface, as in the hypotheses of Theorem 3.6. Specifically,  $f$  is AS and  $\lambda(f) > 1$ . For irreducible invariant curves  $C$ , the implications of Theorem 3.6 are clear:  $C$  is either a rational curve with at most one simple cusp or normal crossing, or  $C$  is a smooth elliptic curve. But the conclusion of the theorem also says much about reducible invariant curves. For instance, in the genus 0 case, the following is well-known (see e.g. [BHPV], page 85)

**Theorem 4.1.** *A connected curve  $C = C_1 \cup \cdots \cup C_k$  has genus 0 if and only if  $C$  is a tree of smooth rational curves  $C_j$ . That is,  $C' \cdot C'' = 1$  for every decomposition of  $C = C' \cup C''$  into connected curves without common components.*

In the case where  $C$  is a connected genus 1 invariant curve, it turns out that we can give an even more specific description. To do this, it is necessary first to refine the argument from Theorem 3.6 and prove

**Theorem 4.2.** *Let  $f : X \dashrightarrow X$  be an AS birational map with  $\lambda(f) > 1$ , and suppose that  $V = f(V)$  is a connected invariant curve with  $g(V) = 1$ . Then by contracting finitely many curves, one may further arrange the following.*

- $V \sim -K_X$  is an anticanonical divisor.
- $I(f^n) \subset V$  for every  $n \in \mathbf{Z}$ .
- Any connected curve strictly contained in  $V$  has genus zero.
- If  $W$  is a connected  $f$ -invariant curve not completely contained in  $V$ , then  $W$  has genus zero, is disjoint from  $V$ , and is equal to a tree of smooth rational curves, each with self-intersection  $-2$ .

We remark that in general, contracting curves will be necessary to achieve the conclusions of this theorem. One can always cause the conclusions to fail trivially by blowing up finitely many consecutive elements in the orbit of a point not contained in  $\bigcup_{n \in \mathbf{Z}} I(f^n)$ .

*Proof.* Let us observe from the outset that contracting exceptional curves for  $f$  and  $f^{-1}$  will not disconnect  $V$ , nor change the fact that  $f(V) = V$ , and since the genus of  $V$  cannot decrease when a curve is contracted, it follows from Theorem 3.6 that  $g(V)$  will remain equal to one.

After contracting exceptional curves, we can assume that the conclusion of Theorem 2.6 is in force. So if  $C$  is a curve containing a point  $p \in I(f^n)$ , then we have that

$$\theta^+ \cdot C = \frac{f^{n*}\theta^+}{\lambda^n} \cdot C = \theta^+ \cdot \frac{f^n C}{\lambda^n} > 0,$$

because the last item in Proposition 2.2 tells us that  $f^{n*}C$  contains the the  $f^{-n}$ -exceptional curve  $f^n(p)$ . Similarly,  $\theta^- \cdot C > 0$  whenever  $C \cap I(f^{-n}) \neq \emptyset$ .

From the proof of Theorem 3.6 and the assumption that  $g(V) = 1$ , we see that  $h^0(K_X) = h^0(K_V) = 1$ . In particular, the line bundle  $K_X + V$  is effective. So assuming that  $K_X \not\sim -V$ , we have that

$$K_X + V \sim D,$$

where  $D$  is a non-trivial effective divisor. We will show that we can contract a set of mutually disjoint components of  $D$  without changing the facts that  $V$  is invariant, that  $g(V) = 1$ , and especially, that  $f$  is AS.

By Corollary 3.2 and the fact that the set of effective divisors is preserved by pullback we have

$$0 \leq f^{n*}D \lesssim D$$

for all  $n \in \mathbf{Z}$ . In addition, Corollary 3.3 implies that every irreducible component of  $D$  is intersection orthogonal to  $\theta^+$  and  $\theta^-$ . From this, we have that the intersection form is

negative definite for divisors supported on  $D$ . Thus the previous inequality actually holds on the level of divisors:

$$0 \leq f^{n*}D \leq D.$$

The second paragraph of the proof implies that  $I(f^n) \cap \text{supp } D = \emptyset$  for all  $n \in \mathbf{Z}$ . So for each irreducible component  $C$  of  $D$  and each  $n \in \mathbf{Z}$ , we have that  $f^n(C) = f_*^n C$  is either a point (i.e. 0) or an irreducible curve dominated by  $D$ . In the latter case, the absence of points of indeterminacy in  $D$  combines with Lemma 2.3 to imply that  $f^n(C)$  will be smooth if and only if  $C$  is.

The irreducible components  $C$  of  $D$  are now seen to fall into two classes: *periodic components*, satisfying  $C = f^n(C)$  for some  $n \in \mathbf{N}$ , and *eventually exceptional* components satisfying  $C \subset \text{supp } E(f^n)$  for  $n \in \mathbf{N}$  large enough. Since a component is periodic for  $f$  if and only if it is periodic for  $f^{-1}$ , it must also be the case that a component is eventually exceptional for  $f$  if and only if it is eventually exceptional for  $f^{-1}$ .

To find a component of  $D$  to contract, we apply the genus formula and the hypothesis  $g(V) = 1$  to arrive at  $D \cdot V = 0$ . Thus

$$0 > D^2 = D \cdot (V + K_X) = D \cdot K_X.$$

We can therefore choose an irreducible component  $C$  of  $D$  satisfying  $C \cdot K_X < 0$ . Because  $C^2 < 0$ , ([BHPV], p91) tells us that such a component must actually be a smooth rational curve with self-intersection  $-1$ —i.e. contractible. Any non-trivial image  $f^n(C) = f_*^n C$  of  $C$  will therefore also be a smooth rational curve. Because  $f_*^n$  is intersection increasing and  $(f_*^n C)^2 < 0$ , we see that  $(f_*^n C)^2$  must also be  $-1$ . Finally, if  $f_*^n C$  is distinct from  $C$ , then  $(f_*^n C + C)^2 < 0$  implies that  $f^n(C) \cap C = \emptyset$ . Hence the entire one-dimensional portion of the orbit of  $C$  can be contracted simultaneously, yielding a smooth surface in which each irreducible curve in the orbit of  $C$  has been replaced by a point.

The map  $f$  descends to a birational map on this new surface. In the case where the contracted curves are eventually exceptional, there exist both points of indeterminacy and exceptional curves that are eliminated by the contraction, but in neither case is a point of indeterminacy or an exceptional curve created. Hence  $f$  remains AS after the contraction. As in the first paragraph of the proof, the connectedness, the invariance, and the genus of  $V$  are unaffected by the contraction (even though, in the case where  $C$  is periodic, it could happen that  $C \subset V$ ). So after contracting, either we have  $V \sim -K_X$ , or we can repeat the preceding argument to contract yet more curves in  $X$ . The dimension of  $\text{Pic}(X)$  is finite and decreases with every contraction, so eventually the process will end and it will then be the case that  $V \sim -K_X$ . That is, we have established the first assertion of the theorem.

Since  $V \sim -K_X$ , the first and fifth conclusions of Proposition 2.2 together with Lemma 3.1 imply for every  $n \in \mathbf{Z}$  that

$$E(f^n) + V \sim f^{n*}V \leq E(f^n) + V,$$

which can only happen if  $f^{n*}V = E(f^n) + V$ . The fifth item in Proposition 2.2 therefore yields additionally that  $I(f^n) \subset V$  for all  $n \in \mathbf{Z}$ . That is, the second assertion in the theorem holds.

Now suppose that  $W$  is a connected curve strictly contained in  $V$ , and let  $W' = V - W$  be the complementary curve. Since  $V$  is connected, we have  $W \cdot W' > 0$ . Hence  $V \sim -K_X$

implies that  $W \cdot (W + K) = W \cdot (-W') < 0$ . So by the genus formula,  $g(W) = 0$ , and the third assertion is proved.

Finally, we consider a connected  $f$ -invariant curve  $W$  not completely contained in  $V$ . If  $W \cap V \neq \emptyset$ , then  $W \cup V$  is a connected  $f$ -invariant curve of genus at least one and not linearly equivalent to  $-K_X$ . Therefore, we can apply the first assertion of the theorem to  $V \cup W$  in place of  $V$ , blowing down curves in  $X$  until  $V \cup W$  descends to a curve linearly equivalent to  $-K_X$ . Again, this process can only be repeated finitely many times, so after contracting more curves if necessary, we can suppose with no loss of generality that an  $f$ -invariant curve  $W \not\subset V$  is actually disjoint from  $V$ . In particular  $W \cdot K_X = 0$  and  $W \cap I(f^n) = \emptyset$  for every  $n \in \mathbf{Z}$ . The latter property implies that  $W = f^*W = f_*W$  is invariant on the level of total, as well as proper, transform. Therefore,

$$\lambda\theta^+ \cdot W = f^*\theta^+ \cdot W = \theta^+ \cdot f_*W = \theta^+ \cdot W,$$

which implies that  $\theta^+ \cdot W = 0$ . Similarly,  $\theta^- \cdot W = 0$ . Since  $\theta^+ + \theta^-$  is nef and has positive self-intersection, we conclude from Theorem 2.1 that the intersection form restricted to  $W$  is negative definite. The last assertion in Theorem 4.2 now follows from the discussion of A-D-E curves in [BHPV, page 92].  $\square$

**Corollary 4.3.** *The curve  $V$  in the conclusion of the Theorem 4.2 is one of the following*

- a smooth elliptic curve;
- a rational curve with an ordinary cusp;
- a rational curve with a single normal crossing;
- a union of two smooth rational curves meeting tangentially;
- a union of two smooth rational curves meeting transversely at two distinct points.
- a union of three smooth rational curves intersecting at a single point;
- a ‘cycle’  $C_1, \dots, C_k$  of two or more smooth rational curves satisfying  $C_j C_{j+1} = 1$  for all  $j \geq 1$ ,  $C_k C_1 = 1$  and  $C_j C_k = 0$  for distinct  $j, k$  otherwise.

By paying a little more attention in the proof of Theorem 4.2 we can extract another piece of information that will prove useful below.

**Corollary 4.4.** *Suppose that the curve  $V$  in the conclusion of Theorem 4.2 contains a cusp, a triple point, or a tangency. Then  $V$  has one of these singularities even before contraction.*

*Proof.* If the corollary is false, then  $V$  develops a cusp, a triple point, or a tangency in the course of contracting a  $-1$  curve  $C$ . Suppose first that  $C \not\subset V$ . If  $C \cdot V > 1$ , then the genus of  $V$  will strictly increase when  $C$  is contracted. Since the genus of an invariant curve cannot exceed one and attains this value even before contraction, we see that in fact  $C \cdot V \leq 1$ . So either  $C$  does not intersect  $V$ , or  $C$  meets  $V$  transversely at some smooth point of  $V$ . In neither case will contracting  $V$  add to or change the singular points of  $V$ . That is, no cusps, triple points, or tangencies can be created when  $C \not\subset V$ .

Now suppose that  $C \subset V$ . Then contracting  $C$  leads to a cusp only if  $C$  is tangent to some other component of  $V$  and contracting  $C$  leads to a tangency only if  $C$  meets two other components of  $V$  at the same point. Hence, the only real concern is that contracting  $C$  might create a triple point. If that happens, then we see that  $C$  meets at least three other

components of  $V$ . Thus

$$C \cdot V = C \cdot C + C \cdot (V - C) \geq -1 + 3 = 2.$$

Once again, combining this estimate with the genus formula implies that the genus of  $V$  will strictly increase when  $C$  is contracted. Since this cannot happen, the proof is complete.  $\square$

If  $V \sim -K_X$  then there is a meromorphic two form  $\eta$  on  $X$ , unique up to constant multiple, with simple poles along  $V$  and no zeroes or poles elsewhere. If  $V$  is also  $f$ -invariant, then

$$(f^*\eta) = f^*(\eta) - E(f) = -f^*V - E(f) = -V + E(f) - E(f) = -V = (\eta).$$

Hence

$$(4) \quad f^*\eta = t\eta$$

for some  $t \in \mathbf{C}$ . The constant  $t$  can be computed by looking at  $f|_V$ , as we will now explain. We recall that the explicit version of the linear transformation  $H^0(X, K_X + V) \rightarrow H^0(V, K_V)$  is the Poincaré residue map ([BHPV], p66), which prescribes to each meromorphic two form on  $X$  with simple poles along  $V$  a meromorphic one form on  $V$ . If  $V$  has local defining function  $h$  on some open set  $U \subset X$ , then we can write

$$\eta = \frac{dh}{h} \wedge \eta' + \tau,$$

on  $U$ , where  $\eta'$  is a holomorphic 1-form and  $\tau$  is a holomorphic 2-form. It turns out that  $\text{res}(\eta) := \eta'|_V$  is independent of the choice of defining function. Since  $h^0(K_V) = 1$  and  $f|_V$  is an automorphism, it follows that

$$(5) \quad f^* \text{res}(\eta) = t \text{res}(\eta)$$

for some  $t \in \mathbf{C}$ .

**Proposition 4.5.** *The constant  $t$  is the same in (4) and (5).*

*Proof.* It's enough to observe that  $f^*$  (applied to meromorphic one and two forms) commutes with the Poincaré residue map. This is clear from the definition of the Poincaré residue, since near any point  $p \in V - I(f^{-1}) - \text{supp } E(f^{-1})$ , we have that  $h \circ f$  remains a local defining function for  $V$  near  $f^{-1}(p)$  and

$$f^*\eta = \frac{d(h \circ f)}{h \circ f} \wedge f^*\eta' + f^*\tau,$$

on  $f^{-1}(U)$ . This shows that  $\text{res}(f^*\eta) = f^* \text{res}(\eta)$  on a dense open subset of  $V$ , and by analytic continuation, the equality holds everywhere on  $V$ .  $\square$

**Corollary 4.6.** *Let  $f : X \dashrightarrow$  be an AS birational map preserving a curve  $V$  with genus one. Then there is a meromorphic two form  $\eta$  on  $X$  satisfying  $f^*\eta = t\eta$  for some  $t \in \mathbf{C}$ . Moreover,  $t$  is a root of unity unless one of the following holds:*

- some rational component of  $V$  has an ordinary cusp;
- two distinct rational components of  $V$  meet tangentially;
- three distinct rational components of  $V$  meet transversely at a single point;

*Proof.* Let  $V$  be an  $f$ -invariant curve with genus one. Let us first assume that  $V$  is as in the conclusion of Theorem 4.2. Then as described above, we may choose  $\eta$  to be a meromorphic two form with divisor  $(\eta) = -V$ , and conclude that  $f^*\eta = t\eta$ . In general, it will be necessary to contract curves in  $X$  to reach this situation. However, if  $\pi : X \rightarrow \tilde{X}$  is the map that accomplishes the contraction, and  $\tilde{\eta}$  is the invariant (up to scale) two form on  $\tilde{X}$ , then it follows that,  $\eta := \pi^*\tilde{\eta}$  is invariant on  $X$  simply because invariance is a pointwise condition for a two form and need only be verified on an open set that avoids the curves contracted by  $\pi$ .

Now for the conclusion concerning the specific value of  $t$ , Corollary 4.4 allows to assume that  $V \sim -K_X$  is one of the curves in the conclusion of Corollary 4.3. We recall some facts about meromorphic one forms in the range of the Poincaré residue operator associated to a curve  $V \sim -K_X$ . It is evident from the definition above that such a form will be holomorphic and non-zero at smooth points of  $v$ . Less obviously, a non-trivial form in the range of  $\text{res}$  will have simple poles at any normal crossing of  $V$  and double poles at other singularities. Let us consider the implications of these facts in two contrasting particular cases.

If  $V = C' \cup C''$  is a union of two smooth rational curves intersecting transversely at two distinct points, then we can choose a uniformizing parameter  $z$  for  $C'$  so that the normal crossings correspond to  $z = 0$  and  $z = \infty$ . Thus forms in the range of the residue operator will be multiples of  $\frac{dz}{z}$  in this coordinate. On the other hand, the restriction of  $f$  to  $C'$  will either preserve or switch  $z = 0$  and  $z = \infty$ ; that is  $f|_{C'} : z \mapsto az$  or  $f|_{C'} : z \mapsto a/z$ . From this, it is clear that  $(f|_{C'})^*\frac{dz}{z} = \pm\frac{dz}{z}$ . In particular,  $t = \pm 1$ .

Suppose instead that  $V = C' \cup C''$  is a union of two smooth rational curves intersecting tangentially at a single point. In this case we choose a parameter  $z$  on  $C'$  so that the intersection occurs at  $z = \infty$ . Thus forms in the range of  $\text{res}$  are multiples of  $dz$  and  $f|_{C'} : z \mapsto az + b$ . It follows that  $t = a$ . The remaining cases in Corollary 4.6 can all be analyzed in a similar fashion with the result that  $t$  is a root of unity unless  $V$  is a rational curve with a cusp, a union of two lines meeting tangentially, or a union of three lines meeting at a single point.  $\square$

**Example 4.7.** For  $a, b, c \in \mathbf{C}$ , the map  $f = f_{abc} : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$  given in homogeneous coordinates by

$$f_{abc} : [x, y, z] \mapsto [x(x + ay + z/b), y(x/a + y + cz), z(bx + y/c + z)]$$

is birational and preserves the curve  $\{xyz = 0\}$ , which is a union of three lines and has genus 1. The inverse of  $f_{abc}$  is the map  $f_{a^{-1}b^{-1}c^{-1}}$ . For generic values of  $a, b, c$ , one can check using the second item in Theorem 2.4 that  $f$  is AS. In this case  $\lambda(f) = 2$  is just the degree of the homogeneous polynomials defining  $f$ . The meromorphic two form, given in affine coordinates by  $dx \wedge dy/xy$  is invariant under  $f$ .

In his thesis [Jac], Jackson gives examples of birational maps  $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$  with genus 1 invariant curves of each of the last four types presented in Corollary 4.3. Such examples seem plentiful. However, we are not presently aware of an example of an AS birational map  $f : X \dashrightarrow X$  with  $\lambda(f) > 1$  that preserves an irreducible genus 1 curve.

5. THE NUMBER OF IRREDUCIBLE COMPONENTS OF AN INVARIANT CURVE

Throughout this section, we take  $X$  to be a rational surface and set

$$h^{1,1} := \dim H^{1,1}(X) = \dim \text{Pic}(X).$$

We take  $f : X \dashrightarrow X$  to be an AS birational map with first dynamical degree  $\lambda(f) > 1$ , and we suppose that  $C \subset X$  is a (not necessarily connected)  $f$ -invariant curve. It would be nice to have an upper bound for the number of components of  $C$  that depended only on  $X$  and not on  $f$ . Our results so far come close to giving such a bound, but in the case where each connected component of  $C$  has genus 0, we are not presently able to rule out the possibility that  $C$  contains arbitrarily many fibers in a rational fibration. Consider, for instance, the following example that shows it is possible to have at least three such fibers invariant.

**Example 5.1.** Let  $f : \mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow \mathbf{P}^1 \times \mathbf{P}^1$  be given in affine coordinates by

$$f(x, y) = \left( x \frac{xy + 3y - 4}{4xy - 6x + 2}, \frac{xy + 3y - 4}{xy^2 + 5y - 6} \right).$$

Then  $f$  is birational with  $I(f) = \{(\infty, 0), (0, \infty), (1, 1), (-1, 2)\}$ . For any choice of three points  $p_1, p_2, p_3 \in I(f)$ , there is a unique hyperbola of the form  $(x - a)(y - b) = c$  passing through  $p_1, p_2, p_3$ , and the four hyperbolas obtained this way constitute the exceptional set of  $f$ . The lines  $\{x = 0\}$ ,  $\{x = 1\}$  and  $\{x = \infty\}$  are all  $f$ -invariant. Moreover, if we take the classes of a horizontal and a vertical line as a basis for  $\text{Pic}(\mathbf{P}^1 \times \mathbf{P}^1)$ , then  $f^*$  is given by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The map  $f$  is not by itself AS : for instance,  $f(\{xy = -2\}) = (1, 1) \in I(f^{-1}) \cap I(f)$ . However, if we set  $g(x, y) = L \circ f$ , where  $L(x, y) = (x, \frac{ay+b}{cy+d})$ , then  $g$  will be AS for generic choices of  $a, b, c, d$ , and all of the other properties of  $f$  that we have just described will be retained for  $g$ . In particular,  $\lambda(g)$  will be the largest eigenvalue 3 of  $f^*$ .

Our aim in this section is to show that, once large unions of fibers in a rational fibration are excluded, we have the desired upper bound on the number of components of  $C$ . We rely on the following variation on a result [BHPV, pp. 111] of Zariski.

**Proposition 5.2.** Let  $V \subset X$  be a curve and  $H \subset \text{Pic}(X)$  be the subspace generated by divisors supported on  $V$ . Suppose that the intersection form is non-positive on  $H$ . Then there is a unique choice of  $k$  effective divisors  $D_1, \dots, D_k$  with the following properties.

- For each  $1 \leq j \leq k$ ,  $\text{supp } D_j$  is a distinct connected component of  $V$ .
- A divisor  $D$  supported on  $V$  has self-intersection 0 if and only if  $D = \sum c_j D_j$  for some  $c_j \in \mathbf{Z}$ .
- Every linear equivalence among divisors supported on  $V$  has the form  $\sum_{j=1}^k c_j D_j \sim 0$  for some  $c_j \in \mathbf{Z}$ .
- If  $k \leq 1$ , then  $V$  has at most  $h^{1,1} - 1$  irreducible components.
- If  $k \geq 2$ , then there is a fibration  $\pi : X \rightarrow \mathbf{P}^1$  such that each  $D_j$  is a (possibly non-generic) fiber of  $\pi$ . Moreover,  $V$  has at most  $h^{1,1} + k - 2$  irreducible components.

*Proof.* Suppose first that  $V$  is connected and that there is a non-trivial divisor  $D$  with  $\text{supp } D \subset V$  and  $D^2 = 0$ . Suppose that  $E \subset V$  is an irreducible component of  $V - \text{supp } D$  such that  $E \cap \text{supp } D \neq \emptyset$ . Replacing  $D$  by  $-D$  if necessary, we may assume that  $D \cdot E > 0$ . Thus

$$(D + nE)^2 = 2nE \cdot D + E^2 > 0$$

for  $n$  large enough, contradicting our assumption that the intersection form is non-positive for divisors supported on  $V$ . It follows that  $\text{supp } D = V$ . As in the proof of Theorem 2.1, we may assume that  $D$  is effective.

If  $D'$  is some other non-trivial effective divisor supported on  $V$  with  $D'^2 = 0$ , then we may choose  $a, b \in \mathbf{N}$  so that  $aD - bD'$  vanishes on some irreducible component of  $V$ . However, our hypothesis on  $V$  and Theorem 2.1 imply that

$$0 \geq (aD' - bD)^2 = 2abD' \cdot D \geq 0.$$

That is,  $(aD - bD')^2 = 0$ . If  $aD - bD'$  were non-trivial, then the argument in the first paragraph would force  $\text{supp } aD - bD' = V$ . By construction, this is not the case, so we conclude  $aD = bD'$ . Furthermore by minimizing  $b/a$ , we arrive at an effective divisor  $D$  supported on  $V$  such that any other divisor  $D'$  supported on  $V$  with  $D'^2 = 0$  is an integer multiple of  $D$ .

Now let us drop the assumption that  $V$  is connected. In this case, our arguments so far yield divisors  $D_1, \dots, D_k$  satisfying the first two conclusions of the proposition. Any linear equivalence involving only divisors supported on  $V$  may be written

$$(6) \quad D_0^+ + D_1^+ + \dots + D_k^+ = D_0^- + D_1^- + \dots + D_k^-$$

where  $D_j^\pm$  are effective divisors with no irreducible components in common and satisfying

- $\text{supp } D_j^\pm \subset \text{supp } D_j$  for  $j = 1, \dots, k$ ; and
- $\text{supp } D_{0,+} \cap \text{supp } D_j = \emptyset$ ,

for  $j = 1, \dots, k$ . If  $D_0^+ \neq 0$ , then intersecting both sides of (6) with  $D_0^+$  and applying our hypothesis on  $V$  implies that

$$0 > D_0^{+2} = D_0^+ \cdot D_0^- \geq 0.$$

Hence  $D_0^+ = 0$ , and similarly  $D_0^- = 0$ . The same argument shows for  $j \geq 1$ , that  $D_j^\pm$  is a multiple of  $D_j$ . Since the divisors  $D_j$  are minimal among effective divisors with self-intersection 0, the multiples are integers. This establishes the third conclusion of the proposition.

To obtain the desired upper bounds on the number of irreducible components of  $V$ , we observe that  $H \neq \text{Pic}(X)$  because  $X$  is rational and must contain curves with positive self-intersection. Hence  $\dim H \leq h^{1,1} - 1$ . The third conclusion of the proposition implies that there are no more than  $\max\{0, k - 1\}$  independent equivalences among the irreducible components of  $V$ . Hence  $V$  has at most  $h^{1,1} - 1 + \max\{0, k - 1\}$  irreducible components.

Finally, if  $k \geq 2$ , then Theorem 2.1 tells us that  $aD_1 \sim bD_2$  for some  $a, b \in \mathbf{N}$ . Since  $\text{supp } D_1 \cap \text{supp } D_2 = \emptyset$ , we can choose a surjective holomorphic function  $\pi : X \rightarrow \mathbf{P}^1$  with divisor  $(\pi) = aD_1 - bD_2$ . By Stein Factorization [BHPV, pp. 32], we can assume that  $\pi$  has connected fibers. In particular, each of the other divisors  $D_j$  must have support equal to a fiber of  $\pi$ , and since we have chosen  $D_j$  to be minimal, we conclude for each  $j$  that some integer multiple of  $D_j$  is linearly equivalent to the generic fiber of  $\pi$ .  $\square$



**Theorem 5.3.** *Suppose that every connected component of  $C$  has genus 0. Then either*

- $C$  has at most  $h^{1,1} + 1$  connected components; or
- there is a holomorphic map  $\pi : X \rightarrow \mathbf{P}^1$ , unique up to automorphisms of  $\mathbf{P}^1$ , such that  $C$  contains exactly  $k \geq 2$  distinct fibers of  $\pi$ , and  $C$  has at most  $h^{1,1} + k - 1$  irreducible components.

Actually, we know of no examples of a birationally invariant curve  $C$  equal to a disjoint union of connected genus 0 curves and comprising more than  $h^{1,1} + 1$  irreducible components.

*Proof.* If  $C$  supports no divisors with positive self-intersection, then the corollary follows immediately from Proposition 5.2. So we assume with no loss of generality that  $C$  supports a divisor  $D$  with positive self-intersection. We may assume that  $\text{supp } D$  is connected, since  $(D_1 + D_2)^2 = D_1^2 + D_2^2$  for divisors with disjoint supports. Hence there is a connected component  $C_0 \subset C$  containing  $\text{supp } D$ , and by Theorem 2.1, the intersection form is negative definite on  $C - C_0$ .

Let  $E_0 \subset \text{supp } D$  be any irreducible component. If  $C_0 - E_0$  supports no divisors with positive self-intersection, then once again the theorem follows from Proposition 5.2. Otherwise, we apply the argument from the first paragraph and obtain a unique connected component  $C_1$  of  $C_0 - E_0$  that supports a divisor with positive self-intersection. Moreover, by Theorem 4.1, the curve  $C_0$  is a tree of smooth rational curves, and there is a unique irreducible component  $E_1$  of  $C_1$  that meets  $E_0$ . We then (uniquely) continue this process, obtaining two finite sequences of curves  $E_j, C_j \subset C$ ,  $0 \leq j \leq n$ , subject to the following conditions.

- $C_j$  supports a divisor with positive self-intersection;
- $C_{j+1}$  is a connected subtree of  $C_j$ .
- $E_j$  is the unique irreducible component of  $C_j$  that meets  $C - C_j$ .

The terminal index  $n$  is the first for which the intersection form  $C_n - E_n$  supports no divisors with positive self-intersection. That is, the intersection form is non-positive on  $C_n - E_n$ .

There are now two possibilities to consider. The first is that the intersection form is negative definite for divisors supported on  $C_n - E_n$ . Because  $C_n$  supports a divisor with positive self-intersection and  $C_n \cap (C - C_n - E_{n-1}) = \emptyset$ , the intersection form is also negative definite for divisors supported on  $C - C_n - E_{n-1}$ . All told, the intersection form is negative for divisors supported anywhere on  $C - E_n - E_{n-1}$ , and the theorem follows from Proposition 5.2.

The other possibility is that  $C_n - E_n$  supports divisors with zero self-intersection. In this case, the intersection form will be non-positive on  $C_n - E_n$ . Let  $H, H' \subset \text{Pic}(X)$  be the subspaces spanned by divisors supported on  $C - E_n$  and  $C$ , respectively. Then  $\dim H < \dim H'$ , because  $H'$  contains elements with positive self-intersection, whereas  $H$  does not. Hence the theorem follows again from the bound on  $\dim H$  given in Proposition 5.2. □

**Theorem 5.4.** *If the invariant curve  $C$  contains a curve of genus 1, then the number of irreducible components of  $C$  is no larger than  $h_{1,1} + 2$ .*

*Proof.* By Theorem 1.1 no connected component of  $C$  has genus larger than 1; and (with our notion of genus) the genus of a sum of two disjoint curves is less than the sum of

their individual genera. It follows that if  $C$  contains a curve of genus 1, then that curve is connected.

Contracting a  $-1$  curve reduces  $h_{1,1}$  by one, while it reduces the number of connected components of  $C$  by at most one. Thus we may apply Theorem 4.2 to conclude that  $C = C' + C''$  is a disjoint union of a connected genus one curve  $C'$  as in the conclusion of Corollary 4.3, and a curve  $C''$  that supports only divisors with negative self-intersection.

Going through the list of possibilities in Corollary 4.3, we see that removing an irreducible component  $E_0$  of  $C'$  leaves us with a (possibly empty) chain  $C_0$  of smooth rational curves. This allows us to employ the argument from Theorem 5.3 and find an irreducible component  $E_1 \subset C_0$  such that the intersection form is non-positive on  $C_0 - E_1$  and therefore, more generally, on  $C - E_0 - E_1$ . Moreover, the only connected components of  $C - E_0 - E_1$  that can support non-trivial divisors with zero self-intersection are the (at most) two components of  $C_0 - E_1$ . By the first and second conclusions of Proposition 5.2, it follows that there are at most two independent divisors with zero self-intersection. Hence  $C - E_0 - E_1$  contains at most  $h^{1,1}$  irreducible components.  $\square$

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