Kodaira Dimension of Subvarieties

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Introduction

In this article we study how the birational geometry of a normal projective variety X is influenced by a normal subvariety $A \subset X$. One of the most basic examples in this context is provided by the following situation. Let $f : X \to Y$ be a surjective holomorphic map with connected fibers between compact connected complex manifolds. It is well known (see, e.g., [7]) that given a general fiber A of f we have

$$\kappa(X) \leq \kappa(A) + \dim Y.$$

This article grew out of the realization that this result should be true with $\dim Y$ replaced by the codimension $\mathrm{cod}_X A$ for a pair (X, A) consisting of a normal subvariety A of a compact normal variety X under weak semipositivity conditions on the normal sheaf of A and the weak singularity condition $\mathrm{cod}_A(A \cap \mathrm{Sing}\, X) \geq 2$. We shall now state our main results in the special case of a submanifold A in a projective manifold X and we also simplify the semipositivity notion.

Theorem 0.1 *Let X be a projective manifold and A a compact submanifold. Then*

$$\kappa(X) \leq \kappa(A) + \mathrm{cod}_X A$$

if one of the following conditions is satisfied

1. *some symmetric power $S^m \mathcal{N}_{A|X}$ of the normal bundle has global sections which generate the bundle almost everywhere;*

2. *$\mathcal{N}_{A|X}$ is nef and A has a good minimal model;*

3. *$\mathcal{N}_{A|X}$ is nef and* $\dim A \leq 3$;

4. $\mathrm{cod}_X A = 1$, *$\mathcal{N}_{A|X}$ is in the closure of the effective cone of A and A has a good minimal model.*

Recall that a normal projective variety Z is a good minimal model if Z is $\mathbb{Q}-$factorial with at most terminal singularities and some multiple mK_Z is generated by global sections.

The above results actually hold more generally, e.g., with A a normal projective subvariety of a normal projective variety X such that $\mathrm{cod}_A(A \cap \mathrm{Sing}\, X) \geq 2$. In this case it is necessary to use the arithmetic Kodaira dimension of X and A. Moreover the effectivity and nefness assumptions can be weakened in the following way; we assume again X and A smooth and $\mathcal{N}_{A|X}$ is a "generically nef" vector bundle. Generically nef means the following: there is a Zariski open set $U \subset X$ with $\mathrm{cod}_X(X \setminus U) \geq 2$, such that $B|U$ is nef, i.e., $B|C$ is nef for every compact curve $C \subset B$.

For technical reasons we have formulated most of the paper in a non-compact setting, namely for *small normal pairs* (X, A). Here X is a normal variety and $A \subset X$ a normal subvariety such that $A \cap \mathrm{Sing}\,(X)$ has codimension at least 2 in A. We require the existence of normal compactifications $\overline{X}$ and $\overline{A}$ (not necessarily $\subset \overline{X}$) such that the boundary components have codimension at least 2. The reason for using this category is that at some point we have to perform birational maps to minimal models and then this language seems appropriate.

We would like to thank the referee for helpful suggestions. In particular, the referee's observation that "generically spanned" implies "generically nef" let us streamline some of our original arguments.

This paper being almost completed, Michael Schneider died in a tragic accident. The scientific community lost a very active mathematician; we lost also a very good friend. We dedicate this work to his memory.

1 Preliminaries

For most purposes of this article singular varieties are not much harder to deal with than smooth varieties, except that some care must be taken with definitions. A normal variety is a connected normal quasi-projective scheme over the complex numbers (or a connected normal quasi-projective complex space).

Let $\mathcal{F}$ be a coherent sheaf on a complex algebraic variety X. Coherent sheaves are of course always understood to be algebraic. By $\widehat{\mathrm{S}}^t\mathcal{F}$ we denote $(\mathrm{S}^t\mathcal{F})^{**}$, the double dual of the t-th symmetric power, $\mathrm{S}^t\mathcal{F}$, of $\mathcal{F}$. When $\mathcal{L}$ is of rank one reflexive sheaf on a normal variety X, we often, for $t \geq 1$, denote $\widehat{\mathrm{S}}^t\mathcal{L}$ by $t\mathcal{L}$.

If A and B are reflexive sheaves, we define

$$A \hat{\otimes} B = (A \otimes B)^{**}.$$

The following is left to the reader.

Lemma 1.1 *Let $\mathcal{F}$ denote a reflexive sheaf on a normal variety V. Assume that there is an embedding $i : V \to \overline{V}$ of V as a Zariski open set in a compact normal variety $\overline{V}$. Then $\mathcal{F}$ extends to a reflexive sheaf $\overline{\mathcal{F}}$ on $\overline{V}$. If $\mathrm{cod}_{\overline{V}}\overline{V} \setminus V \geq 2$ then*

$\widehat{\mathrm{S}}^t\mathcal{F}$ extends to $\widehat{\mathrm{S}}^t\overline{\mathcal{F}}$ for all $t \geq 0$, and moreover $h^0(\widehat{\mathrm{S}}^t\mathcal{F}) = h^0(\widehat{\mathrm{S}}^t\overline{\mathcal{F}})$. Furthermore $i_*\widehat{\mathrm{S}}^t\mathcal{F} \cong \widehat{\mathrm{S}}^t\overline{\mathcal{F}}$.

Given a rank one reflexive sheaf $\mathcal{L}$ on a normal variety V the *Kodaira dimension, $\kappa(\mathcal{L})$ of* $\mathcal{L}$, is defined as follows

1. $\kappa(\mathcal{L}) = -\infty$, if $h^0(\widehat{\mathrm{S}}^t\mathcal{L}) = 0$ for all $t \geq 1$;

2. $\kappa(\mathcal{L}) = 0$, if $h^0(\widehat{\mathrm{S}}^{t_0}\mathcal{L}) = 1$ for some $t_0 \geq 1$ and $h^0(\widehat{\mathrm{S}}^t\mathcal{L}) \leq 1$ for all $t \geq 1$;

3. $\kappa(\mathcal{L})$ is a positive integer k, if $\overline{\lim}_{\{t\geq 1\}} \dfrac{h^0(\widehat{\mathrm{S}}^t\mathcal{L})}{t^k}$ is a positive real number, where $\overline{\lim}$ denotes the limsup;

4. $\kappa(\mathcal{L}) = \infty$ otherwise.

Observe that $\kappa(\mathcal{L})$ is always an integer. In fact, in case (3) $\kappa(\mathcal{L})$ is just the maximal dimension of the image of the maps associated to $H^0(N\mathcal{L})$ where N is any positive integer.

We have the following corollary of the above lemma.

Lemma 1.2 *Let $\mathcal{L}$ denote a reflexive rank one sheaf on a normal variety V. Assume that there is an embedding $i : V \to \overline{V}$ of V as a Zariski open set in a compact normal variety $\overline{V}$. Let $\mathcal{L}$ extend to the reflexive sheaf $\overline{\mathcal{L}}$ on $\overline{V}$. If $\mathrm{cod}_{\overline{V}}(\overline{V} \setminus V) \geq 2$ then for any $t \geq 1$*

$$\kappa(\mathcal{L}) = \kappa(t\mathcal{L}) = \kappa(t\overline{\mathcal{L}}) = \kappa(\overline{\mathcal{L}}) < \infty$$

By a *small normal pair* (X, A) we mean a pair (X, A) where:

1. A is a normal subvariety of a normal variety X; and

2. there is an embedding $A \subset \overline{A}$ of A as a Zariski open set in a normal compact projective variety $\overline{A}$ in such a way that $\mathrm{cod}_{\overline{A}}(\overline{A} \setminus A) \geq 2$; and

3. there is an embedding $X \subset \overline{X}$ of X as a Zariski open set in a normal compact projective variety $\overline{X}$ in such a way that $\mathrm{cod}_{\overline{X}}(\overline{X} \setminus X) \geq 2$; and

4. $\mathrm{cod}_A(A \cap \mathrm{Sing}\, X) \geq 2$.

Note we do not require that $\overline{A}$ equals the closure of A in $\overline{X}$. The main example will be given by a normal projective variety X and a normal projective subvariety A such that $\mathrm{cod}_A(A \cap \mathrm{Sing}\, X) \geq 2$.

The condition that $\overline{X}$ is projective is needed for the basic inequality Theorem 2.1. The condition that $\overline{A}$ is projective is needed for Theorem 3.4. The codimension two conditions are needed for finiteness results. Indeed it is very easy to see that without some such conditions the results are at best meaningless, e.g., let A be a smooth compact curve and let $X := A \times (C \setminus \{x\})$ for some point x on a smooth compact curve C.

Some further notations:

1. Given a small normal pair (X, A), we say that *a reflexive sheaf $\mathcal{F}$ on X is adapted to* (X, A), if $A \not\subset \operatorname{Sing}\mathcal{F}$.

2. A coherent sheaf $\mathcal{F}$ is nef on a normal variety V if $\mathcal{F}$ is nef on every compact irreducible curve $C \subset V$, i.e. $\mathcal{O}_{\mathbb{P}(\mathcal{F}|C)}(1)$ is nef on $\mathbb{P}(\mathcal{F}|C)$.

3. For a coherent sheaf $\mathcal{F}$ we say that $\mathcal{F}$ has rank r if there is a Zariski open and dense set on which it is a locally free sheaf of rank r. In this case we define we define the determinant of $\mathcal{F}$ by $\det(\mathcal{F}) = (\bigwedge^r \mathcal{F})^{**}$.

2 The basic inequality

The following is the basic inequality underlying this paper.

Theorem 2.1 *Let (X, A) denote a small normal pair and let $\mathcal{L}$ be a reflexive rank one coherent sheaf on X adapted to (X, A). Then there is a positive integer c such that for all $t \geq 0$*

$$h^0(t\mathcal{L}) \leq \sum_{k=0}^{ct} h^0(\widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X} \hat{\otimes} t\mathcal{L}_A).$$

In particular, if we have

$$h^0\left(\widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X} \hat{\otimes} t\mathcal{L}_A\right) \leq C\left(\operatorname{rank} \mathrm{S}^k \mathcal{N}^*_{A|X}\right) t^a$$

for some positive constants C, a that do not depend on t, k, then we have

$$\kappa(\mathcal{L}) \leq \operatorname{cod}_X A + a.$$

Proof. Let $\mathcal{J}_k = \mathcal{I}_A^k$. The essential point is to show that there is a positive integer c such that $h^0(t\mathcal{L} \otimes \mathcal{J}_k) = 0$ for $k > ct$. Since there are projective varieties $\overline{A}, \overline{X}$ such that $\operatorname{cod}_{\overline{X}}(\overline{X} \setminus X) \geq 2$ and $\operatorname{cod}_{\overline{A}}(\overline{A} \setminus A) \geq 2$, it follows that we can assume without loss of generality that A and X are compact.

If A is a divisor then choose a very ample curve $C \subset X$, i.e., the intersection of $\dim X - 1$ very ample divisors on X. By choosing the very ample divisors generically we can assume that

1. C is a smooth connected curve lying in $X \setminus \operatorname{Sing} X$ and meeting A transversely in points lying in $A \setminus \operatorname{Sing} A$; and

2. the restriction $\mathcal{L}_C$ is invertible.

Since

$$\deg(t\mathcal{L} \otimes \mathcal{J}_k)_C = t \deg \mathcal{L}_C - kA \cdot C,$$

we see that $h^0((t\mathcal{L} \otimes \mathcal{J}_k)_C) = 0$ for $k > \left(\dfrac{\deg \mathcal{L}_C}{A \cdot C}\right) t$. Since C is very ample this implies that $h^0(t\mathcal{L} \otimes \mathcal{J}_k) = 0$ for $k > \left(\dfrac{\deg \mathcal{L}_C}{A \cdot C}\right) t$. So let $c := k > \left(\dfrac{\deg \mathcal{L}_C}{A \cdot C}\right) t$ and our inequality follows by power series expansion.

If A is not a divisor we proceed as follows. Again it is sufficient to show

$$H^0(C, \widehat{\mathrm{S}}^l \mathcal{N}^*_{A|X} \otimes t\mathcal{L}_C) = 0 \tag{$*$}$$

for $k > ct$, with c not depending on the individual curve C. To verify (*) we choose C again general, blow up X in a neighborhood of C where X and A are smooth and apply the old argument. Q.E.D.

3 $\mathbb{Q}$-effective and generically nef sheaves

We need the rank > 1 version of a $\mathbb{Q}$-effective divisor, i.e., of a $\mathbb{Q}$-effective rank one coherent sheaf. A coherent sheaf $\mathcal{F}$ on a normal variety V is said to be *generically spanned* if the global sections $\Gamma(\mathcal{F})$ span $\mathcal{F}$ over a dense Zariski open set of V. We say that a coherent sheaf $\mathcal{F}$ on a normal variety V is $\mathbb{Q}$-*effective* if there is a positive integer $N > 0$ such that $\widehat{\mathrm{S}}^N \mathcal{F}$ is generically spanned. Note that for a line bundle, $\mathbb{Q}$-effective agrees with the usual notion.

A coherent sheaf $\mathcal{F}$ on a normal variety V is said to be *generically nef* if there is a dense Zariski open set $U \subset V$, with $\mathrm{cod}_V(V \setminus U) \geq 2$, such that $\mathcal{F}_U$ is locally free and nef. To say that a locally free sheaf $\mathcal{F}_U$ is nef means that $\mathcal{F}|C$ is nef for every compact curve $C \subset U$.

We thank the referee for pointing out the following useful fact.

Lemma 3.1 *If $\mathcal{F}$ is a $\mathbb{Q}$-effective coherent sheaf on a normal variety V, then $\mathcal{F}$ is generically nef.*

Proof. Since the complement of the Zariski open dense set V' where $\mathcal{F}$ is not locally free is of codimension ≥ 2, we can, by restricting $\mathcal{F}$ to V', assume without loss of generality that $\mathcal{F}$ is locally free on V.

By assumption there is a Zariski open dense set U' such that $\widehat{\mathrm{S}}^N \mathcal{F}_{U'}$ is spanned by global sections. Let A be an effective ample divisor of $V \setminus U'$ which does not contain any component of $V \setminus U'$. Then $\mathrm{cod}_V A \geq 2$. We claim that $\mathcal{F}_{V \setminus A}$ is nef. To see this let $C \subset V \setminus A$ be a compact curve. Note that since A is ample on $V \setminus U'$, it follows that C cannot be be contained in $V \setminus U'$. Thus $C \cap U' \neq \emptyset$ and $\widehat{\mathrm{S}}^N \mathcal{F}_{C \cap U'}$ is spanned by global sections of $\widehat{\mathrm{S}}^N \mathcal{F}_C$. This easily implies that $\mathcal{F}_C$ is nef. Thus $\mathcal{F}$ is generically nef with $U := V \setminus A$. Q.E.D.

Generic nefness should be a notion generalizing the property of a line bundle to be in the closure of the effective cone. However it is not clear whether for line bundles the two notions coincide. Line bundles which are in the closure of the effective cone are studied in [1], where they are called pseudo-effective.

The proofs of the following lemmas are immediate.

Lemma 3.2 *Let $\mathcal{F}$ be a reflexive sheaf on a normal variety V. Then the following are equivalent:*

1. *$\mathcal{F}$ is $\mathbb{Q}$-effective (respectively generically nef);*

2. $\mathcal{F}_U$ *is* $\mathbb{Q}$*-effective (respectively generically nef) for every dense Zariski open set* $U \subset V$ *with* $\operatorname{cod}_V(V \setminus U) \geq 2$*;*

3. $\mathcal{F}_U$ *is* $\mathbb{Q}$*-effective (respectively generically nef) for some dense Zariski open set* $U \subset V$ *with* $\operatorname{cod}_V(V \setminus U) \geq 2$*.*

Lemma 3.3 *Let* $\mathcal{F}$ *be a* $\mathbb{Q}$*-effective (respectively generically nef) coherent sheaf on a normal variety* V*. Then any coherent quotient sheaf* $\mathcal{T}$ *of* $\mathcal{F}$ *on* V *is also* $\mathbb{Q}$*-effective (respectively generically nef). Further* $\widehat{\mathrm{S}}^b\mathcal{F}$ *is* $\mathbb{Q}$*-effective (respectively generically nef) for all* $b > 0$*.*

Theorem 3.4 *Let* $\mathcal{F}$ *be a torsion free coherent sheaf on a normal variety* V *and assume that* $\mathcal{F}^*$ *is generically nef. Assume that* V *is a Zariski open subset of a projective variety* $\overline{V}$ *such that* $\operatorname{cod}_{\overline{V}}(\overline{V} \setminus V) \geq 2$*, e.g.,* V *is projective. If* $s \in \Gamma(\mathcal{F})$ *is not identically zero, then* s *is nowhere vanishing on any smooth Zariski open set* $U \subset V$ *with* $\operatorname{cod}_V V \setminus U \geq 2$*,* $\mathcal{F}_U$ *locally free, and* $\mathcal{F}^*_U$ *nef. In particular,* $h^0(\mathcal{F}) \leq \operatorname{rank}\mathcal{F}$*.*

Proof. By replacing V by a smooth Zariski open set $U \subset V$ with $\operatorname{cod}_V V \setminus U \geq 2$, $\mathcal{F}$ is locally free, and $\mathcal{F}^*$ nef, we can assume without loss of generality that V is smooth, $\mathcal{F}$ is locally free, $\mathcal{F}^*$ is nef, and that V is a Zariski open subset of a projective variety $\overline{V}$ such that $\operatorname{cod}_{\overline{V}}(\overline{V} \setminus V) \geq 2$.

We must show that s does not vanish on V.

By assumption s gives rise to a non-zero map $\mathcal{F}^* \to \mathcal{O}_V$. Assume that there was a point $x \in V$ with $s(x) = 0$. Since V is a Zariski open subset of a projective variety $\overline{V}$ such that $\operatorname{cod}_{\overline{V}}(\overline{V} \setminus V) \geq 2$, we can choose a smooth irreducible projective curve $C \subset V$ with $x \in C$, with s_C a not identically vanishing section of $\mathcal{F}_C$ that vanishes at x. If $\mathcal{F}^* \to \mathcal{O}_C \to 0$, then we are done. If this is not true then we have $\mathcal{F}^* \to L^* \to 0$ for an ample line bundle L on C. Since a quotient of a nef bundle has to be nef, we obtain a contradiction. Q.E.D.

The above result immediately gives a number of strong consequences.

Theorem 3.5 *Let* (X, A) *be a small normal pair. Assume that the normal sheaf* $\mathcal{N}_{A|X}$ *is generically nef. Assume that* $\mathcal{L}$ *is a rank one reflexive sheaf on* X *adapted to* (X, A)*. If* $\kappa(\mathcal{L}^*_A) > 0$ *then* $\kappa(\mathcal{L}) = -\infty$*.*

Proof. By Theorem 2.1 it suffices to show that

$$h^0(A, t\mathcal{L}_A \hat{\otimes} \widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X}) = 0$$

for $t > 0$ and $k < ct$. Assume that we have a nonzero section

$$\sigma \in H^0(A, t\mathcal{L}_A \hat{\otimes} \widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X}).$$

By assumption, for some $m > 0$ we have a non trivial section $\tau \in H^0(A, -m\mathcal{L}_A)$ with zeros. This leads to an inclusion $mt\mathcal{L}_A \to \mathcal{O}_A$ with zeros and altogether we obtain a nontrivial section with a divisor of zeros in $H^0(A, \widehat{\mathrm{S}}^{km}\mathcal{N}^*_{A|X})$. This contradicts Theorem 3.4. Q.E.D.

Corollary 3.6 *Let X be a normal projective variety and let $A \subset X$ be a smooth projective curve with generically nef normal bundle, e.g., the normal sheaf $\mathcal{N}_{A|X}$ is nef or generically spanned. Assume $A \cap \operatorname{Sing} X = \emptyset$. If $\kappa(X) = \kappa(K_X) \geq 0$ then $K_X \cdot A \geq 0$.*

Another variant of this is the following result.

Theorem 3.7 *Let X be a normal projective variety and let $A \subset X$ be a smooth projective curve with generically nef normal bundle, e.g., the normal sheaf $\mathcal{N}_{A|X}$ is nef or generically spanned. Assume $A \cap \operatorname{Sing} X = \emptyset$. If X is of general type then $K_X \cdot A > 0$.*

Proof. By the previous corollary we only need to exclude the case $K_X \cdot A = 0$. Assume that $K_X \cdot A = 0$. Note that given a torsion free sheaf on A whose dual is generically nef, we have

$$h^0(A, \mathcal{F}) \leq \operatorname{rank} \mathcal{F}$$

by the last conclusion of Theorem 3.4. This will be applied to $(tK_X|A) \otimes \widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X}$ in the standard estimate

$$h^0(X, tK_X) \leq \sum_{k=0}^{ct} h^0((tK_X|A) \otimes \widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X}) \leq \sum_{k=0}^{ct} \operatorname{rank} (\widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X}).$$

For large t this grows like t^{n-1}, $n = \dim X$. Hence $\kappa(X) \leq n-1$, contradicting our hypothesis. Q.E.D.

We now prove three lemmas which will be important for the proof of the main results in the next section.

Lemma 3.8 *Let $\mathcal{L}$ be a reflexive $\mathbb{Q}$-Cartier rank one coherent sheaf on a normal variety V. Assume that V is a Zariski open subset of a projective variety $\overline{V}$ such that $\operatorname{cod}_{\overline{V}}(\overline{V} \setminus V) \geq 2$. If $\mathcal{L}$ is semiample and $\mathcal{F}^*$ is a generically nef sheaf on V, then we have for $t \geq 0$*

$$h^0 \left(\mathcal{F} \hat{\otimes} t\mathcal{L} \right) \leq C \, (\operatorname{rank} \mathcal{F}) \, t^{\kappa(\mathcal{L})}$$

where C is a positive constant that depends only on $(V, \mathcal{L})$ and neither on $\mathcal{F}$ nor on t.

Proof. It is enough to show this for all positive multiples t of a positive integer t_0. Therefore, by passing to a suitable multiple of $\mathcal{L}$ if necessary, we may assume that $\mathcal{L}$ is locally free, that $\mathcal{L}$ is already spanned and moreover that V is compact.

Let $f : V \to W$ be the Stein factorization of the morphism associated to the linear system $|\mathcal{L}|$. Then $\mathcal{L} = f^*(\mathcal{L}')$; we may assume that $\mathcal{L}'$ is very ample. We shall proceed by induction on $d = \dim W$.

The case $d = 0$ is obvious because then $\mathcal{L} = \mathcal{O}_V$.

So suppose $\dim W = d > 0$. We fix a smooth member $H \in |2\mathcal{L}'|$. Let $V_t = f^{-1}(tH)$. Then we have an exact sequence for $t > 0$

$$0 \to H^0(V, \mathcal{L}^t \otimes \mathcal{I}_{V_t} \otimes \mathcal{F}) \to H^0(V, \mathcal{L}^t \otimes \mathcal{F}) \to H^0(V_t, \mathcal{L}^t \otimes \mathcal{F}|V_t).$$

Now our claim will be a consequence of the two following assertions

1. $H^0(V, \mathcal{L}^t \otimes \mathcal{I}_{V_t} \otimes \mathcal{F}) = 0$ and
2. $h^0(V_t, \mathcal{L}^t \otimes \mathcal{F}|V_t) \leq C(\operatorname{rank} \mathcal{F})\ t^{\kappa(\mathcal{L})}$.

For the proof of (1) notice first

$$\mathcal{L}^t \otimes \mathcal{I}_{V_t} \otimes \mathcal{F} = f^*(\mathcal{L}'^{(-t)}) \otimes \mathcal{F}. \qquad (*)$$

Now suppose that s is a non zero section of $\mathcal{L}^t \otimes \mathcal{I}_{V_t} \otimes \mathcal{F}$. Let $U \subset V$ be a smooth Zariski open set with $\mathrm{cod}_V V \setminus U \geq 2$, $\mathcal{F}$ locally free, and $\mathcal{F}^*$ nef. Let $C \subset U$ be a general compact irreducible curve. Take a non-zero section $s' \in H^0(C, f^*(\mathcal{L}'^t|f(C)))$. Since $\mathcal{L}'$ is ample, s' has zeros. Then, using (*), $s \otimes s' \in H^0(C, \mathcal{F}|C))$ is a section with zeros contradicting Theorem (3.4).

As to the proof of (2) we use the fact that V_t is the t-th infinitesimal neighborhood of $V_1 = V_H = f^{-1}(H)$. Hence

$$h^0(V_{tH}, \mathcal{L}^t \otimes \mathcal{F}|V_{tH}) \leq \sum_{\mu=0}^{t-1} h^0(V_H, \mathcal{L}^{t-2\mu} \otimes \mathcal{F}).$$

By induction there is a constant independent of t and $\mathcal{F}$ such that

$$h^0(V_H, \mathcal{L}^{t-2\mu} \otimes \mathcal{F}) \leq C(\operatorname{rank} \mathcal{F})(t - 2\mu)^{\kappa(\mathcal{L}|V_h)}$$

if $\dim W \geq 2$ and

$$h^0(V_H, \mathcal{L}^{t-2\mu} \otimes \mathcal{F}) \leq C(\operatorname{rank} \mathcal{F})\ \deg\mathcal{L}'$$

if $\dim W = 1$. In this last case we substitute C by $C\deg\mathcal{L}'$. Adding up and observing $\kappa(\mathcal{L}_H) = \kappa(\mathcal{L}) - 1$, we obtain

$$h^0(V_{tH}, \mathcal{L}^t \otimes \mathcal{F}|V_{tH}) \leq C(\operatorname{rank} \mathcal{F})\ t^{\kappa(\mathcal{L})}.$$

Q.E.D.

Actually one can prove the last lemma for every reflexive sheaf of rank one but we do not need this. It seems however to be an interesting problem whether Lemma (3.8) holds under more general assumptions, e.g. when $\mathcal{L}$ is nef.

Next we investigate the behavior of generically nef sheaves under birational maps which are isomorphisms outside sets of codimension at least 1 resp. 2. We fix the following situation. Let V and V' be normal varieties, $Y \subset V$ and $Y' \subset V'$ algebraic subsets of codimension at least 1 resp. 2. Let $\varphi : V \rightharpoonup V'$ be birational such that $\varphi : V \setminus Y \to V' \setminus Y'$ is an isomorphism. So φ^{-1} does not contract any divisor. Let $\psi : W \to V$ be birational such that the induced map $\tau : W \to V'$ is a morphism. If $\mathcal{F}$ is a torsion free sheaf on V, we define

$$\varphi_*(\mathcal{F}) = \tau_*(\psi^*(\mathcal{F}))^{**}.$$

In other words, $\varphi_*(\mathcal{F})$ is the "canonical" extension of $\varphi_*(\mathcal{F}|V \setminus Y)$. In this context we can state

Lemma 3.9 *If $\mathcal{F}^*$ is generically nef, then $(\varphi_*\mathcal{F})^*$ is generically nef.*

Proof. Let $U \subset V$ be a smooth Zariski open set with $\mathrm{cod}_V V \setminus U \geq 2$, and $\mathcal{F}$ locally free, and $\mathcal{F}^*$ nef. Since $\mathrm{cod}_{V'}\varphi(V \setminus U) \geq 2$, we see that we can choose a smooth Zariski open set $U' \subset V' \setminus \varphi(V \setminus U)$ with $\mathrm{cod}_{V'} V' \setminus U' \geq 2$ and $(\varphi_*\mathcal{F})^*_{U'} \cong \mathcal{F}^*_{\varphi^{-1}(U')}$ locally free and nef. Q.E.D.

Concerning global sections we have in the same situation with reflexive sheaves $\mathcal{L}$ and $\mathcal{F}$ on V :

Lemma 3.10

$$h^0(V, t\mathcal{L}\hat{\otimes}\mathcal{F}) \leq h^0(V', \varphi_*(t\mathcal{L})\hat{\otimes}\varphi_*(\mathcal{F})).$$

Proof. This is clear since

$$H^0(V, t\mathcal{L}\hat{\otimes}\mathcal{F}) = H^0(V' \setminus Y', \varphi_*(t\mathcal{L}\hat{\otimes}\mathcal{F}))$$

and every section in

$$H^0(V' \setminus Y', \varphi_*(t\mathcal{L}\hat{\otimes}\mathcal{F}))$$

extends to

$$H^0(V', \varphi_*(t\mathcal{L}\hat{\otimes}\mathcal{F})^{**}) = H^0(V', \varphi_*(t\mathcal{L})\hat{\otimes}\varphi_*(\mathcal{F})).$$

Q.E.D.

4 Main Results

If X is a normal variety we again let $\kappa(X)$ denote the arithmetic Kodaira dimension of X, i.e., $\kappa(K_X)$. Note that $\kappa(X) = \kappa(U)$ for any Zariski open set $U \subset X$ such that $\mathrm{cod}_X(X \setminus U) \geq 2$. The arithmetic Kodaira dimension of a normal variety X satisfies $\kappa(\overline{X}) \leq \kappa(X)$ for any desingularization $\overline{X}$ of X. There are easy examples, e.g., a cone $X \subset \mathbb{P}^3$ on a smooth cubic curve where this inequality is strict.

Theorem 4.1 *Let (X, A) be a small normal pair. If $\mathcal{N}_{A|X}$ is $\mathbb{Q}$-effective, then*

$$\kappa(X) \leq \kappa(A) + \mathrm{cod}_X A.$$

Proof. We are going to apply Theorem (2.1) with $\mathcal{L} = K_X$. First notice that since $\mathcal{N}_{A|X}$ is $\mathbb{Q}$-effective, we have an inclusion $\widehat{\mathrm{S}}^{k_0}\mathcal{N}_{A|X}^* \subset \mathcal{O}_A^M$ for a suitable positive integer k_0. Hence

$$h^0(\widehat{\mathrm{S}}^m\mathcal{N}_A^*\hat{\otimes}t\mathcal{L}_A) \leq \dim \widehat{\mathrm{S}}^{k_0}H^0(S^{k_0}\mathcal{N}_A^*\hat{\otimes}\mathcal{L}_A)$$
$$\leq h^0(\widehat{\mathrm{S}}^{k_0}\widehat{\mathrm{S}}^m\mathcal{N}_A^* \otimes k_0t\mathcal{L}_A) \leq C\ (\mathrm{rank}\,\widehat{\mathrm{S}}^m\mathcal{N}_A^*)\ h^0(k_0t\mathcal{L}_A)$$
$$\leq C'\ (\mathrm{rank}\,\widehat{\mathrm{S}}^m\mathcal{N}_A^*)\ t^{\kappa(\mathcal{L}_A)}.$$

Thus we obtain

$$\kappa(X) \leq \kappa(K_X|A) + \mathrm{cod}_X A.$$

Now $K_X|A = K_A \otimes \det N_A^*$ at least outside a set of codimension ≥ 2. Since $\det N_A$ is $\mathbb{Q}$-effective, we conclude (similarly as above) that

$$\kappa(K_X|A) \leq \kappa(A)$$

and our claim follows. Q.E.D.

Actually most parts of Theorem (4.1) hold in more generality, we have

Theorem 4.2 *Let (X, A) be a small normal pair. Let $\mathcal{L}$ be a torsion free sheaf of rank 1 on X and assume $\mathrm{cod}_A(A \cap \mathrm{Sing}\,\mathcal{L}) \geq 2$. Assume that $\mathcal{N}_{A|X}$ is $\mathbb{Q}$-effective. Then*

$$\kappa(\mathcal{L}) \leq \kappa(\mathcal{L}_A^{**}) + \mathrm{cod}_X A.$$

We next discuss the case that A has generically nef normal bundle, e.g., nef normal bundle. To work out the difficulties with this case, let us assume for simplicity that A is a Cartier divisor in X. A main point in Theorem (4.1) was that some power of N_A has a section and that therefore the dual is a subsheaf of $\mathcal{O}_A$. If say N_A is nef, then it might happen that no power of N_A has a section, so we cannot argue in this way. To make this more concrete, suppose that X is a projective manifold, that $\mathcal{L}$ and $\mathcal{F}$ are line bundles on X and that $\mathcal{F}^*$ is nef. Intuitively one would say that $h^0(\mathcal{F} \otimes L) \leq h^0(L)$. Of course this is false for trivial reasons. Take e.g., X to be an abelian variety, $\mathcal{L} \in \mathrm{Pic}^o(X)$ non-torsion and $\mathcal{F} = \mathcal{L}^*$. On the other hand we need only $h^0(\mathcal{F} \otimes \mathcal{L}^t) \leq C\mathrm{rank}\, h^0(\mathcal{L}^t)$ asymptotically.

Lemma (3.8) shows that this is indeed true if $\mathcal{L}$ is semi-ample, in particular for $\mathcal{L} = K_X$ when X is a good minimal model. Lemmas (3.9) and (3.10) show that if our original A has a good minimal model, e.g. via divisorial contraction of an extremal ray (i.e. contractions of a prime divisor) or flips (see e.g., [3]), then we can go to A' and argue there. Thus we have

Theorem 4.3 *Let (X, A) be a small normal pair. Assume $\overline{A}$ has at worst terminal singularities and that $\overline{A}$ admits a good minimal model. If $\mathcal{N}_{A|X}$ is generically nef, then*

$$\kappa(X) \leq \kappa(A) + \mathrm{cod}_X A.$$

Proof. We may assume (X, A) projective. Let $\varphi : A \rightharpoonup A'$ be a birational map to a good minimal model A'. By [6], (0.3.7), there exists a divisor $Y \subset A$ and an algebraic set $Y' \subset A'$ of codimension 2 such that $\varphi | A \setminus Y \to A' \setminus Y'$ is an isomorphism. Hence we can apply (3.9) and (3.10). By our basic inequality we have

$$h^0(tK_X) \le \sum_{k=0}^{ct} h^0(\widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X} \hat{\otimes} tK_A \hat{\otimes} \det t\mathcal{N}^*_{A|X}).$$

Let

$$\mathcal{F}_{t,k} = \widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X} \hat{\otimes} t \det \mathcal{N}_{A|X}.$$

By Lemmas (3.9) and (3.10) we conclude

$$\sum_{k=0}^{ct} h^0(\widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X} \hat{\otimes} tK_A \hat{\otimes} \det t\mathcal{N}^*_{A|X}) = \sum_{k=0}^{ct} h^0(\varphi_*(\mathcal{F}^*_{t,k}) \otimes tK_{A'}))$$

$$\le C \sum_{k=0}^{ct} \operatorname{rank} \varphi_*(\mathcal{F}^*_{t,k}) t^{\kappa(A)}.$$

Here we have used the semi-ampleness of $K_{A'}$ in order to be able to apply Lemma (3.8) and moreover $\kappa(A) = \kappa(A')$. Note also that the constant C is independent of t and k by Lemma (3.8). Since

$$\operatorname{rank} \varphi_*(\mathcal{F}^*_{t,k}) = \operatorname{rank} \mathcal{F}_{t,k} \sim t^{\mathrm{cod}_X A - 1},$$

we conclude finally that

$$h^0(tK_X) \le C t^{\kappa(A) + codA}$$

asymptotically. Q.E.D.

Parts of the proof actually show

Theorem 4.4 *Let (X, A) be a small normal pair and $\mathcal{L}$ a reflexive sheaf on X. Assume that $\mathcal{L}$ is $\mathbb{Q}$-Cartier, that $\mathcal{L}^{**}_A$ is semi-ample and adapted to (X, A). Assume that $\mathcal{N}_{A|X}$ is generically nef. Then*

$$\kappa(\mathcal{L}) \le \kappa(\mathcal{L}^{**}_A) + \mathrm{cod}_X A.$$

Without the semi-ampleness assumption on $\mathcal{L}$, Theorem (4.3) is however false if the normal bundle $\mathcal{N}_{A|X}$ is, say, only nef. In fact, let C be an elliptic curve and let $\mathcal{F} \in \mathrm{Pic}^0(C)$ be a non-torsion point. Set $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{F})$ and let $A \subset X$ be the section given by the epimorphism $\mathcal{O} \oplus \mathcal{F} \to \mathcal{O}$. Set $\mathcal{L} = \mathcal{O}_X(A)$. Then $\mathcal{L}_A = \mathcal{F}^{-1}$ and therefore $\kappa(L_A) = -\infty$. On the other hand $\kappa(L) = 0$. If however we define a refined Kodaira dimension $\tilde{\kappa}$ substituting $-\infty$ by -1 in the definition, then this example does not work.

Remark 4.5 It is worth noting that the same methods yield some easy but very useful results in special situations. For example, assume that A is a positive dimensional connected compact complex submanifold of a connected complex, but not necessarily compact, manifold X. Assume that $h^0(S^k\mathcal{N}^*_{A|X}) = 0$ for all $k > 0$, e.g., assume that the normal bundle of A in X is ample. Then given any holomorphic map $\phi : X \to Y$ from X to a complex space Y with $\dim \phi(A) = 0$, it follows that ϕ maps X to a point. Thus if we have a globally generated line bundle L on a complex manifold X and L is trivial on a positive dimensional compact submanifold with ample normal bundle, it follows that L is the trivial bundle.

Question

1. *Assume $\mathcal{N}_{A|X}$ to be generically nef and $\mathcal{L}$ nef. Is*
$$\kappa(\mathcal{L}) \leq \kappa(\mathcal{L}_A) + \mathrm{cod}_X A?$$
2. *Assume instead of nefness of $\mathcal{L}$ that $\mathcal{L}$ is big and its canonical ring $\bigoplus_m H^0(X, \mathcal{L}^m)$ is finitely generated. Do we have*
$$\kappa(\mathcal{L}) \leq \kappa(\mathcal{L}_A) + \mathrm{cod}_X A?$$

It is easy to see that the answer to part (1) of the question is yes for surfaces.

By [6, 5, 2] every threefold with only terminal singularities is either uniruled or has a good minimal model. Hence we can state (we stick to the most important case that X and A are projective).

Corollary 4.6
1. *Let (X, A) be a small normal pair with X and A a projective threefold such that $\overline{A}$ has only terminal singularities and A is not uniruled. If $\mathcal{N}_{A|X}$ is generically nef, then*
$$\kappa(X) \leq \kappa(A) + \mathrm{cod}_X A.$$
2. *Let A be a non-uniruled connected projective submanifold of a projective manifold X. If $\mathcal{N}_{A|X}$ is nef and $\dim A \leq 3$, then $\kappa(X) \leq \kappa(A) + \mathrm{cod}_X A$.*

It remains to consider the case when A is uniruled. Here we consider only the projective situation. It is necessary to make a slightly stronger assumption on the normal sheaf. On the other hand it is not necessary to assume anything on the singularities of A.

Theorem 4.7 *Let (X, A) be a small normal projective pair, i.e., a small normal pair with X projective. Assume that A is uniruled and has only terminal or canonical singularities. Assume furthermore that $\widehat{\mathrm{S}}^m\mathcal{N}_{A|X} \simeq A\hat{\otimes}B$, where A is a nef reflexive sheaf and B is a $\mathbb{Q}$-effective sheaf for some positive integer m. Then $\kappa(X) = -\infty$.*

Proof. Let (C_t) be a covering family of rational curves. Since $A|C_t$ is nef (possibly with torsion!), $B|C_t$ is nef for general t, and since moreover $K_A \cdot C_t < 0$ (here we need the assumption on the singularities), it follows

$$H^0(\widehat{\mathrm{S}}^k \mathcal{N}^*_{A|X} \hat{\otimes} (tK_A) \hat{\otimes} t\det \mathcal{N}^*_{A|X}) = 0$$

Thus the claim results from our basic inequality. Q.E.D.

Notice that in Theorem (4.7) we do not claim that X is uniruled. If X and A are both smooth, this can be proved:

Theorem 4.8 *Assuming in Theorem (4.7) additionally that X and A are smooth, it follows that X is uniruled.*

Proof. Let (C_t) be a covering family of rational curves. Then $T_A|C_t$ is nef for general t. This is well known and is easily seen by considering the graph of the family. Now the normal bundle sequence and our assumption on the normal bundle imply that $T_X|C_t$ is nef for general t. Hence X is uniruled, since the deformations of C_t fill up X (if C_t consider the normalization $f_t : \mathbb{P}_1 \mapsto C_t$ and deform the morphism f_t, compare e.g., [4]). Q.E.D.

The main difficulty for proving Theorem (4.7) without some hypotheses on the singularities of X and A is the lack of a dimension estimate of the space of maps $\mathrm{Hom}(\mathbb{P}_1, X)$, cp. [4].

Corollary 4.9 *Let (X, A) be a small normal projective pair. Assume $\dim A \leq 3$ and that A has only terminal singularities. Assume that $\widehat{\mathrm{S}}^m \mathcal{N}_{A|X} \simeq A \hat{\otimes} B$ with A nef and B $\mathbb{Q}$-effective. Then*

$$\kappa(X) \leq \kappa(A) + \mathrm{cod}_X A.$$

Of course we conjecture that all the above results hold without restriction on $\dim A$. Here are some more special cases

Proposition 4.10 *Let (X, A) be a small normal projective pair. Then $\kappa(X) \leq \kappa(A) + \mathrm{cod}_X A$, if one of the following conditions holds.*

1. *$\mathcal{N}_{A|X}$ is generically nef and $\dim A = 1$;*
2. *$\mathcal{N}_{A|X} = A \hat{\otimes} B$ with a nef $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor B, moreover X is $\mathbb{Q}$-Gorenstein of dimension n, $\dim A = n - 1$ and $\kappa(X) = n$.*

Proof. (1) First suppose $\dim A = 1$. Then X is smooth in a neighborhood of A, and the normal bundle $\mathcal{N}_A$ is nef. By adjunction we have

$$K_X \cdot A = 2g - 2 + \deg(\mathcal{N}_A^*),$$

and by the nefness of $\mathcal{N}_A$ we get $K_X \cdot A \leq 2g-2, g = g(A)$. If $g = 0$, we obtain $K_X \cdot A < 0$ and therefore $\kappa(X) = -\infty$ by Corollary (3.6). If $g = 1$ we have $K_X \cdot A \leq 0$ and we conclude by Theorem (3.7). If $g \geq 2$, there is nothing to prove.

(2) By Lemma (4.14) below, we have $\kappa(X|A) = n-1$. From the adjunction formula it now follows easily that $\kappa(A) = n-1$.

The following result is easily verified by the reader along the lines of the results of this section.

Theorem 4.11 *Let (X, A) be a small normal projective pair with $A \subset X$ Cartier. Assume that $\mathcal{N}_{A|X}$ is in the closure of the effective cone, that A has only terminal singularities and that A has a good minimal model. Then*

$$\kappa(X) \leq \kappa(A) + 1.$$

In the rest of this section we are discussing applications of Theorem (4.1) and related results.

Theorem 4.12 *Let X be an normal projective variety. Let $\mathcal{E}$ be a vector bundle of rank r on X. Assume that there is a section s of $\mathcal{E}$ that vanishes on an normal variety A such that:*

1. *on $X \setminus \operatorname{Sing} X$, the graph of s is transverse to X; and*

2. $\operatorname{cod}_A(A \cap singX) \geq 2$.

If $\mathcal{E}$ is ample or generically spanned then $\kappa(X) \leq \kappa(A) + \operatorname{rank} \mathcal{E}$.

Proof. Just observe that $\mathcal{N}_{A|X} = \mathcal{E}|A$ is $\mathbb{Q}$-effective. Q.E.D.

Of course similar theorems can be stated with generic nefness or nefness conditions on $\mathcal{E}$, e.g., $\mathcal{E}$ is nef, A has only terminal singularities and $\dim A \leq 3$. The last condition can be restated as $\operatorname{rank} \mathcal{E} \geq \dim X - 3$.

Theorem 4.13 *Let (X, A) be a small normal projective pair. If $\kappa(\det \mathcal{N}_{A|X}) > \kappa(A)$ and if $\mathcal{N}_{A|X}$ is $\mathbb{Q}$-effective, then $\kappa(X) = -\infty$.*

Proof. If $\kappa(X) \geq 0$ then by Theorem (2.1) there would be a not identically zero section of $kK_{X|A}\hat{\otimes}\widehat{\mathrm{S}}^t\mathcal{N}^*_{A|X}$ for some $k > 0$ and some $t \geq 0$. Since $\mathcal{N}_{A|X}$ is $\mathbb{Q}$-effective we conclude that we have a not identically zero section of $kK_{X|A}$ for some $k > 0$. This is absurd because the condition $\kappa(\det \mathcal{N}_{A|X}) > \kappa(A)$ implies that $\kappa(K_{X|A}) = -\infty$ by adjunction. Q.E.D.

As an application consider the situation when we have a connected complex submanifold A of a projective manifold X with $\mathcal{N}_{A|X}$ ample. If $\kappa(A) \neq \dim A$ then we conclude that $\kappa(X) = -\infty$. The ampleness of $\mathcal{N}_{A|X}$ can be reduced in certain situations.

Lemma 4.14 *Let (X, A) be a small normal projective pair. Assume that $A \subset X$ is a divisor whose normal sheaf can be written as $\mathbb{Q}$-Weil divisor as a sum of a nef and an effective divisor. If $\mathcal{L}$ is a rank one reflexive free sheaf on X adapted to (X, A) with $\kappa(\mathcal{L}) = \dim X$, then $\kappa(\mathcal{L}_A) = \dim A$.*

Proof. Write (as $\mathbb{Q}-$divisors) $L = H + E$ with H ample and E effective. Now add all components E_i of E to H which have the property that $E_i|A$ is the sum a nef and an effective $\mathbb{Q}$-divisor. Then we obtain a decomposition

$$L = H' + E'$$

and no component of E' has this property. Therefore A is not a component of E'. Thus $H'|A$ is big and $E'|A$ is effective, so that $L|A$ is big.

Corollary 4.15 *Let (X, A) be a small normal pair with $A \subset X$ be a divisor whose normal sheaf is (as $\mathbb{Q}$-Weil divisor) a sum of an effective and a nef divisor. If X is of general type, then A is of general type, also.*

Our last application concerns covering families of subvarieties.

Theorem 4.16 *Let (X, A) be a small normal projective pair. Assume that tA moves as a cycle in a covering family in X for some $t > 0$. Then*

$$\kappa(X) \leq \kappa(A) + \mathrm{cod}_X A.$$

Proof. We want to prove that $\mathcal{N}_{A|X}$ is $\mathbb{Q}$-effective. Going to the graph of the covering family, we see immediately that $\mathcal{N}_{tA|X}$ is generically spanned. Via the canonical, generically injective map $\mathcal{I}_A^t/\mathcal{I}_A^{t+1} \to S^t(\mathcal{I}/\mathcal{I}^2)$ it follows that $\widehat{\mathrm{S}}^t\mathcal{N}_A$ is generically spanned, hence $\mathcal{N}_A$ is $\mathbb{Q}$-effective.

Corollary 4.17 *Let X be a projective manifold, (A_t) a covering family of positive dimensional subvarieties. Assume that A_t is smooth for general t. If A_t is not of general type (t general), then X is not of general type.*

References

[1] Demailly,J.P.;Peternell,Th.;Schneider,M., *in preparation.*

[2] Kawamata,Y.: *Abundance theorem for minimal threefolds. Inv. Math. 108 (1992), 229-246.*

[3] Kawamata,Y.;Matsuda,K.;Matsuki,K.: *Introduction to the minimal model problem. Adv. Stud. Pure Math. 10 (1987), 283-360.*

[4] Kollár,J.: *Rational curves on algebraic varieties. Springer 1996.*

[5] Miyaoka,Y.: *Abundance conjecture for threefolds:* $\nu = 1$ *case. Comp. Math. 68 (1988), 203-220.*

[6] Mori,S.: *Flip theorem and the existence of minimal models for 3-folds. J. Amer. Math. Soc. 1 (1988), 117-253.*

[7] Ueno,K.: *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Math. 639 (1975), Springer-Verlag, New York.

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