CHAPTER 13

FINS WITH RADIATION

13.1 INTRODUCTION

One of the first papers to treat the radiation mode as the sole means of heat dissipation from the faces of a fin was that of Callinan and Berggren (1959). This paper considered flat and convex tubes with radiation from one side and fin-and-tube double-surface radiators. The radiation interchange between fin and tube was approximated except that interreflections were not considered for the gray body case, and no account was taken of the incident radiation from the fin on the tube. Even at this early stage in the technology pertaining to the investigation of radiative dissipation from fins, an attempt was made to maximize heat rejection on a per unit mass basis.

Chambers and Somers (1959) analyzed the radiation from one side of a circular disk to an absolute zero heat sink, and in this and the next chapter, radiation to an absolute zero heat sink will be referred to as radiation to free space. In a NASA Technical Note, Lieblein (1959) unknowingly covered much of the ground covered by Callinan and Berggren (1959), but he provided the basis for the consideration of an equivalent heat sink temperature which would enable later investigators to take account of a variety of environmental conditions. Lieblein also considered the mass minimization problem and provided radiation fin efficiency curves for both finite- and infinite-length plates for various source-to-sink temperature ratios.

Bartas and Sellars (1960) provided efficiency curves for minimum mass fins, and Nilson and Curry (1960) gave a numerical solution to obtain the mass minimization of a straight (longitudinal) fin of rectangular profile from fin surfaces radiating with an emmisivity of 0.50 to free space. Mackay (1960) published a text that provided a wealth of interesting data that would enable a designer to obtain solutions to both fin analysis and fin synthesis problems involving rectangular profile and optimized profile fins radiating to free space and to nearby surfaces.
Eckert et al. (1960) provided a brief note on the formulation of models for the radiation from fins with mutual irradiation. However, in their study, the radiant energy incident on the fin from the base or prime surface was assumed to be negligible. An analytical solution to the problem of mass minimization was given by Liu (1960). His solution involved determination of the optimum dimensions of the longitudinal fin of rectangular profile radiating to free space with no energy absorption from the environment.

Mackay (1960), taking the lead from Schmidt (1926), developed fin shapes to yield linear-temperature gradients when dissipation from the fin faces is purely by radiation. Although he made no claim that the linear temperature gradient fins were optimum, he did show that these fins were somewhat lighter than their companion rectangular profile fins designed for identical conditions. However, it was Wilkins (1960a,b) who showed that the radiating longitudinal fin of least material is not the one that exhibits a linear temperature gradient. He stated that the problem is one of finding three functions, one of heat flow, one of temperature profile, and one relating to a variable fin thickness that will match the boundary conditions at fin tip and fin base and make the profile area a minimum. Through the use of some clever transformations with regard to temperature and heat flow, he was able to determine the profile required. It is interesting to note that Haley and Westwater (1966) employed Wilkins's procedure to optimize the shape of a spine dissipating heat to a boiling liquid.

Granet and McIlroy (1961) described a procedure that enabled the optimization of any fin whose profile could be expressed as a sum of exponentials. Liu (1961) extended his previous work by giving an analytical solution for the optimum rectangular cooling fin in terms of the beta function. It was Haslet and Lomax (1961) who appeared to be among the first to analyze radiating fins with variable thermal conductivity and variable emissivity on the fin faces. Their treatment pertained to a family of very thin longitudinal fins of rectangular profile. These fins extended symmetrically from a common edge and mutual diffuse radiation was taken into account. At about the same time, Sparrow et al. (1961a) published an analysis that pertained to two longitudinal fins of rectangular profile with a common edge at a general angle with respect to each other. Diffuse radiant interchange between the fins was considered, but no attempt to include the radiant energy incident on the fins from the base (prime) surface was made.

In 1962, Kreith (1962) published his book entitled *Radiation in Space*, which contained a section on radiating fins, and Wilkins (1962a) addressed the problem of optimization with a constant and linear temperature gradient. Wilkins (1962b) also considered a minimum mass radiating fin with internal heat generation.

Sparrow and Eckert (1962) provided a more comprehensive treatment of radiation between fin and base surface in the case of fin-and-tube construction. For blackbody conditions, they included the effect of radiation from the tubes falling on the fins and observed that under practical operating conditions, the radiation from the base surface could cause a significant reduction in the heat dissipation from the fin. For gray body conditions, they set up equations that would account for diffuse reflections, but they did not solve them.

Liu (1962) proposed a general differential equation and solution that applied to
longitudinal fins of arbitrary profile and which considered a possible variation of surface emissivity as a function of fin height.

Sparrow et al. (1961b) developed a model for annular (radial) fins of rectangular profile with blackbody surfaces and included the fin and base radiant interaction. It was shown that the radiant interchange between fin and tube was quite significant, leading to appreciable error if this interaction is not considered. Equations for gray body radiation with constant surface emissivity were presented but were not solved.

Reynolds (1963) pointed out that a typical fin-and-tube space radiator such as the one shown in Fig. 13.1 would have manifold tubes with manifold lengths dependent on the length of the fins. He also considered that the mass of the system, consisting of the tube, the manifolds, the fluid that the tubes and manifolds contain, and any protective armor, may be such that shorter and thicker fins may be more desirable than fins whose design depended solely on individual fin optimization. Reynolds's (1963) work provided a mass optimization for longitudinal fins of rectangular, trapezoidal, and triangular profile spaced (Fig. 13.1) 180° apart on the periphery of a circular tube, which, in turn, was attached to a manifold at each end. The mass optimization included the supporting manifolds.

Stockman and Kramer (1963) considered the variation of thermal conductivity and emissivity as linear functions of temperature in a fin-and-tube configuration assuming

![Diagram](image)

**Figure 13.1** Longitudinal fin–radiator configuration.
one-dimensional heat flow and with radiation to an equivalent heat sink temperature. Stockman and Bittner (1965) provided what was one of the first treatments of two-dimensional heat flow in radiating fins. The study was based on a fin-and-tube configuration with stainless steel cladding on a copper fin and radiation to free space. Kotan and Armas (1965) studied the parabolic profile longitudinal radiating fin and provided an optimization.

Okamoto (1966a,b) also made a two-dimensional study for the rectangular profile fin. He concluded that the one-dimensional model was accurate down to height-to-thickness ratios as low as 3, although he also ignored the base and fin radiation interaction. Hering (1966) considered the specular radiation interaction in the angular space between rectangular plates of rectangular cross section having a common edge. This work was significant because it showed that the total heat loss from the plates that were acting as fins was greater for specular radiation than for the diffuse radiation considered earlier.

Sarabia and Hitchcock (1966) extended the work of Sparrow and Eckert (1962) by solving the problem of gray body interchange between fin and base for a configuration of infinite length. Stockman et al. (1966) enhanced the work of Stockman and Bittner (1965) by comparing one- and two-dimensional solutions and including the radiation interchange between fins and tube (base surface). The study showed that the one-dimensional solutions were in good agreement with the two-dimensional solutions.

Tien (1967) commented on the work of Hering (1966) and provided an approximate third-order polynomial solution for the temperature distribution when specular radiation was present, and Cohen (1969) provided a numerical solution for heat transfer from a bar of variable thermal conductivity by radiation.

Keller and Holdridge (1969) conducted a numerical solution for the steady-state behavior of the annular (radial) fin of trapezoidal profile and provided charts relating the fin efficiency to other dimensionless groups defined in their analysis. Kosshlyaev (1969) calculated the fin efficiency and obtained optimum combinations of thermal and conduction parameters with respect to mass for straight radiating fins on tubes. Donovan and Rohrer (1971) formulated a set of nonlinear integrodifferential equations pertaining to heat dissipation by radiation in an array of longitudinal fins of rectangular profile on a plane surface. These equations were solved numerically, and the results revealed that the fins are most effective when the spacings between them are relatively large and when shorter fins (smaller fin height) of higher thermal conductivity are employed. They considered mutual irradiation and observed that this had an important overall effect on the overall heat exchange process.

Campo and Wolko (1973) investigated the conduction-radiation interplay for a longitudinal fin of rectangular profile dissipating heat to the surroundings at a constant equivalent temperature. They illustrated their mathematical scheme for obtaining the heat transferred by radiation from the fins. Schnurr et al. (1976) used a nonlinear optimization approach to determine the minimum mass design for radiating finned arrays used in space. They considered straight and circular (longitudinal and radial) fins and included fin-to-fin and fin-to-base interactions in their analysis. The results were presented in graphical form and gave optimum geometries for the profiles considered in terms of the dimensionless parameters which they proposed.
Chiou and Na (1977) developed an initial value method for the solution of nonlinear two-point boundary value problems that pertain to the analysis of radiating fins. This method is noniterative, computationally efficient, and gives good agreement with the solutions of identical problems solved by more conventional methods.

Mehta (1978) obtained minimum mass designs for radiating finned arrays which he called heat sinks, using a direct search procedure using pseudorandom numbers. This analysis included fin-to-fin and fin-to-base interactions. Crawford (1978) compared three methods of calculating the heat transfer by radiation from fins of arbitrary shape, and Karam and Eby (1978) showed that the differential equation for the temperature profile when radiation and convection are both present could be simplified considerably if the temperature to the fourth power in the radiation term is replaced by a linear expansion about a term known as the mean temperature. Solution of the linearized steady-state equation was provided and a method was indicated in which the mean temperature was optimized as a function of the fin properties in order to minimize the errors introduced by the process of linearization.

Truong and Mancuso (1980) treated the problem of radiation from an annular (radial) fin whose surfaces had different emissivities. The study included various profile shapes and the results were obtained by the shooting method in conjunction with the Runge–Kutta–Verner fifth- and sixth-order integration method. The results were plotted as a function of dimensionless parameters proposed by the authors.

Delfour et al. (1983) used a finite element method as a first step toward the solution of a minimum mass radiating fin in a satellite application. Colle (1983) provided a general solution for heat transfer in arrays of radiating fins of arbitrary shape, and Chang (1985) obtained an analytical solution for a radiating annular (radial) fin by linearizing the radiating term in the differential equation for the temperature profile. The errors resulting from the linearization process were claimed to have been minimized. Chung and Nguyen (1986) provided a general relationship for the optimized dimensions of longitudinal fins of rectangular, trapezoidal, triangular, and parabolic profile radiating to free space, and Smith (1992) presented a single equation for the profile area of longitudinal fins of rectangular, trapezoidal, and triangular profile as a function of the taper ratio.

In Chapter 1 it was convenient to fix the environments in which extended surfaces were studied merely by assigning a temperature to the environment and a convection coefficient through which heat was transported from the fin faces to the environment. Heat rejection systems for space vehicles usually employ some form of extended surface which must perform in surroundings in which radiation is the only form of heat communication. Similarly, the generation of steam power from fossil fuel has progressively utilized the advantages of radiation over convection in furnace designs.

Although radiation and convection frequently coexist, a study of systems involving radiation alone provides a tidier approach to systems depending wholly or in part on heat transfer by radiation. The mathematic analysis may be based on the modifications of two of the Murray–Gardner assumptions cataloged in Chapter 1. Assumption 3 requires that the fin be exposed to a uniform heat transfer coefficient, and it is apparent that in space vehicles, portions of a fin may be turned toward or away from a radiation heat sink. With this assumption removed, consideration must be given to the purely
radiative heat transport that exists between various points on fin surfaces and their surroundings. The removal of assumption 3 nullifies assumption 10, which requires that the heat transferred be proportional to the temperature excess, \( \theta = T - T_e \), whereas radiation implies a fourth-power temperature dependency.

### 13.2 LONGITUDINAL RADIATING FIN OF RECTANGULAR PROFILE

#### 13.2.1 Radiation to Free Space

Bartas and Sellars (1960) investigated radiation from rectangular fins joining circular tubes as shown in Fig. 13.1. The governing differential equation was formulated using a non-zero surrounding temperature, referred to here as non-free space, and considered the radiation or shape factors that exist between fins and tubes. These interchange factors, designated here as \( F_A \), account for the effects of mutual radiation between a fin and other surfaces and have been studied by Sparrow and Eckert (1962), and Kreith (1962), and many others.

Mackay and Leventhal (1960) derived basic relationships for parameters affecting heat transfer from a plate uniformly heated on one edge. This was the case for radiation to free space where the environmental temperature was taken as absolute zero. They also considered heat input to the fin from sources such as the sun, earth, and other bodies in space.

The terminology and coordinate system for the longitudinal fin of rectangular profile radiating to deep space at \( T_e = 0 \text{ K} \) are shown in Fig. 13.2. The fin receives no heat input on its faces from other bodies in space or from other fins, solar panels, or conduits in the space vehicle and space radiator configuration. Heat enters uniformly at the fin base at \( x = 0 \) and passes from the fin faces by radiation.

Referring to the differential element, \( dx \), in Fig. 13.2, the difference between the heat entering and leaving by conduction is

\[
dq = k \delta L \frac{d^2 T}{dx^2} \ dx
\]

where \( T \) is the absolute temperature in kelvin and \( k \) is the thermal conductivity. In the steady state, this heat must be equal to the heat dissipated by radiation from the faces of the element \( dx \):

\[
dq = 2\sigma \epsilon LT^4 \ dx
\]

where \( \sigma \) is the Stefan–Boltzmann constant,

\[\sigma = 5.669 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4\]

An energy balance over the differential element \( dx \) requires that eqs. (13.1) and (13.2) be equated and after some simplification

\(^1 F_A \text{ for arrangement factor.}\)
Figure 13.2 Terminology and coordinate system for a radiating longitudinal fin of rectangular profile.

\[
\frac{d^2 T}{dx^2} = \frac{2 \sigma \epsilon}{k \delta} T^4
\]  

(13.3)

which is a second-order nonlinear differential equation. Equation (13.3) governs the temperature profile, and its solution can be obtained by successive integration.

Let \( p = \frac{dT}{dx} \), so that

\[
\frac{d^2 T}{dx^2} = \frac{dp}{dx} = \frac{dT}{dx} \frac{dp}{dT} = p \frac{dp}{dT}
\]

Substituting into eq. (13.3) gives

\[
p \frac{dp}{dT} = \frac{2 \sigma \epsilon}{k \delta} T^4
\]

and a separation of variables and integration yields

\[
p = \frac{dT}{dx} = -2 \left( \frac{\sigma \epsilon}{5k \delta} T^4 + C \right)^{1/2}
\]  

(13.4)

where \( C \) is the arbitrary constant of integration. The minus sign in eq. (13.4) is required because the temperature gradient is negative everywhere. The arbitrary constant is evaluated at the fin tip where \( x = b \) and where \( \frac{dT}{dx} = 0 \) and \( T = T_a \). Then eq. (13.4) becomes
\[ \frac{dT}{dx} = -2 \left( \frac{\sigma e}{5k\delta} T^5 - \frac{\sigma e}{5k\delta} T_u^5 \right)^{1/2} \]

or

\[ \frac{dT}{dx} = -2 \left( \frac{\sigma e}{5k\delta} \right)^{1/2} (T^5 - T_u^5)^{1/2} \] (13.5)

The variables in eq. (13.5), are separable, and after algebraic manipulation,

\[ \int_{T_u}^{T_b} \frac{dT}{T^{5/2} \left[ 1 - (T_n/T)^5 \right]^{1/2}} = -2 \left( \frac{\sigma e}{5k\delta} \right)^{1/2} \int_0^b dx \] (13.6)

Let the transformation

\[ u = \left( \frac{T_i}{T} \right)^5 \] (13.7)

be made so that

\[ dT = -\frac{T_n}{5u^{6/5}} du \] (13.8)

and

\[ T^{5/2} = \left( \frac{T_i}{u} \right)^{1/2} \] (13.9)

Moreover, let

\[ Z = \frac{T_b}{T_u} \]

so that at \( x = 0, T = T_u \) and \( u = Z^{-5} \), and at \( x = b, T = T_b \) and \( u = 1 \). These terms and eqs. (13.7) through (13.9) may be put into eq. (13.6) to give

\[ \int_{u=Z^{-5}}^{u=1} u^{-0.7} (1 - u)^{-0.5} \frac{du}{u^{6/5}} = 2 \left( \frac{\sigma e}{5k\delta} \right)^{1/2} \int_0^b dx \]

or

\[ \int_{u=Z^{-5}}^{u=1} u^{-0.7} (1 - u)^{-0.5} du = \left( \frac{20\sigma e T_u^5}{k\delta} \right)^{1/2} \int_0^b dx \] (13.10)

The integral on the left of eq. (13.10) may be evaluated in terms of the beta functions,

\[ \int_{u=Z^{-5}}^{u=1} u^{-0.7} (1 - u)^{-0.5} du = B(a, b) - B(a, b) \] (13.11)

where \( B(a, b) \) is the complete beta function.
\[ B(a, b) = \int_{u=0}^{u=1} u^{a-1}(1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \] (13.12a)

and \( B_n(a, b) \) is the \textit{incomplete beta function},

\[ B_n(a, b) = \int_{u=0}^{u=Z^{-5}} u^{a-1}(1-u)^{b-1} du \] (13.12b)

In eqs. (13.12), the constants \( a \) and \( b \) are numerically equal to \( a = 0.3 \) and \( b = 0.5 \). This suggests the temperature profile is described completely by

\[ B(0.3, 0.5) - B_n(0.3, 0.5) = b \left( \frac{20\sigma\epsilon T_0^3}{k\delta} \right)^{1/2} \] (13.13)

The heat dissipated by the fin is obtained from

\[ q = -k\delta \frac{dT}{dx} \bigg|_{x=0} \]

and using eq. (13.5) for the temperature gradient at \( x = 0 \) where \( T = T_b \) gives

\[ q_b = 2k\delta L \left( \frac{\sigma\epsilon}{5k\delta} \right)^{1/2} \left( T_b^5 - T_0^5 \right)^{1/2} \] (13.14)

As in Chapter 1, the fin efficiency may be defined as the ratio of the actual heat dissipation to the ideal heat dissipation if the entire fin were to operate at the base temperature. With the ideal heat dissipation, \( q_{id} = 2\sigma\epsilon bL T_b^4 \),

\[ \eta = \frac{2k\delta L \left( \frac{\sigma\epsilon}{5k\delta} \right)^{1/2} \left( T_b^5 - T_0^5 \right)^{1/2}}{2\sigma\epsilon bL T_b^4} \]

or by using \( Z = T_b/T_0 \), after algebraic adjustment,

\[ \eta = \frac{2 \left(1/Z^3 - 1/Z^8\right)^{1/2}}{b \left(20\sigma\epsilon T_0^3/k\delta\right)^{1/2}} \] (13.15)

The ratio \( Z \) relates the temperatures at the base and tip of the fin. As shown by eq. (13.13), there is a relationship between the parameter \( Z \) and the parameter

\[ \psi = b \left( \frac{20\sigma\epsilon T_0^3}{k\delta} \right)^{1/2} \]

This relationship, evaluated by using the hypergeometric function (Abramowitz and Stegun, 1964) which, in turn, is related to the incomplete beta function, is given in Fig. 13.3 with \( \psi \) plotted against \( Z \) in accordance with eq. (13.13). The fin efficiency is plotted in Fig. 13.4 using the values obtained from Fig. 13.3 in eq. (13.15). Unlike the convective case, the fin efficiency is a function of the dimensions and thermal properties of the fin (the parameter \( \psi \)) as well as the temperatures at the extremes.
Figure 13.3  Fin parameter $\psi$ as a function of $Z$ for a longitudinal fin of rectangular profile radiating to free space.

of the fin (the parameter $Z$). For this reason, an analysis of the performance of the radiating fin requires a trial-and-error solution.

**Example 13.1: Radiation to Free Space (Fin Analysis).** A longitudinal fin of rectangular profile is 4 m long, 50 cm high, and 0.635 cm thick and is fabricated of magnesium ($k = 152 \text{ W/m-K}$). Its surface has been treated so that its emissivity is 0.85. For a base temperature of 350 K ($77^\circ \text{C}$) and radiation to free space, determine (a) the tip temperature, (b) the efficiency, and (c) the heat dissipation.

**SOLUTION.** (a) For the tip temperature, observe that everything in the parameter $\psi$ is known except $T_0$:

$$\psi \equiv b\left(\frac{20\sigma eT_0^3}{k\delta}\right)^{1/2} = 0.50 \left[\frac{20(5.669 \times 10^{-8})(0.85)}{152(0.00635)}\right]^{1/2} T_0^{3/2}$$

or

$$\psi = 4.996 \times 10^{-1} T_0^{3/2}$$
**Figure 13.4** Radiation fin efficiency for a longitudinal fin of rectangular profile radiating to free space.

*Trial I.* Assume that $T_a = 300$ K (27°C). Then

$$\psi = 4.996 \times 10^{-4}(300)^{3/2} = 2.596$$

and

$$Z = \frac{T_b}{T_a} = \frac{350}{300} = 1.167$$

From Fig. 13.3 read $Z = 1.448$ at $\psi = 2.596$ and note that the $Z$ value of 1.167 does not match the $Z$ value of 1.448.

*Trial II.* Reduce $T_a$ to $T_a = 273$ K (0°C). Then

$$\psi = (4.996 \times 10^{-4})(273)^{3/2} = 2.254$$

and
\[ Z = \frac{T_b}{T_w} = \frac{350}{273} = 1.282 \]

From Fig. 13.3, read \( Z = 1.302 \) at \( \psi = 2.254 \) and note that the \( Z \) value of 1.282 does not match the \( Z \) value of 1.302. However, the agreement is close.

**Trial III.** Assume that \( T_w = 270 \, \text{K} (-3 \, ^\circ\text{C}) \). Then

\[ \psi = (4.996 \times 10^{-4})(270)^{3/2} = 2.217 \]

and

\[ Z = \frac{T_b}{T_w} = \frac{350}{270} = 1.296 \]

From Fig. 13.3, read \( Z = 1.292 \) at \( \psi = 2.217 \). This is close enough to be considered as the solution and

\[ T_w = 270 \, \text{K} \]

**b) For the fin efficiency, use Fig. 13.4 and read at \( Z = 1.296 \),**

\[ \eta = f(Z) = 0.524 \]

**c) For the heat dissipation determine the ideal heat dissipation first:**

\[ q_{id} = 2\sigma e b L r_b^4 \]
\[ = (2)(5.669 \times 10^{-8})(0.85)(4)(0.50)(350)^4 \]
\[ = 2892.4 \, \text{W} \]

and this makes the heat dissipation

\[ q_b = \eta q_{id} \]
\[ = (0.524)(2892.4) \]
\[ = 1515.6 \, \text{W} \]

**Example 13.2: Radiation to Free Space (Fin Synthesis).** A longitudinal fin of rectangular profile is 5 m long and 0.635 cm thick and is fabricated of aluminum \((k = 202 \, \text{W/m-K})\. Its surface has been treated so that its emissivity is 0.90. For a base temperature of 300 K \((27^\circ\text{C})\), determine the fin height required for a 1350-W dissipation.

**SOLUTION.** Work on a 1-m-length basis. With \( T_b = 300 \, \text{K} \) and \( q_b = 1350/5 = 270 \, \text{W} \),

\[ q_{id} = 2\sigma e b L r_b^4 \]
\[ = (2)(5.669 \times 10^{-8})(0.90)(1.00)(300)^4 b \]
or
\[ q_a = 826.5b \text{ W} \]

Moreover,
\[ \eta = \frac{q_b}{q_a} = \frac{270}{826.5b} = \frac{0.3267}{b} \]

and
\[ \psi = b \left( \frac{20\sigma \epsilon T_a^3}{k \delta} \right)^{1/2} \]
\[ = \left[ \frac{(20)(5.669 \times 10^{-8})(0.90)}{(202)(0.00635)} \right]^{1/2} b T_a^{3/2} \]

or
\[ \psi = 8.9192 \times 10^{-4} b T_a^{3/2} \]

**Trial I.** Assume that \( b = 0.40 \text{ m} \) and obtain
\[ \eta = \frac{0.3267}{0.40} = 0.817 \]

Read from Fig. 13.4 at \( \eta = 0.817 \) a value of \( Z = 1.092 \). Then
\[ T_a = \frac{T_b}{Z} = \frac{300}{1.092} = 274.7 \text{ K} \]

Now calculate \( \psi \):
\[ \psi = (8.9192 \times 10^{-4})(0.40)(274.7)^{3/2} = 1.624 \]

From Fig. 13.3, read \( Z = 1.153 \) at \( \psi = 1.624 \), and note that the \( Z \) value of 1.092 does not match the \( Z \) value of 1.153.

**Trial II.** Assume that \( b = 0.60 \text{ m} \) and obtain
\[ \eta = \frac{0.3267}{0.60} = 0.545 \]

Then read from Fig. 13.4 at \( \eta = 0.545 \) a value of \( Z = 1.283 \) and calculate \( T_a \):
\[ T_a = \frac{T_b}{Z} = \frac{300}{1.283} = 233.8 \text{ K} \]

Now calculate \( \psi \):
\[ \psi = (8.9192 \times 10^{-4})(0.60)(233.8)^{3/2} = 1.913 \]

From Fig. 13.3, read \( Z = 1.218 \) at \( \psi = 1.913 \) and note that the \( Z \) value of 1.283 does not match the \( Z \) value of 1.218.
Trial III. Assume that \( b = 0.50 \) m and obtain

\[
\eta = \frac{0.3267}{0.50} = 0.653
\]

Then read from Fig. 13.4 at \( \eta = 0.653 \) a value of \( Z = 1.186 \) and calculate \( T_o \):

\[
T_o = \frac{T_o}{Z} = 300 = 252.9 \text{ K}
\]

Now calculate \( \psi \):

\[
\psi = (8.9192 \times 10^{-4})(0.50)(252.9)^{3/2} = 1.795
\]

From Fig. 13.3, read \( Z = 1.186 \) at \( \psi = 1.795 \) and note that a match has been obtained. The fin height for a dissipation of 270 W/m is 0.50 m or 50 cm.

### 13.2.2 Radiation to Non-Free Space

The longitudinal fin of rectangular profile radiating to a sink that is not at or near absolute zero and with radiant interchange to and from other bodies has been treated by Lieblein (1959), Bartas and Sellars (1960), and Mackay and Leventhal (1960). Both the Lieblein and Bartas and Sellars works considered these environmental effects as part of a fictitious sink temperature. Mackay and Leventhal lumped these effects into what eventually was incorporated into an environmental parameter. The Mackay and Leventhal procedure is considered here.

Refer to the longitudinal fin of rectangular profile in Fig. 13.2 and consider the radiant heat exchange between the differential element of fin surface, \( L \, dx \), and the surroundings. This radiant heat exchange will be composed of two terms,

\[
K_1 T^4 L \, dx
\]

where the constant \( K_1 \) embraces all factors that modify the fin temperature as multipliers and where both sides of the fin dissipate, and

\[
K_2 L \, dx
\]

where the constant \( K_2 \) consists of all terms that do not multiply the fin temperature and may include solar and/or terrestrial radiation, radiant interchange factors between the fin, and other elements in the configuration and the temperature of the surroundings.

The total radiant heat dissipated by the differential fin element will be

\[
dq = (K_1 T^4 - K_2) L \, dx \quad (13.16)
\]

In accordance with the steady-state heat balance, this heat can be equated to the difference in heat entering and leaving the element \( dx \) by conduction. Thus

\[
k \delta L \frac{d^2 T}{dx^2} \, dx = (K_1 T^4 - K_2) L \, dx
\]
or

\[
\frac{d^2 T}{dx^2} = \frac{K_1}{k\delta} T^4 - \frac{K_2}{k\delta} \tag{13.17}
\]

Equation (13.17) is the differential equation governing the temperature profile on the fin. By a procedure similar to that employed for radiation to free space, a single integration gives

\[
\frac{dT}{dx} = -\left(\frac{2K_1}{5k\delta} T^5 - \frac{2K_2}{k\delta} T + C\right)^{1/2} \tag{13.18}
\]

where \( C \) is the constant of integration and where the minus sign assures a temperature gradient that is everywhere negative. The arbitrary constant can be evaluated at \( x = b \), where \( dT/dx \) is set equal to zero (\( dT/dx = 0 \)) and \( T = T_a \):

\[
C = -\frac{2K_1}{5k\delta} T_a^5 + \frac{2K_2}{k\delta} T_a^2
\]

Upon substitution of this value of \( C \) into eq. (13.18), one obtains

\[
\frac{dT}{dx} = -\left[\frac{2K_1}{5k\delta} (T^5 - T_a^5) - \frac{2K_2}{k\delta} (T - T_a)\right]^{1/2}
\]

or after algebraic adjustment,

\[
\frac{dT}{dx} = -\left(\frac{2K_1 T_a^5}{5k\delta}\right)^{1/2} \left(\frac{T}{T_a} - 1\right)^{1/2} \gamma \tag{13.19}
\]

where

\[
\gamma = \left[\left(\frac{T}{T_a}\right)^4 + \left(\frac{T}{T_a}\right)^3 + \left(\frac{T}{T_a}\right)^2 + \frac{T}{T_a} + 1 - \frac{5K_2}{K_1 T_a^4}\right]^{1/2}
\]

Now make the transformation

\[
v = \frac{T}{T_a} - 1 \tag{13.20}
\]

so that

\[
\frac{T}{T_a} = 1 + v^2 \tag{13.21}
\]

and

\[
dT = 2T_a v dv \tag{13.22}
\]

Equations (13.20) through (13.22) may be substituted into eq. (13.19), and after algebraic manipulation to separate the variables, the result is
\[ \int_{v_a}^{v_b} \frac{dv}{[(1 + v^2)^4 + (1 + v^2)^3 + (1 + v^2)^2 + (1 + v^2) + 1 - 5K_2/K_1T_a^4]^{1/2}} \]

\[ = -\int_0^b \left( \frac{K_1T_a^3}{10k\delta} \right)^{1/2} dx = -b \left( \frac{K_1T_a^3}{10k\delta} \right)^{1/2} \quad (13.23) \]

where the limits are

\[ v(x = 0) = v_b = (Z - 1)^{1/2} \]

where \( Z = T_b/T_a \) and where

\[ v(x = b) = v_a = 0 \]

When the limits of integration on the left of eq. (13.23) are reversed, the minus sign on the right may be removed. The integral on the left may be evaluated graphically by plotting the reciprocal of the denominator against the variable \( v \), as shown in Fig. 13.5. The area under each curve will be equal numerically to the parameter on the right of eq. (13.23). Note that the limits of integration are \( v_b = (Z - 1)^{1/2} \) and \( v_a = 0 \). Associated with the limit, \( v_b \) is a value of \( Z \). Hence one may actually plot

\[ b \left( \frac{K_1T_a^3}{10k\delta} \right)^{1/2} = f(Z) \]

This has been carried out in the preparation of Fig. 13.6.

The use of Fig. 13.6 is limited because the ordinate and family of parameters are functions of the fin tip temperature \( T_a \). The ordinate may be adjusted by multiplication by \((5Z^3)^{1/2}\): \( \zeta = b \left( \frac{K_1T_a^3}{2k\delta} \right)^{1/2} = b(5Z^3)^{1/2} \left( \frac{K_1T_a^3}{10k\delta} \right)^{1/2} \) which may be referred to as the profile number for the rectangular fin.

The family parameter may be adjusted by multiplying by \(1/5Z^4\) so that the new value becomes

\[ \frac{K_2}{K_1T_b^4} = \frac{1}{5Z^4} \frac{5K_2}{K_1T_a^4} \]

Thus Fig. 13.6 may be adjusted by multiplying values read from the curves by factors containing the parameter \( Z \). Such a procedure leads to the development of Fig. 13.7, which facilitates design and performance calculations.

The heat transferred from the fin faces will be equal to the quantity of heat entering the fin at its base:

\[ q_b = -k\delta L \left. \frac{dT}{dx} \right|_{x=0} \]
Figure 13.5 Curves for the graphical evaluation of eq. (13.23) for a longitudinal fin of rectangular profile radiating to non-free space.

The derivative is given by eq. (13.19) and is evaluated at \( x = 0 \) where \( T = T_b \). The heat transferred becomes

\[
q_b = k \delta L \left[ \frac{2K_1 T_0^5}{5k \delta} (Z - 1) \left( \Phi - \frac{5K_2}{K_1 T_a^4} \right) \right]^{1/2}
\]  

(13.24)

where

\[
\Phi = Z^4 + Z^3 + Z^2 + Z + 1
\]
Figure 13.6  Parameter relationship for a longitudinal fin of rectangular profile radiating to
non-free space.

Rearrangement of terms in eq. (13.24) gives

$$\frac{q_b}{k \delta L T_b} = b \left( \frac{K_1 T_b^2}{2k\delta} \left[ \frac{4(Z - 1)}{5Z^3} \right] \left( \Phi - 5Z^2 \frac{K_2}{K_1 T_b^4} \right) \right)^{1/2}$$

(13.25)

Observe that the parameters
Figure 13.7 Parameter relationship for a longitudinal fin of rectangular profile radiating to non-free space.

\[
\left( \frac{K_1 T_b^3}{2k\delta} \right)^{1/2} \quad \text{and} \quad \frac{K_2}{K_1 T_b^4}
\]

are the same as those used in plotting the temperature parameter in Fig. 13.7. Indeed, values of \(K_2/K_1 T_b^4\) may be selected and values of

\[
b \left( \frac{K_1 T_b^3}{2k\delta} \right)^{1/2}
\]

may be obtained from Fig. 13.7 at a particular value of \(Z\). The heat transfer parameter

\[
\frac{q_b}{k\delta L T_b}
\]

may then be plotted as shown in Fig. 13.8.

The fin efficiency is the ratio of the actual heat dissipated to the ideal heat dissipated if the entire fin were to operate at the base temperature \(T_b\) with no heat received from the environment \((K_2 = 0)\). The actual heat dissipation is given by eq. (13.25). With \(K_1 = 2\sigma\epsilon\),

\[
\eta = \frac{q_b}{2\sigma \epsilon b L T_b^4}
\]

This may be modified to accommodate the ordinate and abscissa values shown in Fig. 13.8. Multiplying numerator and denominator by \(b/k\delta L T_b\) yields
\[ \eta = \frac{q_b b / k \delta L T_b}{2 \sigma \epsilon b^2 T_0^3 / k \delta} = \frac{q_b h / k \delta L T_b}{b \left( 2 \sigma \epsilon T_0^3 / k \delta \right)^{1/2}} \]  
(13.26)

It is seen that the numerator is the ordinate and the denominator is twice the square of the abscissa of Fig. 13.8. The fin efficiency is plotted in Fig. 13.9.

**Example 13.3: Radiation to Non-Free Space (Fin Analysis).** It is the intent of this example to show that Figs. 13.7, 13.8, and 13.9 can be used for the free-space case. A good basis for comparison is Example 13.1. All that is necessary is to set \( K_2 = 0 \) so that \( K_2 / K_1 T_0^2 = 0 \).

A longitudinal fin of rectangular profile, 4 m long, 50 cm high, and 0.635 cm thick is fabricated of magnesium \((k = 152 \text{ W/m-K})\). Its surface has been treated so that its emissivity is 0.85. For a base temperature of 350 K (377°C), determine (a) the tip temperature, (b) the efficiency, and (c) the heat dissipation.

**SOLUTION.** (a) For the tip temperature with \( K_1 = 2 \sigma \epsilon \), form the profile number
Figure 13.9 Efficiency of a longitudinal fin of rectangular profile radiating to non-free space.

\[
\zeta = b \left( \frac{K_1 T_b^3}{2k\delta} \right)^{1/2}
\]

\[
= 0.50 \left[ \frac{(5.669 \times 10^{-8})(0.85)(350)^3}{(152)(0.00635)} \right]^{1/2}
\]

\[
= 0.732
\]

From Fig. 13.7, at \(\zeta = 0.732\) and \(K_2/K_1 T_b^4 = 0\), read

\[
Z = \frac{T_b}{T_d} = 1.30
\]
and

\[ T_b = \frac{T_b}{Z} = \frac{350}{1.30} = 269.2 \text{ K} \]

This compares favorably with \( T_a = 270 \text{ K} \) obtained in Example 13.1.

(b) For the fin efficiency, use Fig. 13.9 with \( \frac{K_2}{K_1 T_b^4} = 0 \) and read at \( \zeta = 0.732 \),

\[ \eta = 0.520 \]

which compares favorably to \( \eta = 0.524 \) obtained in Example 13.1.

(c) For the heat dissipation use Fig. 13.8 with \( \frac{K_2}{K_1 T_b^4} = 0 \) and at \( \zeta = 0.732 \),

\[ \frac{q_b}{k \delta L T_b} = 0.560 \]

so that

\[ q_b = \frac{(0.560)k \delta L T_b}{b} \]

\[ = \frac{(0.560)(152)(0.00635)(4)(350)}{0.50} \]

\[ = 1513.4 \text{ W} \]

which compares with \( q_b = 1515.6 \text{ W} \) obtained in Example 13.1.

Example 13.4: Radiation to Non-Free Space (Fin Synthesis). A longitudinal fin of rectangular profile is 1.524 m long and 0.3175 cm thick. It is fabricated of aluminum \((k = 202.5 \text{ W/m-K})\). Its surface has been treated so that its emissivity is 0.88, and it is positioned in such a manner as to make the constant \( K_2 = 621 \text{ W/m}^2 \). For a base temperature of 420 K \((147^\circ \text{C})\), determine the fin height required for a heat dissipation of 655 W.

SOLUTION. First, with \( K_2 = 621 \text{ W} \),

\[ K_1 = 2\sigma \epsilon = (2)(5.669 \times 10^{-8})(0.88) = 9.997 \times 10^{-8} \]

and

\[ \frac{K_2}{K_1 T_b^4} = \frac{621}{(9.997 \times 10^{-8})(420)^4} = 0.200 \]

Then

\[ \frac{q_b b}{k \delta L T_b} = \frac{655b}{(202.5)(0.003175)(1.524)(420)} = 1.592b \]

and

\[ \zeta = b \left( \frac{K_2 T_b^3}{2k \delta} \right)^{1/2} = b \left[ \frac{(9.997 \times 10^{-8})(420)^3}{(2)(202.5)(0.003175)} \right]^{1/2} = 2.398b \]
Trial I. Assume that \( b = 0.50 \) m and obtain

\[
\frac{q_b b}{k \delta L T_b} = (1.592)(0.50) = 0.796
\]

and at this value, Fig. 13.8 gives \( \zeta = 1.02 \). But

\[
b = \frac{\zeta}{2.398} = \frac{1.02}{2.398} = 0.425
\]

and the two values of \( b \) do not match.

Trial II. Assume that \( b = 0.30 \) m and obtain

\[
\frac{q_b b}{k \delta L T_b} = (1.592)(0.30) = 0.478
\]

and Fig. 13.8 gives \( \zeta = 0.68 \). But

\[
b = \frac{\zeta}{2.398} = \frac{0.68}{2.398} = 0.284
\]

and the two values of \( b \) almost match. One more try is in order.

Trial III. Assume that \( b = 0.25 \) m and obtain

\[
\frac{q_b b}{k \delta L T_b} = (1.592)(0.25) = 0.398
\]

and Fig. 13.8 gives \( \zeta = 0.59 \). But

\[
b = \frac{\zeta}{2.398} = \frac{0.59}{2.398} = 0.247
\]

and the two values of \( b \) match.

The fin height should be \( b = 0.875 \) m.

13.3 LONGITUDINAL RADIATING FINS OF TRAPEZOIDAL AND TRIANGULAR PROFILE

The terminology and coordinate system for the longitudinal fin of trapezoidal profile is displayed in Fig. 13.10. The triangular profile is an extension of the trapezoidal profile when the tip thickness \( \delta_x = 0 \). Note that the coordinate system has its origin at the fin tip and the height coordinate is taken positive in a direction toward the fin base.

Mackay and Bouch (1961) were among the first to treat the radiating longitudinal fin of trapezoidal profile. As in previous cases, the differential equation for the temperature distribution is obtained from a heat balance on the differential height element \( dx \). The difference in heat entering and leaving by conduction is

\[
dq = kL \frac{dT}{dx} \left[ \frac{\delta(x)}{dx} \right] dx
\]

(13.27)

This is the same as the heat leaving the element by radiation:
\[ = 0.796 \]
\[ = 0.478 \]
\[ = 0.284 \]
\[ y \text{ is in order.} \]
\[ = 0.398 \]
\[ = 0.247 \]

**TRAPEZOIDAL**

A longitudinal fin of trapezoidal profile is an extension of the trapezoidal coordinate system that has its origin at a direction toward the fin base. To treat the radiating longitudinal the differential equation for the balance on the differential height by conduction is

\[ \int dx \]  
(13.27)

radiation:

\[ dq = (K_1 T^4 - K_2) L \, dx \]  
(13.28)

The resulting nonlinear differential equation is

\[ \frac{k}{d} \frac{d}{dx} \left[ \delta(x) \frac{dT}{dx} \right] = K_1 T^4 - K_2 \]  
(13.29)

From the geometry in Fig. 13.10 it is seen that

\[ \delta(x) = \delta_a + (\delta_b - \delta_a) \frac{x}{b} = 2 \Lambda \left( \frac{\delta_a}{2\Lambda} + x \right) \]  
(13.30)

where the **fin taper** is

\[ \Lambda = \frac{\delta_b - \delta_a}{2b} \]

Let

\[ z = \frac{T}{T_a} \]

and substitute eq. (13.30) into eq. (13.29). After dividing by \( K_1 T_a^4 \) and algebraic adjustment, one obtains
\[ k \frac{d}{dx} \left[ \frac{2\Lambda}{K_1 T_a^4} \left( \frac{\delta_a}{2\Lambda} + x \right) \frac{dT}{dx} \right] = z^4 - \frac{K_2}{K_1 T_a^4} \quad (13.31) \]

Then let
\[ K_3 = \frac{K_1 T_a^4}{2\Lambda k} \quad (13.32a) \]

and
\[ K_4 = \frac{\delta_a}{2\Lambda} \quad (13.32b) \]

Noting that
\[ \frac{dT}{dx} = T_a \frac{dz}{dx} \]

eqs. (13.32) may be substituted into eq. (13.31) to give
\[ \frac{d}{dx} \left[ \frac{1}{K_3} (K_4 + x) \frac{dz}{dx} \right] = z^4 - \frac{K_2}{K_1 T_a^4} \quad (13.33) \]

Now introduce the variable
\[ w = K_3 (K_4 + x) \quad (13.34) \]

Its derivative is \( dw/dx = K_3 \), and \( dz/dx \) can be represented in terms of \( dz/dw \) as
\[ \frac{dz}{dx} = \frac{dz}{dw} \frac{dw}{dx} = K_3 \frac{dz}{dw} \]

With this in eq. (13.33), the result is
\[ \frac{d}{dw} \left( w \frac{dz}{dw} \right) - z^4 + \frac{K_2}{K_1 T_a^4} = 0 \]

Because \( z \) is a function of \( w \), this may be written, with primes denoting derivatives, as
\[ w f''(w) - f'(w) - [f(w)]^4 + \frac{K_2}{K_1 T_a^4} = 0 \quad (13.35) \]

Equation (13.35) is the governing differential equation for the temperature profile of the fin. Its solution may be obtained via a numerical integration procedure that has starting points as shown in Table 13.1. The numerical integration of eq. (13.35) will yield a set of values and curves of \( z = f(w) \) as a function of \( w \). Typical plots as functions of \( K_2/K_1 T_a^4 \) and \( w_a = K_3 K_4 \) (or zero for the triangular profile) are shown in Fig. 13.11. At the same time, the computer may be used to generate values of \( f'(w) \), \( f''(w) \), and \( \int_{w_a}^{w} dw \), all as functions of \( w \). These will be seen to be most useful.
Figure 13.11  Relationship of $f(w)$ to $w$ for longitudinal fins of trapezoidal and triangular profiles.
### TABLE 13.1 Starting Points for Numerical Integration of Eq. (13.35).

<table>
<thead>
<tr>
<th>Item</th>
<th>Trapezoidal Profile: $\delta_a$ Is Finite</th>
<th>Triangular Profile: $\delta_a = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$ at $x = 0$</td>
<td>$w = w_a = K_1 K_4$ from eq. (13.34)</td>
<td>$w = w_a = 0$ from eqs. (13.34) and (13.32b)</td>
</tr>
<tr>
<td>$z = f(w)$ at $x = 0$</td>
<td>$f'(w_a) = 0$ because $z = T_a/T_b = 1$</td>
<td>$f'(w_a) = 1$ because $z = T_a/T_b = 1$</td>
</tr>
<tr>
<td>$f''(w)$ at $x = 0$</td>
<td>$f''(w_a) = 1$ because $dz/dx = 0$</td>
<td>$f''(w_a) = 1 - K_3/K_1 T_b^4$ from eq. (13.35)</td>
</tr>
<tr>
<td>$f'''(w)$ at $x = 0$</td>
<td>$f'''(w_a) = \frac{1}{w_a} \left( 1 - \frac{K_2}{K_1 T_b^4} \right)$ from eq. (13.35)</td>
<td>$f'''(w_a) = 2$ using L'Hôpital's rule</td>
</tr>
</tbody>
</table>

The derivative of eq. (13.34) is $dw/dx = K_3$. Separation of the variables gives

$$K_3 \int_0^b dx = \int_{w_a}^{w_b} dw \quad (13.36)$$

where the upper limit $w_b$ can be defined by eq. (13.34):

$$w_b = K_3 (K_4 + b) \quad (13.37)$$

The integration as indicated by eq. (13.36) may be performed:

$$K_3 b = \int_{w_a}^{w_b} dw \quad (13.38)$$

or with $K_3$ in terms of $b$ by eq. (13.32a) using the definition of $\Lambda$ and $f(w_a) = T_a/T_b$:

$$K_3 = \frac{K_1 T_b^2 b^2}{k \delta_b} \frac{1}{[f(w_b)]^4 (1 - \lambda)} \quad (13.39)$$

where $\lambda$ is defined as the taper ratio\(^3\):

$$\lambda = \frac{\delta_a}{\delta_b} \quad (13.40)$$

The first term on the right-hand side in eq. (13.39) may be designated as the profile number for the trapezoidal or triangular fin:

\(^3\)Not the fin taper, $\Lambda$. 

which is dimensionless. Then a combination of eqs. (13.38), (13.39), and (13.41) gives

\[ \zeta = \left[ f(w_b) \right] \left( 1 - \lambda \right) \int_{w_i}^{w_b} dw \]

(13.42)

The combining of eqs. (13.30), (13.32a), (13.37) (13.38), and (13.40) gives

\[ \frac{1}{\lambda} = \frac{\delta_n}{\delta_i} = 1 + \frac{1}{w_i} \int_{w_i}^{w_b} dw \]

(13.43)

Equation (13.43) may be substituted into eq. (13.42) to give a complete representation of the profile number in terms of parameters generated by the computer:

\[ \zeta = \frac{\left[ f(w_b) \right] \left( \int_{w_i}^{w_b} dw \right)^2}{w_i + \int_{w_i}^{w_b} dw} \]

(13.44)

The heat flowing into the fin may be obtained from

\[ q_b = k_b L \frac{dT}{dx} \bigg|_{x=b} \]

or in terms of the transformed variable \( w \),

\[ q_b = k_b L T_i K_i |f'(w_b)| \]

(13.45)

The fin efficiency is

\[ \eta = \frac{q_b}{q_{ha}} = \frac{k_b T_i K_i |f'(w_b)|}{K_i b T_i^2} \]

(13.46)

and a combination of eqs. (13.30), (13.38), (13.44), and (13.46) provides an expression for the fin efficiency in terms of the parameters generated by the computer:

\[ \eta = \frac{f'(w_b) \left( w_i + \int_{w_i}^{w_b} dw \right)}{\left[ f(w_b) \right] \left( \int_{w_i}^{w_b} dw \right)} \]

(13.47)

The numerical solution to eq. (13.35) is the key to the development of design charts for the fin efficiency as a function of the profile number. The reasoning is:

1. The value of \( w_i = K_i K_1 \), which is used as a starting point, basically fixes the fin taper.
2. The numerical solutions, which are based on a fixed value of \( K_2/K_1 T_a^4 \), yield at a value of \( Z = T_b/T_a = f(w_b) \) a parameter that is more useful:

\[
\frac{K_2}{K_1 T_b^4} = \frac{1}{Z^4} \frac{K_2}{K_1 T_a^4}
\]

This is the environmental parameter used in Figs. 13.7 to 13.9 for the rectangular profile.

3. For any value of \( Z = f(w_b) \) and \( w_a = K_3 K_4 \), the environmental parameter \( K_2/K_1 T_a^4 \) is fixed, and the profile number and fin efficiency may be calculated from eqs. (13.44) and (13.47), respectively.

4. Design plots with the fin efficiency as the ordinate as a function of the profile number as the abscissa may be obtained for several values as the environmental parameter \( K_2/K_1 T_a^4 \). Each set of plots is for a particular value of \( \lambda = \delta_a/\delta_b \).

Figures 13.12 through 13.15 are the result of this procedure. These figures are for the trapezoidal fin with \( \lambda = 0.25, 0.50, \) and \( 0.75 \). Because of computational difficulties, the triangular profile (Fig. 13.15) with a theoretical \( \lambda = 0.00 \) was obtained using \( \lambda = 0.01 \). The curves designated theoretical optimum have been inserted to show the ideal for a design. These optima occur when the slope of each of the curves is \(-\frac{1}{3}\), and this is derived in Section 14.2.2.

**Example 13.5: Radiation to Free and Non-Free Space.** A longitudinal fin of trapezoidal profile is fabricated of a lampblack-coated steel fin (\( k = 30 \text{ W/m-K} \), \( \varepsilon = 0.95 \)) is 15.24 cm high, 0.9525 cm thick at the base, 0.4763 cm thick at the tip, and 1.3716 m long, and the base temperature is 445 K. How much heat is dissipated by the fin (a) to free space, and (b) if the fin receives 845 W from the surroundings?

**SOLUTION.** Here, in both cases,

\[
\lambda = \frac{\delta_a}{\delta_b} = \frac{0.4763}{0.9525} = 0.50
\]

\[
K_1 = 2\sigma \varepsilon = (2)(5.669 \times 10^{-8})(0.95) = 1.077 \times 10^{-7}
\]

and

\[
\zeta = \frac{K_1 T_a^4 b^2}{k \delta_b} = \frac{(1.077 \times 10^{-7})(445)^3(0.1524)^2}{(30)(0.009525)} = 0.771
\]

(a) For free space, use Fig. 13.13 and read at \( \zeta = 0.771 \) and \( K_2/K_1 T_a^4 = 0 \)

\[
\eta = 0.548
\]
Figure 13.12 Fin efficiency for a longitudinal radiating fin of trapezoidal profile with a taper ratio of 0.75.

and then

\[ q_v = K_1 L b T_b^3 \eta \]
\[ = (1.077 \times 10^{-7})(1.3716)(0.1524)(445)^4(0.548) \]
\[ = 483.8 \text{ W} \]

(b) If the fin receives 845 W from the surroundings, the environmental parameter will be

\[ \frac{K_2}{K_1 T_b} = \frac{845}{(1.077 \times 10^{-7})(445)^4} = 0.20 \]

and from Fig. 13.13, read at \( \zeta = 0.771 \) and \( K_2/K_1 T_b^4 = 0.20 \).
Figure 13.13  Fin efficiency for a longitudinal radiating fin of trapezoidal profile with a taper ratio of 0.50.

\[ \eta = 0.432 \]

and then

\[ q_b = K_1 L b T_b^4 \eta \]

\[ = (1.077 \times 10^{-7})(1.3716)(0.1524)(445)^4(0.432) \]

\[ = 381.4 \text{ W} \]

13.4 USE OF THE CASCADE ALGORITHM

Smith (1992) has used the cascade algorithm

\[ Y_{in} = \frac{r_{22} + r_{21}(q_u/T_u)}{r_{11} + r_{12}(q_u/T_u)} \]  \hspace{1cm} (7.6)
to predict the performance of longitudinal radiating fins of rectangular, trapezoidal, and triangular profile. His analysis method divides the fin under consideration into 100 subfins and uses eq. (6.6) to effectively cascade the 100 subfins into an overall single entity.

The heat transfer coefficient employed derives from the radiative dissipation from an element of fin with surface area $2L \, dx$ (two-sided dissipation)

$$ dq_r = \sigma \varepsilon F_t (2L \, dx) (T_{av}^4 - T_s^4) $$

where $T_{av}$ is the average temperature of the fin surface element. For a small body in a large enclosure (the fin in outer space), the shape factor $F_t = 1$, and the emissivity factor $F_r = \varepsilon$. Thus with $T_s$ effectively at 0 K,

$$ dq_r = 2 \varepsilon \sigma L T_{av}^4 \, dx $$

Figure 13.14  Fin efficiency for a longitudinal radiating fin of trapezoidal profile with a taper ratio of 0.25.
Figure 13.15 Fin efficiency for a longitudinal radiating fin of triangular profile.

Fin efficiency & dimensionless

Profile number $K_1/L_b$, dimensionless

$K_2/L_b$, dimensionless

Theoretical optimum
and a radiative heat transfer coefficient may be proposed to fit

\[ dq_r = h_r dS T^{3}_{av} \]

where \( dS = 2L \, dx \) and where

\[ h_r = \sigma \epsilon T^{3}_{av} \]

Environmental effects can be accommodated by adjusting \( h_r \) to the general form

\[ h_r = \sigma \epsilon \left( T^{3}_{av} - \frac{K_2}{T^{\lambda}_{av}} \right) \]

so that for any of the three profiles considered,

\[ m = \left( \frac{h \sin \kappa}{k} \right)^{1/2} \]

where \( \kappa = \text{arctan} \frac{\delta_0}{2b} \) is the taper angle.

Smith's computational procedure is described in the flowchart of Fig. 13.16. Note that the analyst specifies all pertinent thermal requirements and physical and thermal parameters, as well as a starting point for the analysis. Moreover, Smith has been able to match the Mackay–Bacha (1962) curves, and Figs. 13.17 to 13.20 show some of his results.

Figure 13.17 gives the heat dissipation from a longitudinal radiating fin of rectangular profile as a function of fin height and environmental parameter, \( K_2/K_1 T^{\lambda}_{av} \). Fin parameters for this case were \( \delta = 1 \, \text{cm}, \, L = 1 \, \text{m}, \, T_{av} = 400 \, \text{K}, \, \epsilon = 0.85, \, k = 209.4 \, \text{W/m-K}, \) and density, \( \rho = 0.26 \, \text{kg/cm}^3 \). A presentation of the temperature profile for this fin with \( b = 50 \, \text{cm} \) is provided in Fig. 13.18, where the abscissa reckons distance from the fin tip.

Smith has also attempted to optimize the radiating fin. Figure 13.19 considers the same rectangular fin but without a fin height or thickness specification. Several profile areas are considered and Fig. 13.19 clearly shows that for each profile area, there is a maximum heat dissipation associated with a particular width. Figure 13.20 compares three profiles and shows that for a given profile area, the triangular profile outperforms both the rectangular profile and trapezoidal profiles, with taper ratio \( \lambda = 0.50 \). It also clearly shows that, an optimum may be obtained at a particular fin width.

The discussion concerning optimum values will continue in Chapter 14.

13.5 LONGITUDINAL RADIATING FIN WITH CONSTANT-TEMPERATURE GRADIENT

Mackay (1960) took the idea from the convective case that the longitudinal radiating fin of least material would be a fin with a constant-temperature gradient and studied a radiating fin with a linear temperature profile

\[ T = T_a + \frac{\lambda}{b} (T_b - T_a) \]  \hspace{1cm} (13.48)
with a constant-temperature gradient

\[ \frac{dT}{dx} = \frac{T_b - T_a}{b} \]  

(13.49)

This fin does not represent the contour of least material, but it entails only a slight increase in weight over the profile described by Wilkins (1960a). However, it does not have a requirement that the tip temperature be zero.

Suppose that the fin with constant-temperature gradient possesses the arbitrary profile shown in Fig. 13.21. The origin of the height coordinate \( x \) is at the fin tip and the heat conducted past any plane located at \( x \) will be

\[ q_x = 2kf_2(x)L \frac{dT}{dx} \]
Figure 13.17  Heat dissipation from a longitudinal radiating fin of rectangular profile.

Figure 13.18  Temperature profiles for a longitudinal radiating fin of rectangular profile (Fig. 13.17).
Figure 13.19  Heat dissipation from a longitudinal radiating fin of rectangular profile as a function of fin width for several profile areas.

Figure 13.20  Comparison of the heat dissipation from various longitudinal radiating fin profiles.
Figure 13.21  Longitudinal radiating fin of arbitrary profile for analysis of the constant-temperature gradient.

If this is substituted into eq. (13.49), the result is

$$q_b = \frac{2kf_2(x)L(T_b - T_o)}{b}$$  \hspace{1cm} (13.50)

The heat represented by eq. (13.50) must eventually be dissipated to the surroundings by radiation. Thus

$$\int_0^T dq = \int_0^L (K_1T_1 - K_2)L dx$$

and this may be expressed in terms of temperature by once again using eq. (13.49):

$$q = \int_{T_o}^T \frac{K_1T_1^4 - K_2}{T_b - T_o}Lb dT$$  \hspace{1cm} (13.51)

When eqs. (13.50) and (13.51) are equated, integrated, and simplified, the result is

$$\frac{2kf_2(x)(T_b - T_o)}{b} = \frac{K_1b(T_b^5 - T_o^5)}{5(T_b - T_o)} - \frac{K_2b(T - T_o)}{T_b - T_o}$$  \hspace{1cm} (13.52)

This may be rearranged using $z = T / T_o$ and $Z = T_b / T_o$ to provide a solution for the fin profile function:

$$f_2(x) = \frac{K_1b^3T_b^5(z^5 - 1)}{10kZ^5(Z - 1)^2} - \frac{K_2b^2Z(z - 1)}{2kT_b(Z - 1)^2}$$  \hspace{1cm} (13.53)
From eq. (13.48),

$$z = \frac{T}{T_a} = 1 + \frac{\chi}{b} \left( \frac{T_b}{T_a} - 1 \right) = 1 + \frac{\chi}{b} (Z - 1) \quad (13.54)$$

and this may be substituted into eq. (13.53), which, upon simplification, gives

$$f_2(x) = \frac{K_1 b^2 T_b}{10k Z^3 \Phi^2} \left[ \left( 1 + \frac{\chi}{b} \Phi \right)^5 - 1 - \frac{5K_2 Z^4}{K_1 T_b^4} \frac{\chi}{b} \Phi \right] \quad (13.55)$$

where $\Phi = Z - 1$. At the fin base, where $x = b$ and $z = Z = T_b/T_a$, eq. (13.55) reduces to

$$f_2(b) = \frac{\delta_b}{2} = \frac{K_1 b^2 T_b^3}{10k Z^3 \Phi^2} \left( Z^5 - 1 - \frac{5K_2 Z^4}{K_1 T_b^4} \Phi \right) \quad (13.56)$$

and at the fin tip where $x = 0$ and $z = T_a/T_a = 1$, L'Hôpital's rule may be employed to show that $f_2(x) = 0$. The fin profile is described by eq. (13.55) in terms of the environmental extremes, environmental effects, and certain fin parameters. The profile is constrained to have a narrowing characteristic from base to tip and a zero thickness at the tip. The base-to-tip temperature ratio $Z$ must be chosen discriminatingly. Its limit may be determined by focusing on the portion of the fin close to the tip, where $x/b$ is small and by working with eq. (13.55). The fin profile must always be positive; hence the term in brackets must always be positive. This requires that

$$\left( 1 + \frac{\chi}{b} \Phi \right)^5 > 1 + \frac{5K_2 Z^4}{K_1 T_b^4} \left( \frac{x}{b} \Phi \right)$$

and the term on the left may be subjected to a binomial expansion. Because $x/b$ is small, a good approximation is obtained by considering only the first two terms of the expansion, and thus

$$1 + 5 \frac{x}{b} \Phi + \cdots > 1 + \frac{5K_2 Z^4}{K_1 T_b^4} \left( \frac{x}{b} \Phi \right)$$

Cancellation of the common terms gives the inequality

$$\frac{K_2 Z^4}{K_1 T_b^4} < 1$$

from which it is observed that

$$Z_{\text{max}} = \left( \frac{K_1 T_b^4}{K_2} \right)^{1/4} \quad (13.57)$$

and

$$T_{a, \text{min}} = \left( \frac{K_2}{K_1} \right)^{1/4} \quad (13.58)$$
The approximate limits on the tip temperature are given by eqs. (13.57) and (13.58). Observe that for free space where \( K_2 = 0, T_o = 0 \), which is the approximate temperature of free space. The heat radiated by the fin with constant-temperature gradient is given by eq. (13.51). Integration between \( T_o \) and \( T_b \) and simplification with \( Z = T_b/T_o \) yields

\[
q_b = \frac{K_1 L b T_o^4 (Z^5 - 1)}{5\Phi} - K_2 L b \tag{13.59}
\]

The ideal heat dissipation is \( q_{id} = K_1 L b T_b^4 \), and the fin efficiency is

\[
\eta = \frac{q_b}{q_{id}} = \frac{Z^5 - 1}{5Z^4\Phi} - \frac{K_2}{K_1 T_b^4} \tag{13.60}
\]

The profile number is useful in design and performance calculations and may be obtained from eq. (13.56), which may be written as

\[
\delta_n = \frac{K_1 b^2 T_b^3}{k} \frac{Z^5 - 1 - (5K_2 Z^4/K_1 T_b^4)\Phi}{5Z^4\Phi^2}
\]

so that

\[
\zeta = \frac{K_1 b^2 T_b^3}{k\delta_n} = \frac{5Z^4\Phi^2}{Z^5 - 1 - (5K_2 Z^4/K_1 T_b^4)\Phi} \tag{13.61}
\]

Figure 13.22 shows the efficiency of the fin with constant-temperature gradient as a function of the profile number. The dashed line represents the approximate limit given by eq. (13.57). Figure 13.23 is a plot of the profile number as a function of the base-to-tip temperature-excess ratio \( Z \), as given by eq. (13.61). The several curves in these figures are due to different values of the environmental parameter \( K_2/K_1 T_b^4 \). Figures 13.22 and 13.23 contain all the data necessary for design and performance calculations for the fin with a constant-temperature gradient.

### 13.6 PARABOLIC RADIATING PROFILES

Kotan and Armas (1965) investigated radiation from longitudinal fins of concave and convex parabolic profiles whose cross-sectional area functions for unit fin length are given, respectively, by

\[
A(x) = A_b \left(1 - \frac{x}{b}\right)^2 \tag{13.62a}
\]

and

\[
A(x) = A_b \left(1 - \frac{x}{b}\right)^{1/2} \tag{13.62b}
\]

where \( A_b = b_o L \). The differential equation for the temperature ratio \( \gamma = T/T_o \) is obtained by considering a differential length element \( dx \). The difference between the
heat entering and leaving the element by conduction is equated to the heat dissipated from the faces of the element by radiation. In dimensionless form with \( X = x/b \) and \( \zeta, K_1 \) and \( K_2 \) as defined previously, the differential equation for the concave parabolic fin is

\[
(1 - X)^2 \frac{d^2 y}{dx^2} + 2(1 - X) \frac{dy}{dX} - \zeta \left( y^4 - \frac{K_2}{K_1 T_b^4} \right) = 0 \quad (13.63a)
\]

and for the convex parabolic fin

\[
(1 - X)^{1/2} \frac{d^2 y}{dx^2} + \frac{1}{2(1 - X)^{1/2}} \frac{dy}{dX} - \zeta \left( y^4 - \frac{K_2}{K_1 T_b^4} \right) = 0 \quad (13.63b)
\]
Kohan and Armas (1965) obtained computer solutions for eqs. (13.63) with the boundary conditions

\[ y(X = 0) = 1 \quad \text{and} \quad \frac{dy}{dX} \bigg|_{X=1} = 0 \]
Solutions for the heat dissipation and fin efficiency were also obtained.

13.7 RADIAL RADIATING FINS

The general case of the radial fin of trapezoidal profile, which allows for the treatment of specific cases of the rectangular and triangular profiles, depends on a profile function

\[ f_z(r) = \frac{\delta_a}{2} + \frac{R}{2}(\delta_b - \delta_a) \]  
(13.64)

where

\[ R = \frac{r_a - r}{r_a - r_b} \]

When \( \delta_b = \delta_a \) and \( \delta_a = 0 \), the specific cases of the fins of rectangular and triangular profile occur. The terminology and coordinate system for the radial fin of trapezoidal profile are shown in Fig. 13.24. Observe that the radial height coordinate has its origin at the center of the fin and is positive in the outward direction. The difference in heat conduction into and out of the element \( dr \) is

\[ dq = k \frac{d}{dr} \left\{ 2\pi r [2 f_z(r)] \frac{dT}{dr} \right\} dr \]

which equals the heat dissipation from the faces of the element \( dr \):

\[ dq = 2\pi r (K_1 T_4 - K_2) \, dr \]

where \( K_1 = 2\pi \epsilon \) if the radiation occurs from both faces of the fin. Thus

\[ T_a, T_b, f_z(r), -f_z(r) \]

\[ \delta_a, \delta_b, r_a, r_b, dr \]

**Figure 13.24** Coordinate system for a radiating radial fin of trapezoidal profile.
\[ k \frac{d}{dr} \left[ r^2 \frac{d}{dr} \left( 2f_z(r) \frac{dT}{dr} \right) \right] - K_1 r T_4 + K_2 r = 0 \] (13.65)

The taper and radius ratios have been defined previously:
\[ \lambda = \frac{\delta_h}{\delta_b} \] (13.40)

and
\[ \rho = \frac{r_h}{r_o} \] (1.36)

Equation (13.64) can be written in terms of these parameters:
\[ \delta(r) = 2f_z(r) = \frac{\delta_b \rho}{1 - \rho} \left[ \frac{1}{\rho} - \lambda + \frac{r}{r_h} (\lambda - 1) \right] \] (13.66)

Then
\[ 2f_z'(r) = \frac{\delta_b \rho (\lambda - 1)}{(1 - \rho)r_h} \]

and differentiation of eq. (13.65) gives
\[ r \delta_b \rho \left[ \frac{1}{\rho} - \lambda + \frac{r}{r_h} (\lambda - 1) \right] \frac{d^2 T}{dr^2} \]
\[ + \left[ \frac{r \delta_b \rho (\lambda - 1)}{(1 - \rho)r_h} + \frac{\delta_b \rho}{1 - \rho} \left[ \frac{1}{\rho} - \lambda + \frac{r}{r_h} (\lambda - 1) \right] \right] \frac{dT}{dr} \]
\[ - \frac{K_1}{k} T^4 + \frac{K_2}{k} r = 0 \] (13.67)

With \( \gamma = T/T_b \) and \( \xi = r/r_h \),
\[ \frac{dT}{dr} = T_b \frac{dy}{d\xi} \frac{d\xi}{dr} = \frac{T_b}{r_h} \frac{dy}{d\xi} \]

and
\[ \frac{d^2 T}{dr^2} = \frac{T_b}{r_h^2} \frac{d^2 y}{d\xi^2} \]

With substitution and rearrangement, eq. (13.67) becomes
\[ \left[ \frac{1}{\rho} - \lambda + \xi (\lambda - 1) \right] \frac{d^2 y}{d\xi^2} \]
\[ + \left[ \frac{1 - \gamma^2 \rho}{\rho \xi} - 2(\lambda - 1) \right] \frac{dy}{d\xi} - \frac{\xi (1 - \rho)}{\rho} \left( \gamma^4 - \frac{K_2}{K_1 T_b^4} \right) = 0 \] (13.68)

where
\[ \zeta = \frac{K_1 T_0^4 r_a^4}{k h} \]

is the profile number for the radial radiating fin.

The governing differential equation for the temperature profile is eq. (13.68). It is nonlinear and must be solved numerically for the dimensionless temperature \( \theta \) as a function of the dimensionless ratio \( \xi \) for assumed values of the parameters \( \zeta, \rho, \lambda, \) and \( K_2/K_1 T_b^4 \). The solution is subject to the condition that \( d\theta/d\xi = dT/dr \) is equal to zero at \( \xi = r_a/r_b = 1/\rho \).

The heat transferred from the fin may be evaluated by applying Fourier's law at the fin base,

\[ q_b = -2\pi k \delta_b r_b \frac{dT}{dr} \bigg|_{r=r_b} \]

or in terms of the dimensionless parameters,

\[ q_b = -2\pi k \delta_b T_b \frac{d\theta}{d\xi} \bigg|_{\xi=1} \]  \hspace{1cm} (13.69)

The fin efficiency will be this actual heat flow divided by the ideal heat flow with no heat gained from the surroundings:

\[ \eta = \frac{q_b}{q_{id}} = \frac{-2\pi k \delta_b T_b \frac{d\theta}{d\xi} \bigg|_{\xi=1}}{\pi K_1 T_b^4 (r_a^2 - r_b^2)} = \frac{2\rho^2 \frac{d\theta}{d\xi} \bigg|_{\xi=1}}{\zeta(1 - \rho^2)} \]  \hspace{1cm} (13.70)

The numerical solution of eq. (13.68) yields the values of \( \theta \) and \( d\theta/d\xi \) as a function of \( \xi \). Typical plots of the numerical data for a fin 10.16 cm by 20.32 cm by 0.635 cm thick at the base, fabricated of a metal with a thermal conductivity of 106.4 W/m·K and an emissivity of 0.90 radiating from one face, are given in Fig. 13.25. Note that the curves are for several taper ratios \( \lambda \) and for several values of \( K_2/K_1 T_b^4 \). For these dimensions, the profile number is \( \zeta = 0.0147 \) and \( \rho = 0.50 \).

The value of the derivative \( d\theta/d\xi \) at \( \xi = 1.0 \) is always negative, and the fin efficiency can be computed using eq. (13.70) with particular values of \( \rho, \lambda, \zeta, \) and the environmental parameter. Unfortunately, the large number of variables requires a multiplicity of curves for the presentation of the fin efficiency. Table 13.2 lists and summarizes the curves included here.

**Example 13.6: Radiation from Radial Fins (Fin Analysis).** A radial fin has a base temperature of 445 K and is fabricated of a material having thermal conductivity \( k = 82.1 \) W/m·K. The fin is 10.16 cm inside diameter by 25.4 cm outside diameter by 0.3125 cm thick at the base and radiates from one face, which has an emissivity of 0.809. Solar energy is received by the dissipating face at a rate of 1303.7 W/m² and the fin material has an absorptivity of 0.276 at the wavelength of solar radiation. Determine the heat dissipation from radial fins of (a) rectangular profile, (b) trapezoidal profile with a tip width of 0.1588 cm, and (c) triangular profile.
Figure 13.25  Some temperature profiles obtained from a computer solution of eq. (13.70). For these profiles, $\zeta = 0.0147$ and $\rho = 0.5$. 
TABLE 13.2 Summary of Radial Fin Efficiency Data

<table>
<thead>
<tr>
<th>Figure</th>
<th>$\lambda$</th>
<th>$\frac{K_2}{K_1 T_b^4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.26</td>
<td>1.00</td>
<td>0.00</td>
</tr>
<tr>
<td>13.27</td>
<td>1.00</td>
<td>0.20</td>
</tr>
<tr>
<td>13.28</td>
<td>1.00</td>
<td>0.40</td>
</tr>
<tr>
<td>13.29</td>
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<td>0.00</td>
</tr>
<tr>
<td>13.30</td>
<td>0.75</td>
<td>0.20</td>
</tr>
<tr>
<td>13.31</td>
<td>0.75</td>
<td>0.40</td>
</tr>
<tr>
<td>13.32</td>
<td>0.50</td>
<td>0.00</td>
</tr>
<tr>
<td>13.33</td>
<td>0.50</td>
<td>0.20</td>
</tr>
<tr>
<td>13.34</td>
<td>0.50</td>
<td>0.40</td>
</tr>
<tr>
<td>13.35</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>13.36</td>
<td>0.00</td>
<td>0.20</td>
</tr>
<tr>
<td>13.37</td>
<td>0.00</td>
<td>0.40</td>
</tr>
</tbody>
</table>

SOLUTION. For the given fin and the conditions imposed,

$$K_1 = \sigma \epsilon = 5.669 \times 10^{-3} \times (0.809) = 4.588 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4$$

and

$$K_2 = 1303.7 \alpha = (1303.7)(0.276) = 359.8 \text{ W/m}^2$$

Environmental parameter:

$$\frac{K_2}{K_1 T_b^4} = \frac{359.8}{(4.588 \times 10^{-8})(445)^4} = 0.20$$

Profile number:

$$\zeta = \frac{K_1 T_b^3 r_b^2}{k \delta_b} = \frac{(4.588 \times 10^{-8})(445)^3(0.0508)^2}{(21.1)(0.003175)} = 0.040$$

Radius ratio:

$$\rho = \frac{r_b}{r_o} = \frac{d_b}{d_i} = \frac{10.16}{25.40} = 0.40$$

Surface area of one face (all profiles):

$$S = \pi (r_o^2 - r_b^2) = \pi [(0.1270)^2 - (0.0508)^2] = 0.0426 \text{ m}^2$$

Ideal heat dissipation (all profiles):

$$q_{id} = K_1 S T_b^4 = (4.588 \times 10^{-8})(0.0426)(445)^4 = 76.6 \text{ W}$$

(a) Rectangular profile, $\lambda = 1$. From Fig. 13.27 at $\rho = 0.40$ and $\zeta = 0.04$, read
Figure 13.26  Radiation fin efficiency of a radial fin of rectangular profile. Taper ratio, $\lambda = 1.00$; environmental factor, $K_2/K_1 T_e^4 = 0.00$. 

Profile number, dimensionless
Figure 13.27  Radiation fin efficiency of a radial fin of rectangular profile. Taper ratio, $\lambda = 1.00$; environmental factor, $K_2/K_1 T^4 = 0.20$. 
and then

\[ q_b = \eta q_{id} = (0.676)(76.6) = 51.8 \text{ W} \]

(b) Trapezoidal profile, \( \delta = 0.1588 \text{ cm} \):

\[ \lambda = \frac{\delta_b}{\delta_p} = \frac{0.1588}{0.3175} = 0.50 \]

From Fig. 13.33 at \( \rho = 0.40 \) and \( \zeta = 0.04 \), read

\[ \eta = 0.666 \]

and then

\[ q_b = \eta q_{id} = (0.666)(76.6) = 51.0 \text{ W} \]

(c) Triangular profile, \( \lambda = 0 \). From Fig. 13.36 at \( \rho = 0.40 \) and \( \zeta = 0.04 \), read

\[ \eta = 0.643 \]

and then

\[ q_b = \eta q_{id} = (0.643)(76.6) = 49.3 \text{ W} \]

Example 13.7: Radiation from Radial Fins (Fin Synthesis). A radial fin of rectangular profile is to have a base temperature of 450 K and is fabricated of a material having thermal conductivity \( k = 86 \text{ W/m K} \) and an emissivity of \( \epsilon = 0.81 \). The fin is to have an inner diameter of 10.16 cm and a thickness of 0.635 cm and is required to radiate 76 W from one face to free space. Determine the outer diameter required.

SOLUTION. For the given fin and the conditions imposed,

\[ K_1 = \sigma \epsilon = (5.669 \times 10^{-8})(0.81) = 4.592 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4 \]

and

\[ K_2 = 0 \quad \text{free space} \]

Environmental parameter:

\[ \frac{K_2}{K_1 T_b^4} = 0 \]

Profile number:

\[ \zeta = \frac{K_1 T_b^4 r_b^2}{k \delta_b} = \frac{(4.592 \times 10^{-8})(450)^3(0.0508)^2}{86(0.00615)} = 0.020 \]
Figure 13.28  Radiation fin efficiency of a radial fin of rectangular profile. Taper ratio, $\lambda = 1.00$; environmental factor, $K_2/K_1T_y^\infty = 0.40$. 
Figure 13.29  Radiation fin efficiency of a radial fin of trapezoidal profile. Taper ratio, $\lambda = 0.75$, environmental factor, $K_2 / K_1 T_{\infty}^4 = 0.00$. 
Figure 13.30: Radiation fin efficiency of a radial fin of trapezoidal profile. Taper ratio, $\lambda = 0.75$; environmental factor, $K_e/K_r = 0.20$. 

Radius ratio, $\rho = \rho_s/\rho_a$, dimensionless
Profile number, dimensionless

Efficiency %, dimensionless
Figure 13.31 Radiation fin efficiency of a radial fin of trapezoidal profile. Taper ratio, $\lambda = 0.75$; environmental factor, $K_2/K_1T_0^4 = 0.40$. 

Profile number, dimensionless:
- 0.015
- 0.02
- 0.025
- 0.03
- 0.035
- 0.04
- 0.045
- 0.05
- 0.055
- 0.06
- 0.065
- 0.07
- 0.075
- 0.08

Radius ratio, $\rho = r_0/r_2$, dimensionless:
- 0.25
- 0.3
- 0.35
- 0.4
- 0.45
- 0.5
- 0.55
- 0.6
- 0.65
- 0.7
- 0.75
- 0.8
Figure 13.32  Radiation fin efficiency of a radial fin of trapezoidal profile. Taper ratio, $\lambda = 0.50$; environmental factor, $K_2/K_1T_b^* = 0.00$. 
Figure 13.33  Radiation fin efficiency of a radial fin of trapezoidal profile. Taper ratio, $\lambda = 0.50$; environmental factor, $K_z/K_i T_0^4 = 0.20$. 

Profile number, dimensionless

$0.01$

$0.02$  
$0.03$

$0.04$

$0.06$

$0.08$

$0.10$

$0.15$

$0.20$

$0.25$

$0.30$

$0.35$

$0.40$

$0.45$

$0.50$

$0.55$

$0.60$

$0.65$

$0.70$

$0.75$

$0.80$

Radius ratio, $r_i/r_g$, dimensionless

Radiation fin efficiency, $\gamma$
Figure 13.34  Radiation fin efficiency of a radial fin of trapezoidal profile. Taper ratio, $\lambda = 0.50$; environmental factor, $K_2 / K_1 T_b^4 = 0.40$. 
Figure 13.35  Radiation fin efficiency of a radial fin of triangular profile. Taper ratio, $\lambda = 0.00$; environmental factor, $K_2/K_1 T_b^4 = 0.00$. 
Figure 13.36  Radiation fin efficiency of a radial fin of triangular profile. Taper ratio, $\lambda = 0.00$; environmental factor, $K_2/K_1 T_n^4 = 0.20$. 
Trial I. Assume an outer diameter $d_o = 25.4$ cm. The radius ratio is

$$\rho = \frac{r_b}{r_o} = \frac{d_b}{d_o} = \frac{10.16}{25.40} = 0.40$$

From Fig. 13.26 at $\rho = 0.40$ and $\zeta = 0.020$, read

$$\eta = 0.916$$

With

$$S = \pi (r_o^2 - r_i^2) = \pi [(0.1270)^2 - (0.0508)^2] = 0.0426 \text{ m}^2$$

and with the ideal heat dissipation

$$q_{id} = K_1 ST_i^4 = (4.592 \times 10^{-8})(0.0426)(450)^4 = 80.1 \text{ W}$$

the required value of $\eta$ will be

$$\eta = \frac{q_b}{q_{id}} = \frac{76}{80.2} = 0.948 > 0.916 \quad \text{(no check)}$$

Trial II. Assume an outer diameter $d_o = 26.5$ cm. The radius ratio is

$$\rho = \frac{r_b}{r_o} = \frac{d_b}{d_o} = \frac{10.16}{26.50} = 0.383$$

From Fig. 13.26 at $\rho = 0.383$ and $\zeta = 0.020$, read

$$\eta = 0.905$$

With

$$S = \pi (r_o^2 - r_i^2) = \pi [(0.1325)^2 - (0.0508)^2] = 0.0470 \text{ m}^2$$

and with the ideal heat dissipation

$$q_{id} = K_1 ST_i^4 = (4.592 \times 10^{-8})(0.0470)(450)^4 = 88.6 \text{ W}$$

the required value of $\eta$ will be

$$\eta = \frac{q_b}{q_{id}} = \frac{76}{88.6} = 0.858 < 0.905 \quad \text{(no check)}$$

Trial III. Assume an outer diameter $d_o = 26.0$ cm. The radius ratio is

$$\rho = \frac{r_b}{r_o} = \frac{d_b}{d_o} = \frac{10.16}{26.00} = 0.391$$

From Fig. 13.26 at $\rho = 0.391$ and $\zeta = 0.020$, read

$$\eta = 0.902$$
Figure 13.37 Radiation fin efficiency of a radial fin of triangular profile. Taper ratio, $\lambda = 0.00$; environmental factor, $K_2/K_1T_b^4 = 0.40$. 

Profile number, dimensionless

0.01

0.02

0.03

0.04

0.05

Radius ratio, $\rho = \rho_0/\rho_b$, dimensionless

0.06

0.07

0.08

0.09

0.10

0.15

0.20

0.30

0.40

0.50

0.60

0.70

0.80

0.90

1.0

0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80
With

\[ S = \pi (r_a^2 - r_b^2) = \pi [(0.1300)^2 - (0.0508)^2] = 0.0450 \text{ m}^2 \]

and with the ideal heat dissipation

\[ q_{id} = K_i S T_b^4 = (4.592 \times 10^{-5})(0.0450)(450)^4 = 84.7 \text{ W} \]

the required value of \( \eta \) will be

\[ \eta = \frac{q_b}{q_{id}} = \frac{76}{84.7} = 0.897 \approx 0.902 \quad \sqrt{\text{1}} \]

The required outer diameter is 26.0 cm.

### 13.8 CLOSURE

In this chapter, the analysis and design of fins with radiation as the sole heat transfer mode between the fin surface and the surroundings for both free space and non-free space conditions has been considered. In Chapter 14, attention turns to the optimization of these fins as well as those fins that operate in a combined convection-radiation mode.

### 13.9 NOMENCLATURE

**Roman Letter Symbols**

- \( A \): cross-sectional area, m\(^2\)
- \( A(x) \): cross-sectional area function, m\(^2\)
- \( a \): fin tip location, m
- \( B(a, b) \): beta function, dimensionless
- \( b \): fin height, m
- \( C \): constant, dimensionless
- \( d \): differential or derivative, dimensionless
- \( F \): factor, dimensionless
- \( f \): function, dimensionless
- \( h \): heat transfer coefficient, W/m\(^2\) · K
- \( K \): constants, dimensions vary
- \( k \): thermal conductivity, W/m-K
- \( L \): fin length, m
- \( m \): fin performance parameter, m\(^{-1}\)
- \( p \): derivative of temperature with respect to height, K/W
- \( q \): heat flow, W
- \( R \): radius ratio, dimensionless
- \( r \): radius, m; radial coordinate, m
- \( S \): surface area, m\(^2\)
- \( T \): temperature, K
\( u \) temperature ratio, dimensionless
\( v \) transformed temperature variable, dimensionless
\( w \) transformed variable, K
\( X \) height, dimensionless
\( x \) height coordinate, m
\( Y \) thermal admittance, W/K
\( y \) temperature ratio, dimensionless
\( Z \) temperature ratio, dimensionless
\( z \) temperature ratio, dimensionless

**Greek Letter Symbols**
\( \Gamma \) gamma function, dimensionless
\( \Delta \) change in, dimensionless
\( \delta \) fin thickness, m
\( \delta(x) \) thickness function, m
\( \delta(r) \) thickness function, m
\( \epsilon \) emissivity, dimensionless
\( \xi \) profile function, dimensionless
\( \eta \) fin efficiency, dimensionless
\( \kappa \) taper angle, rad
\( \Lambda \) fin taper, dimensionless
\( \lambda \) thickness ratio, dimensionless
\( \xi \) radius function, dimensionless
\( \rho \) radius ratio, dimensionless
\( \sigma \) Stefan–Boltzmann constant, W/m\(^2\) · K\(^4\)
\( \Upsilon \) combination of terms
\( \upsilon \) combination of terms, dimensionless
\( \Phi \) combination of terms, dimensionless
\( \psi \) combination of terms, dimensionless

**Roman Letter Subscripts**
\( a \) arrangement; fin tip
\( av \) average
\( b \) fin base
\( id \) ideal
\( in \) input condition
\( max \) maximum value
\( min \) minimum value
\( r \) radiation
\( s \) surroundings
\( u \) incomplete beta function
\( x \) \( x \)-coordinate direction
Greek Letter Superscript
\( \varepsilon \)  emissivity

Symbolic Superscripts
\( ' \)  first derivative
\( '' \)  second derivative