Fluid Dynamics: Boundary Layers

Analogous Equations

- if two normalized (dimensionless) equations take the same form the equations are *analogous*

Reynolds Analogy

The momentum and energy *boundary layer equations* are *analogous* if there is a negligible pressure gradient \( (dp^*/dx^* \sim 0) \) and the \( Pr \sim 1 \)

\[
\begin{align*}
\frac{u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} &= \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^*^2} \\
\frac{u^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} &= \frac{1}{Re_L} \frac{\partial^2 T^*}{\partial y^*^2}
\end{align*}
\]

Given *equivalent* boundary conditions, the solutions take the same form

\[
\begin{align*}
\frac{u^*}{\partial x^*} &= \frac{\partial T^*}{\partial y^*} \\
\frac{\partial u^*}{\partial y^*}|_{y^*=0} &= \frac{\partial T^*}{\partial y^*}|_{y^*=0} \\
C_f \left( \frac{Re}{2} \right) &= Nu
\end{align*}
\]
Fluid Dynamics: Boundary Layers

Reynolds Analogy

Defining a new non-dimensional number

\[ St = \frac{h}{\rho VC_p} = \frac{Nu}{RePr} \]

Stanton number

The *Reynolds Analogy* is defined as

(when \( Pr = 1 \))

\[ \frac{C_f}{2} = St \]

The *Reynolds Analogy* implies that under certain conditions (no pressure gradient, \( Pr = 1 \)) if the velocity parameters are known than the heat transfer parameters can be determined (and vice versa)

Modified Reynolds Analogy: Chilton-Colburn (empirical)

\[ \frac{C_f}{2} = St Pr^{2/3} = j_H \Rightarrow 0.6 < Pr < 60 \]

Colburn \( j \) factor

laminar flows: valid for \( dp^*/dx^* \sim 0 \)
turbulent flows: generally valid without restriction on \( dp^*/dx^* \)
External Convection: Overview

• Determining Heat Transfer Coefficients

  – determining heat transfer coefficients requires an accurate knowledge of the **flow field**

  – few (pseudo-)analytical solutions exist (especially for turbulent flow)
    • similarity solutions, etc.

  – heat transfer coefficient relations are largely **empirical** and are presented based on the **Nusselt number**

\[
Nu = \left. \frac{dT^*}{dy^*} \right|_{y=0} = \frac{hL}{k_f} = f(x^*, Re_L, Pr) \quad \text{local Nusselt number} \\
\overline{Nu} = \frac{\bar{h}L}{k_f} \quad \text{average Nusselt number}
\]

  – The Nusselt number (and heat transfer coefficient) are functions of the fluid properties \((\nu, \rho, \alpha, c, k_f)\)

    • the effect of variable properties may be considered by evaluating all properties at the **film temperature**

\[
T_f = \frac{T_s + T_\infty}{2}
\]

    • most accurate solutions often require iteration on the film properties

D. B. Go
Fluid Dynamics: Boundary Layers

- Transition

Laminar and turbulent boundary layers have different heat transfer characteristics
  - turbulent mixing typically increases heat transfer

Critical Reynolds number approximates the location where the flow transitions from laminar to turbulent flow

\[ \text{Re}_{x,c} \equiv \frac{Vx_c}{\nu} \]

- \( \text{Re}_{x,c} \approx 10^5 \) external (flat plate) flow
- \( \text{Re}_{D,c} \approx 10^3 \) internal (duct) flow

D. B. Go
Transition leads to a significant increase in the local heat transfer coefficient
External Convection: Overview

External Flows

- boundary layers develop freely \textit{without constraint} (compare to a internal/duct flow)
- boundary layer may be laminar, laminar and turbulent, or entirely turbulent
- simplest external flow: \textit{flat plate in parallel flow}

Determining external flow conditions

compute: \[ \text{Re}_L = \frac{u_\infty L}{\nu} \]

compare to \textit{critical Reynolds number} \( \text{Re}_{x,c} \sim 10^5 \text{ external (flat plate) flow} \)

- laminar flow along length of flat plate
- transition to turbulent flow at critical length

\[ \frac{x_c}{L} = \frac{\text{Re}_{x,c}}{\text{Re}_L} \]
External Convection: Overview

- **Transition to Turbulence**
  - critical Reynolds number affected by free stream turbulence and surface roughness of plate
    - nominally
      \[ \text{Re}_{x,c} \sim 10^5 \text{ laminar free stream & smooth plate} \]
  - if the boundary layer is “tripped” at the leading edge:
    - flow is turbulent along entire length of flat plate

\[ \text{Re}_{x,c} = 0 \rightarrow x_c = 0 \]

**Average parameters**

\[
\bar{\tau}_{s,L} = \frac{1}{L} \left( \int_{0}^{x_c} \tau_{s,x,\text{lam}} \, dx + \int_{x_c}^{L} \tau_{s,x,\text{turb}} \, dx \right) \\
\bar{h}_{L} = \frac{1}{L} \left( \int_{0}^{x_c} h_{x,\text{lam}} \, dx + \int_{x_c}^{L} h_{x,\text{turb}} \, dx \right)
\]
External Convection: Overview

- **Thermal Conditions at the Surface** (idealized)
  - uniform heat flux $q_s''$
  - uniform surface temperature $T_s$
  - unheated starting length

 delays thermal boundary layer growth

\[ x < \xi \Rightarrow T_s = T_\infty \]
\[ x > \xi \Rightarrow T_s \text{ or } q_s'' \]