1-D Steady Conduction: Plane Wall

Governing Equation:

\[ \frac{d^2T}{dx^2} = 0 \]

Dirichlet Boundary Conditions:

\[ T(0) = T_{s,1} \quad ; \quad T(L) = T_{s,2} \]

Solution:

\[ T(x) = T_{s,1} + \left( T_{s,2} - T_{s,1} \right) \frac{x}{L} \]

temperature is not a function of \( k \)

Heat Flux:

\[ q''_x = -k \frac{dT}{dx} = \frac{k}{L} \left( T_{s,1} - T_{s,2} \right) \]

heat flux/flow are a function of \( k \)

Heat Flow:

\[ q_x = -kA \frac{dT}{dx} = \frac{kA}{L} \left( T_{s,1} - T_{s,2} \right) \]

Notes:

- \( A \) is the cross-sectional area of the wall perpendicular to the heat flow
- both heat flux and heat flow are uniform \( \Rightarrow \) independent of position \((x)\)
- temperature distribution is governed by boundary conditions and length of domain \( \Rightarrow \) independent of thermal conductivity \((k)\)
1-D Steady Conduction: Cylinder Wall

Governing Equation:

\[
\frac{1}{r} \frac{d}{dr} \left( kr \frac{dT}{dr} \right) = 0 \Rightarrow \frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0
\]

Dirichlet Boundary Conditions:

\[T(r_1) = T_{s,1} ; \quad T(r_2) = T_{s,2}\]

Solution:

\[T(r) = \frac{T_{s,1} - T_{s,2}}{\ln\left(\frac{r_1}{r_2}\right)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}\]

Heat Flux:

\[q''_r = -k \frac{dT}{dr} = \frac{k(T_{s,1} - T_{s,2})}{r \ln(r_2/r_1)}\]

Heat Flow:

\[q_r = -kA \frac{dT}{dr} = 2\pi r L q''_r = \frac{2\pi L k(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}\]

\[q'_r = \frac{q_r}{L} = \frac{2\pi k(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}\]

Notes:

- heat flux is non-uniform ➔ function of position \((r)\)
- both heat flow and heat flow per unit length are uniform ➔ independent of position \((r)\)
1-D Steady Conduction: Spherical Shell

**Governing Equation:**

\[
\frac{1}{r^2} \frac{d}{dr} \left( kr^2 \frac{dT}{dr} \right) = 0 \Rightarrow \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0
\]

**Dirichlet Boundary Conditions:**

\[ T(r_1) = T_{s,1} ; \quad T(r_2) = T_{s,2} \]

**Solution:**

\[ T(r) = T_{s,1} - \left( T_{s,1} - T_{s,2} \right) \frac{1 - (r_1/r)}{1 - (r_1/r_2)} \]

**Heat Flux:**

\[ q''_r = -k \frac{dT}{dr} = \frac{k(T_{s,1} - T_{s,2})}{r^2 \left[ \left( 1/r_1 \right) - \left( 1/r_2 \right) \right]} \]

**Heat Flow:**

\[ q_r = -kA \frac{dT}{dr} = 4\pi r^2 q''_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{\left( 1/r_1 \right) - \left( 1/r_2 \right)} \]

**Notes:**
- heat flux is not uniform \( \Rightarrow \) function of position \( (r) \)
- heat flow is uniform \( \Rightarrow \) independent of position \( (r) \)
**Table 3.3** One-dimensional, steady-state solutions to the heat equation with no generation

<table>
<thead>
<tr>
<th></th>
<th>Plane Wall</th>
<th>Cylindrical Wall&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Spherical Wall&lt;sup&gt;a&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heat equation</td>
<td>$\frac{d^2T}{dx^2} = 0$</td>
<td>$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0$</td>
<td>$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0$</td>
</tr>
<tr>
<td>Temperature distribution</td>
<td>$T_{s,1} - \Delta T \frac{x}{L}$</td>
<td>$T_{s,2} + \Delta T \frac{\ln (r/r_2)}{\ln (r_1/r_2)}$</td>
<td>$T_{s,1} - \Delta T \left[ \frac{1 - (r_1/r)}{1 - (r_1/r_2)} \right]$</td>
</tr>
<tr>
<td>Heat flux ($q''$)</td>
<td>$k \frac{\Delta T}{L}$</td>
<td>$k \frac{\Delta T}{r \ln (r_2/r_1)}$</td>
<td>$k \frac{\Delta T}{r^2 [(1/r_1) - (1/r_2)]]}$</td>
</tr>
<tr>
<td>Heat rate ($q$)</td>
<td>$kA \frac{\Delta T}{L}$</td>
<td>$\frac{2\pi L k \Delta T}{\ln (r_2/r_1)}$</td>
<td>$\frac{4\pi k \Delta T}{(1/r_1) - (1/r_2)}$</td>
</tr>
<tr>
<td>Thermal resistance ($R_{t,\text{cond}}$)</td>
<td>$\frac{L}{kA}$</td>
<td>$\frac{\ln (r_2/r_1)}{2\pi L k}$</td>
<td>$(1/r_1) - (1/r_2)$</td>
</tr>
</tbody>
</table>

<sup>a</sup>The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.
Thermal Circuits: Composite Plane Wall

Circuits based on assumption of
(a) isothermal surfaces normal to $x$ direction

or
(b) adiabatic surfaces parallel to $x$ direction

\[
R_{tot} = \frac{L_E}{k_E A} + \frac{L_F}{2 L_F} + \frac{k_G A}{2 L_G} + \frac{L_H}{k_H A} 
\]

Actual solution for the heat rate $q$ is bracketed by these two approximations
Thermal Circuits: Contact Resistance

In the real world, two surfaces in contact do not transfer heat perfectly

\[
\begin{align*}
R_{t,c} &= \frac{T_A - T_B}{q_x''} \\
&\Rightarrow R_{t,c} = \frac{R''_{l,c}}{A_c}
\end{align*}
\]

Contact Resistance: values depend on materials (A and B), surface roughness, interstitial conditions, and contact pressure \(\Rightarrow\) typically calculated or looked up

Equivalent total thermal resistance: \(R_{tot} = \frac{L_A}{k_A A_c} + \frac{R''_{l,c}}{A_c} + \frac{L_B}{k_B A_c}\)
**Figure 2.4** Range of thermal conductivity for various states of matter at normal temperatures and pressure.
**Figure 2.5** The temperature dependence of the thermal conductivity of selected solids.
## Fins: The Fin Equation

- **Solutions**

### Table 3.4: Temperature distribution and heat loss for fins of uniform cross section

<table>
<thead>
<tr>
<th>Case</th>
<th>Tip Condition ((x = L))</th>
<th>Temperature Distribution (\theta/\theta_b)</th>
<th>Fin Heat Transfer Rate (q_f)</th>
</tr>
</thead>
</table>
| A    | Convection heat transfer: \(h\theta(L) = -kd\theta/dx|_{x=L}\) | \[
\frac{\cosh m(L - x) + (h/mk) \sinh m(L - x)}{\cosh mL + (h/mk) \sinh mL}
\] | \[
M \frac{\sinh mL + (h/mk) \cosh mL}{\cosh mL + (h/mk) \sinh mL}
\] (3.70) |
| B    | Adiabatic \(d\theta/dx|_{x=L} = 0\) | \[
\frac{\cosh mL - x}{\cosh mL}
\] | \[
M \tanh mL
\] (3.75) |
| C    | Prescribed temperature: \(\theta(L) = \theta_L\) | \[
\frac{(\theta_L/\theta_b) \sinh mx + \sinh m(L - x)}{\sinh mL}
\] | \[
M \frac{\cosh mL - \theta_L/\theta_b}{\sinh mL}
\] (3.77) |
| D    | Infinite fin \((L \to \infty)\): \(\theta(L) = 0\) | \[
e^{-mx}
\] | \[
M
\] (3.79) |

\[
\theta = T - T_\infty \quad m^2 = hP/kA_c
\]
\[
\theta_b = \theta(0) = T_b - T_\infty \quad M = \sqrt{hP/kA_c} \theta_b
\]
Fins: Fin Performance Parameters

• **Fin Efficiency**
  – the ratio of *actual amount* of heat removed by a fin to the *ideal amount* of heat removed if the fin was an *isothermal body at the base temperature*
    • that is, the ratio the actual heat transfer from the fin to ideal heat transfer from the fin if the fin had no conduction resistance

\[
\eta_f = \frac{q_f}{q_{f,\text{max}}} = \frac{q_f}{hA_f\theta_b}
\]

• **Fin Effectiveness**
  – ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin

\[
\varepsilon_f = \frac{q_f}{q_{f,\text{max}}} = \frac{q_f}{hA_{c,b}\theta_b} = \frac{R_{t,b}}{R_{t,f}}
\]

• **Fin Resistance**
  – defined using the temperature difference between the base and fluid as the driving potential

\[
R_{t,f} = \frac{\theta_b}{q_f} = \frac{1}{hA_f\eta_f}
\]
**Fins: Efficiency**

<table>
<thead>
<tr>
<th>Fin Type</th>
<th>Efficiency Equation</th>
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</thead>
<tbody>
<tr>
<td><strong>Straight Fins</strong></td>
<td></td>
</tr>
</tbody>
</table>
| Rectangular    | \( \eta = \frac{\tanh mL}{mL_E} \)  
|                | \( L_E = L + \theta/2 \) |
|                | \( A_T = tL \)       |
| Triangular     | \( \eta = \frac{1}{mL} \left( \frac{2mL}{I_f(2mL)} \right) \)  
|                | \( I_f = 2mL \)       |
| Parabolic      | \( \eta = \frac{2}{\left[4(mL)^2 + 1\right]^{1/2} + 1} \)  
|                | \( y = 0.5(2x) - xD^2 \) |

**Table 3.5** Efficiency of common fin shapes
Fins: Efficiency

Circular Fin
Rectangular
\[ A_r = 2\pi (r_2^2 - r_1^2) \]
\[ r_2 = r_1 + (d/2) \]
\[ V = \pi (r_2^2 - r_1^2)h \]

Pin Fins
Rectangular
\[ A_p = \pi DL_c \]
\[ L_c = L + (D/4) \]
\[ V = (\pi D^2/4)L \]

Triangular
\[ A_t = \frac{\pi D}{2} [L^2 + (D/2)^2]^{1/2} \]
\[ V = (\pi/12)D^2L \]

Efficiency
Circular Fin
\[ \eta = C_2 \frac{K_h(r_1)h(r_2) - I_0(r_1)K_0(r_2)}{I_1(r_1)K_1(r_2) + K_0(r_1)I_0(r_2)} \] (3.91)
\[ C_2 = \frac{(2r/m)}{(r_2^2 - r_1^2)} \]

Pin Fins
\[ \eta = \tanh \frac{ML_c}{mL_c} \] (3.95)

Triangular
\[ \eta = \frac{2I_1(2mL)}{mL_1(2mL)} \] (3.96)
Fins: Arrays

- **Arrays**
  - total surface area
    \[ A_t = NA_f + A_b \quad N \equiv \text{number of fins} \]
  - exposed base surface (prime surface)
    \[ A_b \equiv \text{exposed base surface} \]
  - total heat rate
    \[ q_t = N\eta_f hA_f \theta_b + hA_b \theta_b = \eta_o hA_t \theta_b = \frac{\theta_b}{R_{t,o}} \]
  - overall surface efficiency
    \[ \eta_o = 1 - \frac{NA_f}{A_t} \left( 1 - \eta_f \right) \]
  - overall surface resistance
    \[ R_{t,o} = \frac{\theta_b}{q_t} = \frac{1}{hA_t \eta_o} \]
**Fins: Thermal Circuit**

- **Equivalent Thermal Circuit**

  ![Equivalent Thermal Circuit Diagram](image)

- **Effect of Surface Contact Resistance**

  \[
  q_t = \eta_{o(c)} hA_f \theta_b = \frac{\theta_b}{R_{t,o(c)}}
  \]

  \[
  \eta_{o(c)} = 1 - \frac{N A_f}{A_t} \left(1 - \frac{\eta_f}{C_1}\right)
  \]

  \[
  C_1 = 1 - \eta_f hA_f \left(\frac{R''_{t,c}}{A_{c,b}}\right)
  \]

  \[
  R_{t,o} = \frac{1}{hA_t \eta_{o(c)}}
  \]
Fins: Overview

• Fins
  – extended surfaces that enhance fluid heat transfer to/from a surface in large part by increasing the effective surface area of the body
  – combine conduction through the fin and convection to/from the fin
    • the conduction is assumed to be one-dimensional

• Applications
  – fins are often used to enhance convection when $h$ is small (a gas as the working fluid)
  – fins can also be used to increase the surface area for radiation
  – radiators (cars), heat sinks (PCs), heat exchangers (power plants), nature (stegosaurus)
# Fins: The Fin Equation

- Solutions

## Table 3.4 Temperature distribution and heat loss for fins of uniform cross section

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<td>$e^{-mx}$</td>
<td>$M$</td>
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$\theta = T - T_\infty$  
$m^2 = hP/k\ell^2$  
$\theta_b = \theta(0) = T_b - T_\infty$  
$M = \sqrt{hP\ell^2/\theta_b}$

D. B. Go