ON GROTHENDIECK'S APPROACH TO STABILITY

Abstract. In this note we state and give an elementary proof of a simplified version of Grothendieck's theorem that implies, and in fact is equivalent to, the strong version the Fundamental Theorem of Stability Theory.

This version of Grothendieck's theorem came out of discussions at Notre Dame Model Theory seminar in March 2017 during reading Itaï Ben Yaacov's paper [1].

In the paper [1] Itaï Ben Yaacov observed that the Fundamental Theorem of Stability Theory (FTST for short) follows from Grothendieck's double limit theorem (see [2, Theorem 6]), and in fact Grothendieck's Theorem implies a strong version of FTST where a formula φ is assumed to be stable in a given structure M and not in the whole theory. We refer to [3, Section 2] for the strong version of FTST.

1. On extensions of separately continuous functions

Let A, B, C be topological spaces. Recall that a map $f\colon A\times B \to C$ is called *separately continuous* if for every $a \in A$ the map $b \mapsto f(a, b)$ is continuous on B, and for every $b \in B$ the map $a \mapsto f(a, b)$ is continuous on A.

In this note we deal with maps $f\colon A \times B \to 2$, where $2 = \{0, 1\}$ with discrete topology.

Definition 1.1. Let A, B be sets and $f\colon A\times B \to 2$ be a map. We say that f *has the order property* if there are sequences $(a_i)_{i\in\omega} \in A$ and $(b_j)_{j\in\omega} \in B$ such that

(a) either for all $i \neq j$ we have $f(a_i, b_j) = 1$ if and only if $i < j$;
(b) or for all $i \neq j$ we have $f(a_i, b_j) = 1$ if and only if $i > j$.

The order property can be restate in terms of Grothendieck's double limit property.

Definition 1.2 (Grothendieck's double limit property)**.** Let A, B be sets and $f\colon A\times B \to 2$ be a map. We say that f has *Grothendieck's double limit property* if for any sequences $(a_i)_{i\in\omega} \in A$ and $(b_j)_{j\in\omega} \in B$ we have

$$\lim_{i\to\infty}\lim_{j\to\infty} f(a_i, b_j) = \lim_{j\to\infty}\lim_{i\to\infty} f(a_i, b_j),$$

provided the limits on booth sides exist.

Lemma 1.3. *A map $f\colon A\times B \to 2$ has Grothendieck's double limit property if and only if f does not have the order property.*

Proof. Easy. □

Proposition 1.4. *Let $\bar{A}, \bar{B}$ be compact topological spaces. If $\bar{F}\colon \bar{A}\times \bar{B} \to 2$ is a separately continuous map then $\bar{F}$ has Grothendieck's double limit property.*

Proof. Let $(a_i)_{i\in\omega} \in \bar{A}$ and $(b_j)_{j\in\omega} \in \bar{B}$ be sequences such that both limits $\lim_{i\to\infty}\lim_{j\to\infty}\bar{F}(a_i,b_j)$ and $\lim_{j\to\infty}\lim_{i\to\infty}\bar{F}(a_i,b_j)$ exist.

Let $\alpha \in \bar{A}$ be an accumulation point of $(a_i)_{i\in\omega}$ and $\beta \in \bar{B}$ be an accumulation point of $(b_j)_{j\in\omega}$.

Then for all $i\in\omega$ we have $\bar{F}(a_i,\beta) = \lim_{j\to\infty}\bar{F}(a_i,b_j)$; and also for all $j\in\omega$ we have $\bar{F}(\alpha,b_j) = \lim_{i\to\infty}\bar{F}(a_i,b_j)$. Hence

$$\lim_{i\to\infty}\lim_{j\to\infty}\bar{F}(a_i,b_j) = \lim_{i\to\infty}\bar{F}(a_i,\beta) = \bar{F}(\alpha,\beta)$$
$$= \lim_{j\to\infty}\bar{F}(\alpha,b_j) = \lim_{j\to\infty}\lim_{i\to\infty}\bar{F}f(a_i,b_j).$$

□

Working with maps of the form $f\colon A\times B\to C$ we adopt the standard convention: for $a\in A$ we will denote by f_a the map $f_a\colon B\to C$ defined as $f_a(b) = f(a,b)$; and for for $b\in B$ we will denote by f^b the map $f^b\colon A\to C$ defined as $f^b(a) = f(a,b)$.

Theorem 1.5 (simplified version of Grothendieck's theorem)**.** *Let $\bar{A}, \bar{B}$ be compact topological spaces with dense subsets $A\subseteq\bar{A}$, $B\subseteq\bar{B}$. Let $f\colon A\times B\to 2$ be a separately continuous map. Assume:*

(i) for every $a\in A$ the map $f_a\colon B\to 2$ extends to a continuous map $\bar{f}_a\colon \bar{B}\to 2$; and
(ii) For every $b\in B$ the map $f^b\colon A\to 2$ extends to a continuous map $\bar{f}^b\colon \bar{A}\to 2$.

Then the following conditions are equivalent.

(1) There is a separately continuous map $\bar{F}\colon \bar{A}\times\bar{B}$ extending f.
(2) The map $f\colon A\times B\to 2$ has Grothendieck's double limit property.

Proof. (1) ⇒ (2). Follows from Proposition 1.4.

(2) ⇒ (1). We will denote by F the partial map from $\bar{A}\times\bar{B}$ to 2 with the domain $\mathrm{dom}(F) = (\bar{A}\times B)\cup(A\times\bar{B})$ defined as

$$F(a,b) = \begin{cases} \bar{f}^b(a) & \text{if } (a,b)\in\bar{A}\times B \\ \bar{f}_a(b) & \text{if } (a,b)\in A\times\bar{B}. \end{cases}$$

It is not hard to see that, in order to show (1), it is sufficient to show that for any $\alpha \in \bar{A}$, and any $\beta \in \bar{B}$ the following property holds:

($\star$) there are open $\alpha \in U \subseteq \bar{A}$ and $\beta \in V \subseteq \bar{B}$ such that the function F_α is constant on $V \cap B$, the function F^β is constant on $U \cap A$ and these constant values are equal.

Assume ($\star$) fails for some $\alpha \in \bar{A}$, $\beta \in \bar{B}$.

It follows then that for any open $\alpha \in U \subseteq \bar{A}$ and $\beta \in V \subseteq \bar{B}$ there are $a \in U \cap A$ and $b \in V \cap B$ with $F(\alpha, b) \neq F(a, \beta)$.

Notice that by conditions (i) and (ii) we have that for any $a \in A$ the set $\{y \in \bar{B} \colon F(a, y) = 1\}$ is closed and open in $\bar{B}$, and similarly for any $b \in B$ the set $\{x \in \bar{A} \colon F(x, b) = 1\}$ is closed and open in $\bar{A}$.

It follows then that for any finite $B_0 \subseteq B$ the set $\{x \in \bar{A} \colon F_x \restriction B_0 = F_\alpha \restriction B_0\}$ is an open neighborhood of α in $\bar{A}$, and similarly for any finite $A_0 \subseteq A$ the set $\{y \in \bar{B} \colon F^y \restriction A_0 = F^\beta \restriction A_0\}$ is an open neighborhood of β in $\bar{B}$.

Using the density of A in $\bar{A}$ and B in $\bar{B}$ we obtain that for any finite $B_0 \subseteq B$ and $A_0 \subseteq A$ there are $a \in A$ and $b \in B$ such that $F_\alpha \restriction B_0 = F_a \restriction B_0$, $F^\beta \restriction A_0 = F^b \restriction B_0$ and $F_\alpha(b) \neq F^\beta(a)$.

By induction on $i \in \omega$ we build a sequence $(a_i, b_i)_{i \in \omega} \in A \times B$ such that for any $i \in \omega$ we have

- $F^\beta \restriction \{a_j \colon j < i\} = F^{b_i} \restriction \{a_j \colon j < i\}$;
- $F_\alpha \restriction \{b_j \colon j < i\} = F_{a_i} \restriction \{b_j \colon j < i\}$;
- $F_\alpha(b_i) \neq F^\beta(a_i)$.

Passing to a subsequence if needed we can in addition assume that the value of F_α is constant on $(b_i)_{i<\omega}$, say 0, and value of F^β is constant on $(a_i)_{i<\omega}$, say 1.

Then for $i < j$ we have $f(a_i, b_j) = F^{b_j}(a_i) = F^\beta(a_i) = 1$, and similarly for $i > j$ we have $f(a_i, b_j) = F_{a_i}(b_j) = F_\alpha(b_j) = 0$. Thus f has the order property and, by Lemma 1.3, does not have Grothendieck's double limit property. ☐

Remark 1.6. A similar proof shows that (1) and (2) are equivalent if we replace 2 by the interval $[0, 1]$ or by any metrizable compact space.

2. An application to stability.

Let M be a structure and $\varphi(x, y)$ be a formula.

It is more convenient to view M as a two-sorted structure with sorts $X = M^x$, $Y = M^y$ and $\varphi(x, y)$ as a formula with the variable x of the sort X and the variable y of the sort Y.

Remark 2.1. Notice that in the above setting model theory disappears. We have a bi-partite graph with two sets of vertices X and Y and the edge relation given by the formula φ. Everything below can be done in this graph-theoretic setting.

By *a φ-formula* we mean a formula $\theta(x)$ that is a finite Boolean combination of formulas $\{\varphi(x,b)\colon b \in Y\}$; and by *a φ^*-formula* we mean a formula $\theta(y)$ that is a finite Boolean combination of formulas $\{\varphi(a,y)\colon a \in X\}$.

By *a φ-type* we mean a maximal consistent set of φ-formulas, and we denote by $S_x^{\varphi}(M)$ the set of all φ-types. Similarly by *a φ^*-type* we mean a maximal consistent set of φ^*-formulas, and we denote by $S_y^{\varphi^*}(M)$ the set of all φ^*-types.

A φ-type $p(x)$ is called *definable* if there is a φ^*-formula $\theta_p^*(y)$ such that for any $b \in M^y$ we have $\varphi(x,b) \in p(x)$ if and only if $M \models \theta_p^*(b)$. Similarly, a φ^*-type $q(y)$ is called *definable* if there is a φ-formula $\theta_q(x)$ such that for any $a \in M^x$ we have $\varphi(a,y) \in q(y)$ if and only if $M \models \theta_q(a)$.

As usual, for $a \in X$ we will denote by $\mathrm{tp}_{\varphi}(a/M)$ the φ-type of a over M, i.e. the set of all φ-formulas that a satisfies, and similarly for $b \in Y$ we will denote by $\mathrm{tp}_{\varphi^*}(b/M)$ the φ^*-type of b over M.

We say that the formula $\varphi(x,y)$ *has the order property in M* if there are sequences $(a_i)_{i\in\omega} \in M^x$ and $(b_j)_{j\in\omega} \in M^y$ such that

(a) either for all $i \neq j$ we have $M \models \varphi(a_i,b_j)$ if and only if $i < j$;
(b) or for all $i \neq j$ we have $M \models \varphi(a_i,b_j)$ if and only if $i > j$.

And we say that φ is *stable in M* if it does not have the order property in M.

Remark 2.2. Notice that φ may be stable in M but unstable in an elementary extension of M. Thus stability in a structure is weaker than stability in a theory.

We can now state the strong version of the fundamental theorem of stability theory.

Theorem 2.3 (FTST, strong verion)**.** *Let $\varphi(x,y)$ be a formula in a first order structure M. Assume $\varphi(x,y)$ is stable.*

Then every $p \in S_x^{\varphi}(M)$ is definable and every $q \in S_y^{\varphi^*}(M)$ is definable as well.

Moreover let $p(x) \in S_x^{\varphi}(M)$ and $q(y) \in S_y^{\varphi^*}(M)$. Assume p is defined by a φ^*-formula $\theta_p^*(y)$ and q is defined by a φ-formula $\theta_q(x)$. Then $\theta_q(x) \in p$ if and only if $\theta_p^*(y) \in q$.

Remark 2.4. Notice that the moreover part is the above theorem is exactly "heir=coheir" property.

In the rest of this section we show that Theorem 1.5 implies Theorem 2.3.

Let $E_X(x_1, x_2)$ be the equivalence relation on X defined as

$$E_X(a_1, a_2) \text{ if and only if } M \models \varphi(a_1, b) \leftrightarrow \varphi(a_2, b) \text{ for all } b \in Y;$$

and similarly let $E_Y(y_1, y_2)$ be the equivalence relation on Y defined as

$$E_Y(b_1, b_2) \text{ if and only if } M \models \varphi(a, b_1) \leftrightarrow \varphi(a, b_2) \text{ for all } a \in X.$$

It is easy to see that $\varphi(x, y)$ respects both E_X and E_Y.

Let $X' = X/E_X$, $Y' = Y/E_Y$ and $\varphi' \subseteq X' \times Y'$ be the relation induced by φ. It is not hard to see that Theorem 2.3 holds in M if and only it holds in the structure $M' = (X', Y'; \varphi')$. Thus we can replace M by M' if needed and assume that $a_1 \neq a_2 \in X$ implies $\text{tp}_\varphi(a_1/M) \neq \text{tp}_\varphi(a_2/M)$; and similarly $b_1 \neq b_2 \in Y$ implies $\text{tp}_{\varphi^*}(b_1/M) \neq \text{tp}_{\varphi^*}(b_2/M)$. Thus we have embeddings $\text{tp}_\varphi \colon X \hookrightarrow S_x^\varphi(M)$, $\text{tp}_{\varphi^*} \colon Y \hookrightarrow S_y^{\varphi^*}(M)$, and will consider X as a subset of $S_x^\varphi(M)$ and Y as a subset of $S_y^{\varphi^*}(M)$.

Recall that both spaces $S_x^\varphi(M)$ and $S_y^{\varphi^*}(M)$ are compact with respect to the Stone topology, X is dense in $S_x^\varphi(M)$ and Y is dense in $S_y^{\varphi^*}(M)$.

Let $f \colon X \times Y \to 2$ be the map defined as $f(a, b) = 1$ if and only if $M \models \varphi(a, b)$. It is not hard to see that f is separately continuous (with respect to the topologies induced by Stone topologies), and the conclusion of Theorem 2.3 is equivalent to the existence of a separately continuous map $\bar{F} \colon S_x^\varphi(M) \times S_y^{\varphi^*}(M) \to 2$ extending f.

For each $a \in X$ the map $\bar{f}_a \colon S_y^{\varphi^*}(M) \to 2$ defined as $\bar{f}_a(q) = 1$ if and only if $\varphi(a, y) \in q$ is a continuous extension of f_a, and for each $b \in Y$ the similarly defined map $\bar{f}^b \colon S_x^\varphi(M) \to 2$ is a continuous extension of f^b.

Thus all assumptions of Theorem 1.5 are satisfied and the main part of Theorem 2.3 follows.

For the moreover part notice that

$$\theta_q(x) \in p \Leftrightarrow \bar{F}_p(q) = 1 \Leftrightarrow \bar{F}(p, q) = 1 \Leftrightarrow \bar{F}^q(p) = 1 \Leftrightarrow \theta_p^*(y) \in q.$$

References

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