Homework Assignment # 11, Nov. 28

1. Let M be a smooth manifold.

(a) Show that if $f\colon M \to \mathbb{R}$ is a smooth function, then its differential df is a smooth section of the cotangent bundle T^*M. Hint: smoothness of a section s is a local property and hence to check smoothness it suffices to check that the composition $\Phi_\alpha \circ s$ is smooth for local trivializations Φ_α of the vector bundle.

(b) Let $\mu\colon \mathbb{R} \times M \to M$ be an action of the group $\mathbb{R}$ on M which is smooth in the sense that μ is a smooth map. For any point $p \in M$, let $\gamma_p\colon \mathbb{R} \to M$ be the smooth path given by $\gamma_p(s) = \mu(s,p)$ and let $V_p \in T_pM$ be the tangent vector of this path at $s = 0$, i.e., $V_p := \gamma_V'(0)$ (in other words, V_p is the image of $1 \in \mathbb{R}$ under the differential $(\gamma_p)_*\colon \mathbb{R} = T_0\mathbb{R} \to T_pM$). For $M = S^2$, and the $\mathbb{R}$-action on S^2 given by rotating the sphere around the z-axis draw a picture of the vector field V (represented by arrows at a few points of S^2).

(c) Show that the section V of TM given by $M \ni p \mapsto V_p \in T_pM$ is smooth. Hint: to check smoothness at a point p choose a smooth chart (U, ϕ) with $p \in U$ and show that there is some $\epsilon > 0$ and an open subset V such that $\mu((-\epsilon, \epsilon) \times V) \subset U$.

We remark that a smooth $\mathbb{R}$-action on a manifold M is called a *flow*, and the associated vector field V is the *infinitesimal generator* of this flow.

2. The goal of this problem is to prove the *Vector Bundle Construction Lemma* we stated in class.

(a) Let E be a set which is the union of subsets V_α, $\alpha \in A$. Given a topology $\mathcal{T}_\alpha$ on each subset V_α, show that this determines a topology $\mathcal{T}$ on E by defining a subset $V \subset E$ to be open (with respect to $\mathcal{T}$) if and only if $V \cap V_\alpha$ is an open subset of V_α (with respect to the topology $\mathcal{T}_\alpha$).

(b) Show that the subspace topology on V_α agrees with $\mathcal{T}_\alpha$ provided for every $V \subset V_\alpha$ which belongs to T_α the intersection $V \cap V_\beta \subset V_\beta$ is open (with respect to the topology $\mathcal{T}_\beta$).

(c) Prove the following *Vector Bundle Construction Lemma:* Let M be a topological space, E_p a collection of vector spaces parametrized by $p \in M$, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Let E be the set

$$E := \coprod_{p \in M} E_p = \{(p, v) \mid p \in M,\ v \in E_p\}$$

and define $\pi\colon E \to M$ by $\pi(p,v) = p$. For each $\alpha \in A$, let $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ be a bijection with the following properties:

(i) The diagram

$$\begin{array}{ccc} E_{|U_\alpha} := \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times \mathbb{R}^n \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U_\alpha & \end{array}$$

is commutative, where π_1 is the projection onto the first factor.

(ii) The restriction of Φ_α to $E_p = \pi^{-1}(p)$ is a vector space isomorphism between E_p and $\{p\} \times \mathbb{R}^n = \mathbb{R}^n$.

(iii) For $\alpha, \beta \in A$, the composition

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^n \xrightarrow{\Phi_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\Phi_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is continuous.

Then there is a topology on E such that $\pi \colon E \to M$ is a vector bundle with local trivializations Φ_α.

(d) Prove that the construction of Part (c) gives a *smooth* vector bundle with *smooth* local trivializations Φ_α if we assume in addition that M is a smooth manifold and that the transition maps $\Phi_\beta \circ \Phi_\alpha^{-1}$ are smooth.

3. The vector bundles over a fixed topological space M are the objects of a category; the morphisms from a vector bundle $E \to M$ to a vector bundle $F \to M$, called *vector bundle maps*, are defined to be the continuous maps $f \colon E \to F$ that commute with the projection maps to M and whose restriction $f_{|E_p} \colon E_p \to F_p$ to each fiber E_p is a linear map. The composition of vector bundle maps is again a vector bundle map, and hence we have a category of vector bundles over M.

(a) Show that a vector bundle map $f \colon E \to F$ is a vector bundle isomorphisms if and only if the restriction $f_{|E_p} \colon E_p \to F_p$ is a linear isomorphism. Hint: the issue is to prove continuity of the inverse of the vector bundle bundle map f. Continuity is a *local* property, making it possible to check continuity using local trivializations. Use the fact that the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ given by $A \mapsto A^{-1}$ is continuous (which was problem # 4 in the first homework assignment).

(b) Show that the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ given by $A \mapsto A^{-1}$ is smooth.

(c) Prove the analog of (a) for *smooth* vector bundle maps. More precisely, let M be a smooth manifold and consider the category whose objects are smooth vector bundles over M, and whose morphisms are smooth vector bundle map, i.e., vector bundle maps

$f\colon E \to F$ which are smooth (as maps between smooth manifolds). Show that a smooth vector bundle map $f\colon E \to F$ is a vector bundle isomorphisms if and only if the restriction $f_{|E_p}\colon E_p \to F_p$ is a linear isomorphism for all $p \in M$. Hint: same hint as for part (a), just replace "continuous" by "smooth" throughout.

4. We recall that a point of the real projective space $\mathbb{RP}^n$ is a line L in the Euclidean space $\mathbb{R}^{n+1}$, i.e., L is a 1-dimensional subspace of $\mathbb{R}^{n+1}$. Moreover, the *tautological real line bundle* $p\colon E \to \mathbb{RP}^n$ has total space

$$E = \{(L, v) \mid L \in \mathbb{RP}^n,\ v \in L \subset \mathbb{R}^{n+1}\} \subset \mathbb{RP}^n \times \mathbb{R}^{n+1},$$

equipped with the subspace topology for the product topology on $\mathbb{RP}^n \times \mathbb{R}^{n+1}$. The projection map p sends a point $(L, v) \in E$ to $L \in \mathbb{RP}^n$ and the fiber $E_L = p^{-1}(L) = \{(L, v) \mid v \in L\}$ for a fixed $L \in \mathbb{RP}^n$ can obviously be identified with L. In particular, E_L has a vector space structure. The tautological complex line bundle over the complex projective space $\mathbb{CP}^n$ is defined analogously.

(a) Show that the tautological real line bundle $E \to \mathbb{RP}^1 \approx S^1$ is isomorphic to the Möbius band (whose total space we defined to be the quotient space $[0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$). Hint: Use problem 3(a).

(b) Show that $E \to \mathbb{RP}^1$ is not isomorphic to the trivial line bundle $F = \mathbb{RP}^1 \times \mathbb{R}$. Hint: Argue that the topological spaces $E \setminus$ zero-section and $F \setminus$ zero-section are not homeomorphic (the zero-section of a vector bundle is the subspace of the total space consisting of the zero-vectors of all fibers).

(c) Show that the tautological complex line bundle $E \to \mathbb{CP}^1$ is not isomorphic to the trivial complex line bundle $F = \mathbb{CP}^1 \times \mathbb{C} \to \mathbb{CP}^1$. Hint: Show that $E \setminus$ zero-section is homeomorphic to $\mathbb{C}^2 \setminus \{0\}$ and argue that $E \setminus$ zero-section and $F \setminus$ zero-section are not homeomorphic by contemplating their fundamental group.