Homework Assignment # 5, due Sept. 26

1. Let (X, x_0), (Y, y_0) be pointed topological spaces. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the Cartesian product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ of the fundamental groups of (X, x_0) and (Y, y_0).

2. We recall that two continuous maps $f, g \colon X \to Y$ are *homotopic* if there is a continuous map $H \colon I \times X \to Y$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ (such a map H is called a *homotopy* from f to g). We also recall that the objects of the *homotopy category* hoTop are topological spaces, and that the morphisms in hoTop from a space X to a space Y are the *homotopy classes* of continuous maps $X \to Y$.

(a) Show that homotopy is an equivalence relation.

(b) Show that the map

$$\mathrm{hoTop}(Y, Z) \times \mathrm{hoTop}(X, Y) \longrightarrow \mathrm{hoTop}(X, Z) \qquad \text{given by} \qquad ([g], [f]) \mapsto [g \circ f]$$

is well-defined.

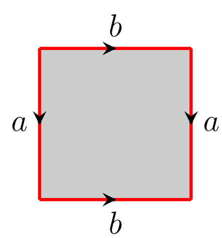
(c) Show that the composition map defined in (b) is associative, and check whatever else is needed to show that hoTop is a category.

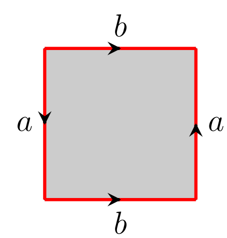
3. A subspace $A \subset X$ of a topological space X is called a *retract of* X if there is a continuous map $r \colon X \to A$ whose restriction to A is the identity.

(a) Show that S^1 is not a retract of D^2. Hint: Show that the assumption that there is a continuous map $r \colon D^2 \to S^1$ which restricts to the identity on S^1 leads to a contradiction by contemplating the induced map r_* of fundamental groups.

(b) Brouwer's Fixed Point Theorem states that every continuous map $f \colon D^n \to D^n$ has a fixed point, i.e., a point x with $f(x) = x$. Prove this for $n = 2$. Hint: show that if f has no fixed point, then a retraction map $r \colon D^2 \to S^1$ can be constructed out of f.

4. We recall that the torus T and the Klein bottle K can both be described as quotient spaces of the square by identifying edges with the same label as shown in the following pictures.

$T =$ 

$K =$ 

Each edge of the square can be interpreted as a path in the square, which projects to a based *loop* in the quotient space T resp. K. Here the base point x_0 of the quotient space is the projection of each of the four vertices of the square. Abusing language, let us denote by a resp. b the elements in the fundamental group of T (resp. K) represented by these loops. Prove the following identities:

$$aba^{-1}b^{-1} = 1 \in \pi_1(T, x_0) \qquad abab^{-1} = 1 \in \pi_1(K, x_0).$$