Homework Assignment # 6, due Oct. 3

1. Show that the following five topological spaces are all homotopy equivalent:

   1. the circle S^1,
   2. the open cylinder $S^1 \times \mathbb{R}$,
   3. the annulus $A = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$,
   4. the solid torus $S^1 \times D^2$,
   5. the Möbius strip

Hint: show that each of the spaces (2)-(5) contains a subspace S homeomorphic to the circle S^1 which is a deformation retract of the bigger space it is contained in.

2. Let H, G_1, G_2 be groups, and $j_1 \colon H \to G_1$, $j_2 \colon H \to G_2$ be homomorphisms. Show that the amalgamated product $G_1 *_H G_2$ is the pushout of $G_1 \xleftarrow{f_1} H \xrightarrow{f_2} G_2$, that is, show that there are group homomorphisms $i_1 \colon G_1 \to G_1 *_H G_2$ and $i_2 \colon G_2 \to G_1 *_H G_2$ such that the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{j_2} & G_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ G_1 & \xrightarrow{i_1} & G_1 *_H G_2 \end{array}$$

is a pushout diagram.

3. Use the Van Kampen Theorem to show $\pi_1(S^n, x_0) = \{1\}$ for $n > 1$. Hint: Recall that $S^n \setminus \{(0, \ldots, 0, 1)\}$ is homeomorphic to $\mathbb{R}^n$ by the stereographic projection.

4.

(a) Use the Seifert van Kampen Theorem to calculate the fundamental group of the Klein bottle.

(b) What is the abelianization of that group?