Homework Assignment # 8, due Oct. 31

1. We recall that the stereographic projection provides a homeomorphism between the open subsets $U_\pm := S^n \setminus \{(\mp 1, 0, \dots, 0)\}$ of S^n and $\mathbb{R}^n$ (compare problem (4) of assignment # 3). More explicitly, the stereographic projection is the map

$$\psi_\pm \colon U_\pm \longrightarrow \mathbb{R}^n \qquad \text{is defined by} \qquad \psi_\pm(x_0, \dots, x_n) := \frac{1}{1 \pm x_0}(x_1, \dots, x_n),$$

and its inverse $\psi_\pm^{-1} \colon \mathbb{R}^n \to U_\pm$ is given by the formula

$$\psi_\pm^{-1}(y_1, \dots, y_n) = \frac{1}{||y||^2 + 1}(\pm(1 - ||y||^2), 2y_1, \dots, 2y_n).$$

In particular, the two charts (U_+, ψ_+), (U_-, ψ_+) form an atlas for S^n.

(a) Show that $\{(U_+, \psi_+), (U_-, \psi_-)\}$ is a *smooth* atlas for S^n.

(b) Show that the atlas above is smoothly compatible with the smooth atlas we've discussed in class, consisting of the charts $(U_i^\pm, \phi_i^\pm)$, where $U_i^\pm \subset S^n$ consists of the vectors $(x_0, \dots, x_n) \in S^n$ such that $x_i > 0$ resp. $x_i < 0$, and $\phi_i^\pm(x_0, \dots, x_n) = (x_0, \dots, \widehat{x_i}, \dots, x_n)$.

2. Let $U_i := \{[z_0, z_1, \dots, z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$ for $i = 0, \dots, n$ and let

$$\phi_i \colon U_i \longrightarrow \mathbb{C}^n \qquad \text{be defined by} \qquad \phi_i([z_0, z_1, \dots, z_n]) = \left(\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i}\right).$$

(a) Show that (U_i, ϕ_i) is a chart for $\mathbb{CP}^n$.

(b) Show that $\{(U_i, \phi_i) \mid i = 0, \dots, n\}$ is a smooth atlas for $\mathbb{CP}^n$.

3. Show that the Cartesian product of $M \times N$ of smooth manifolds of dimension m resp. n is again a smooth manifold of dimension $m + n$.

4. Suppose $G \times M \to M$ is an action of a group G on a topological manifold M of dimension n.

(a) Show that the quotient space M/G is a topological manifold of dimension n provided the action satisfies the following two assumptions:

  (i) the action is properly discontinuous in the sense that for every $x \in M$ there is an open neighborhood U such that $U \cap gU = \emptyset$ for any $g \in G$, $g \neq e$;

  (ii) if $x, x' \in M$ are not in the same G-orbit, then there are open neighborhoods $U \ni x$ and $U' \ni x'$ such that $U \cap gU' = \emptyset$ for all $g \in G$.

Warning: the terminology "properly discontinuous" is not defined uniformly in the literature: some authors include property (ii) in the definition of "properly discontinuous, see for example the discussion in the Stackexchange website `math.stackexchange.com/questions/1082834`.

(b) Show that if G is finite and the action is free, then property (ii) is satisfied (recall that in problem 4(a) of the take-home-exam you've already proved that for a free action of a finite group property (i) is satisfied).

(c) Let G be a group that acts on a manifold M satisfying the hypotheses of part (a). Under the additional assumptions that M is a smooth manifold and the action is smooth (i.e., for every $g \in G$ the map $M \to M$ given by $x \mapsto gx$ is smooth), show that the quotient space M/G is a smooth manifold.