Homework Assignment # 11, due Dec. 6

1. (10 points) For a smooth manifold M, the *category* $\mathcal{V}\mathrm{ect}(M)$ of smooth vector bundles over M is defined as follows.

- The objects of $\mathcal{V}\mathrm{ect}(M)$ are the smooth vector bundles over M.
- For smooth vector bundles E, F over M, a morphism $f\colon E \to F$, called a *vector bundle morphism* is a smooth map $f\colon E \to F$ with commutes with the projection maps to M and whose restriction $f_p\colon E_p \to F_p$ to each fiber E_p is a linear map.

The identity map id_E of a vector bundle E is a vector bundle morphism, and the composition of two vector bundle morphisms is again a vector bundle morphism. Hence this is indeed a category. The goal of this exercise is to prove a simple criterion for a vector bundle morphism $f\colon E \to F$ to be an isomorphism in $\mathcal{V}\mathrm{ect}(M)$.

(a) Show that the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ given by $A \mapsto A^{-1}$ is smooth.

(b) Show that a vector bundle morphism $f\colon E \to F$ is an isomorphism in the category $\mathcal{V}\mathrm{ect}(M)$ if and only if $f_p\colon E_p \to F_p$ is an isomorphism of vector spaces for every $p \in M$. Hint: The issue is to prove smoothness of the inverse of the vector bundle map f. Smoothness is a *local property*, making it possible to check smoothness using local trivializations.

2. (10 points) We recall that the projective space $\mathbb{RP}^n$ is a smooth manifold of dimension n whose underlying set is the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. In particular, each point $p \in \mathbb{RP}^n$ determines tautologically a 1-dimensional subspace $E_p \subset \mathbb{R}^{n+1}$. Let E be the disjoint union $E = \amalg_{p\in\mathbb{RP}^n} E_p$ of the vector spaces E_p. More explicitly,

$$E = \{([x], v) \mid [x] \in \mathbb{RP}^n, v \in \langle x \rangle\},$$

where $x \in \mathbb{R}^{n+1} \setminus \{0\}$, $\langle x \rangle \subset \mathbb{R}^{n+1}$ is the one-dimensional subspace spanned by x, and $[x] \in \mathbb{RP}^n$ is the corresponding point in the projective space.

(a) Use the Vector Bundle Construction Lemma to show that E is a smooth vector bundle of rank 1 over $\mathbb{RP}^n$ (which is called the *tautological line bundle over* $\mathbb{RP}^n$; *line bundle* is a synonym for *vector bundle of rank* 1). Hint: Construct local trivializations of E restricted to $U_i = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$.

(b) Show that the complement of the zero section in E is diffeomorphic to $\mathbb{R}^{n+1} \setminus \{0\}$.

3. (10 points) For any smooth manifold M the *de Rham differential* (also called *exterior differential*) is the unique map $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) d is linear.

(ii) For a function $f \in C^\infty(M) = \Omega^0(M)$ the 1-form $df \in \Omega^1(M) = C^\infty(M, T^*M)$ is the usual differential of f.

(iii) d is a graded derivation with respect to the wedge product; i.e.,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta \qquad \text{for } \omega \in \Omega^k(M),\ \eta \in \Omega^l(M).$$

(iv) $d^2 = 0$.

Show that for $M = \mathbb{R}^n$ the de Rham differential of the k-form

$$\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \text{for } f \in C^\infty(\mathbb{R}^n),\ i_1 < i_2 < \cdots < i_k$$

is given explicitly by the formula

$$d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We note that *every* k-form $\eta \in \Omega^k(\mathbb{R}^n)$ can be written uniquely in the form

$$\eta = \sum_{i_1 < \cdots < i_k} f_{i_1, \dots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for smooth functions $f_{i_1,\dots,i_k} \in C^\infty(\mathbb{R}^n)$.

4. (10 points) Let M, N be smooth manifolds and $F\colon M \to N$ a smooth map. Then a differential form $\omega \in \Omega^k(N)$ leads to a form $F^*\omega \in \Omega^k(M)$, called the *pullback of* ω *along* F which is defined by

$$(F^*\omega)_p(v_1, \dots, v_k) := \omega_p(F_* v_1, \dots, F_* v_k) \qquad \text{for } p \in M,\ v_1, \dots, v_k \in T_pM.$$

In more detail: the k-form $F^*\omega$ is a section of the vector bundle $\mathrm{Alt}^k(TM; \mathbb{R})$, and hence it can be evaluated at $p \in M$ to obtain an element $(F^*\omega)_p$ in the fiber of that vector bundle over p, which is $\mathrm{Alt}^k(T_pM; \mathbb{R})$. In other words, $(F^*\omega)_p$ is an alternating multilinear map

$$(F^*\omega)_p\colon \underbrace{T_pM \times \cdots \times T_pM}_{k} \longrightarrow \mathbb{R},$$

and hence it can be evaluated on the k tangent vectors $v_1, \dots, v_k \in T_pM$ to obtain a real number $(F^*\omega)_p(v_1, \dots, v_k)$. On the right hand side to the equation defining $F^*\omega$, the map $F_*\colon T_pM \to T_{F(p)}N$ is the differential of F. Hence the alternating multilinear map $\omega_{F(p)} \in \mathrm{Alt}^k(T_{F(p)}N;\mathbb{R})$ can be evaluated on $F_*v_1, \dots, F_*v_k$ to obtain the real number $\omega_p(F_*v_1, \dots, F_*v_k)$.

Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Show that

$$F^*(dx^1 \wedge \cdots \wedge dx^n) = \det(dF_x)\; dx^1 \wedge \cdots \wedge dx^n$$

Here $dF_x\colon \mathbb{R}^n \to \mathbb{R}^n$ is the differential of F at the point $x \in \mathbb{R}^n$.

5. (10 points) Show that the exterior derivative for differential forms on $\mathbb{R}^3$ corresponds to the classical operations of *gradient* resp. *curl* resp. *divergence*. More precisely, show that there is a commutative diagram

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathcal{V}\mathrm{ect}(\mathbb{R}^3) & \xrightarrow{\text{curl}} & \mathcal{V}\mathrm{ect}(\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\
\| & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}$$

Here $\mathcal{V}\mathrm{ect}(\mathbb{R}^3)$ is the space of vector fields on $\mathbb{R}^3$, and we recall that grad, curl and divergence are given by the formulas

$$\begin{aligned}
\mathrm{grad}(f) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\mathrm{curl}(f_1, f_2, f_3) &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
\mathrm{div}(f_1, f_2, f_3) &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}
\end{aligned}$$

Here we identify a vector field on $\mathbb{R}^3$ with a triple (f_1, f_2, f_3) of smooth functions on $\mathbb{R}^3$. The vertical isomorphisms are given by

$$\begin{aligned}
(f_1, f_2, f_3) &\mapsto f_1 dx + f_2 dy + f_3 dz \\
(f_1, f_2, f_3) &\mapsto f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \\
f &\mapsto f dx \wedge dy \wedge dz
\end{aligned}$$