Homework Assignment # 2, due Sept. 13, 2019

1. (10 points) Show that the quotient space D^n/S^{n-1} is homeomorphic to the sphere S^n. Hint: produce a bijective map f relating these spaces by writing down an explicit formula, paying attention to have this map go the "natural direction" to make proving its continuity simple. Then show that f is a homeomorphism by verifying that domain and range of f satisfy the conditions that make continuity of f^{-1} automatic.

2. (10 points) Consider the following topological spaces

- The subspace $T_1 := \{v \in \mathbb{R}^3 \mid \operatorname{dist}(v, S) = r\} \subset \mathbb{R}^3$ equipped with the subspace topology, where $S = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $0 < r < 1$.
- The product space $T_2 := S^1 \times S^1$ equipped with the product topology.
- The quotient space $T_3 := ([-1, 1] \times [-1, 1])/\sim$ equipped with the quotient topology, where the equivalence relation is generated by $(s, -1) \sim (s, 1)$ and $(-1, t) \sim (1, t)$.

Show that these three spaces are homeomorphic. Hint: It suffices to produce two homeomorphisms between pairs of these spaces. First construct suitable continuous bijections, making sure to pick maps going in a direction that makes it easy to verify continuity using the Continuity Criterions for maps to/from subspaces, product spaces resp. quotient spaces. Then verify that continuity of the inverse is automatic since domain resp. codomain is compact resp. Hausdorff.

3. (10 points) Show that a closed subspace C of a compact topological space X is compact.

4. (10 points) Let X be a topological space which is the union of two subspaces X_1 and X_2. Let $f\colon X \to Y$ be a (not necessarily continuous) map whose restriction to X_1 and X_2 is continuous.

(a) Show f is continuous if X_1 and X_2 are open subsets of X.

(b) Show f is continuous if X_1 and X_2 are closed subsets of X.

(c) Give an example showing that in general f is *not continuous*.

Remark. This result is needed to verify that various constructions (e.g., concatenations of paths) in fact lead to *continuous* maps. In a typical situation, we have continuous maps $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ which agree on $X_1 \cap X_2$ and hence there is a well-defined map

$$f\colon X \longrightarrow Y \qquad \text{given by} \qquad f(x) = \begin{cases} f_1(x) & x \in X_1 \\ f_2(x) & x \in X_2 \end{cases}$$

The above result then helps to show that this map is continuous.

5. (10 points) Use the Heine-Borel Theorem to decide which of the topological groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are compact. Provide proofs for your statements. Hint: A strategy often useful for proving that a subset C of $\mathbb{R}^n$ is closed is to show that C is of the form $f^{-1}(C')$ for some closed subset $C' \subset \mathbb{R}^k$ (often C' consists of just one point) and some continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^k$.