Homework Assignment # 4, due Sept. 27

1. (10 points) Let Σ, Σ' be compact connected 2-manifolds. Show that the Euler characteristic of the connected sum $\Sigma \# \Sigma'$ is given by the following formula:

$$\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2.$$

Do **not** use the Classification Theorem for 2-manifolds or the fact that every compact connected 2-manifold has a combinatorial description as a polygon with edge identifications.

Warning: change of convention for concatenation! I've changed my mind about the order of paths when concatenating path: from now on I plan to conform with the convention Hatcher uses. My motivation is this: in english we write from left to right; so if we want to think of paths as letters, and their concatenations as words (which is the key point of problem # 5), then is much more natural to read concatenations from left to right. In other words, if α, β are paths in X with $\alpha(1) = \beta(0)$, then $\alpha * \beta$ is the path obtained by first running through α, then β.

2. (10 points) Let $\alpha, \beta, \gamma \colon I \to X$ be paths in a topological space X. Assume that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$ which guarantees that the concatenated paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ can be formed. Show that these two paths are homotopic (relative endpoints). Verifying this shows that if α, β, γ are loops based at $x_0 \in X$ representing elements $a = [\alpha]$, $b = [\beta]$, $c = [\gamma]$ in $\pi_1(X; x_0)$, then $a(bc) = (ab)c$. In other words, this proves associativity of multiplication in $\pi_1(X; x_0)$, one of the last things to verify in order to prove that $\pi_1(X; x_0)$ is indeed a group.

Hint: Show that both paths can be written as a composition

$$I \xrightarrow{\phi} [0,3] \xrightarrow{\alpha \widehat{*} \beta \widehat{*} \gamma} X \qquad \text{with} \quad (\alpha \widehat{*} \beta \widehat{*} \gamma)(s) := \begin{cases} \alpha(s) & 0 \le s \le 1 \\ \beta(s-1) & 1 \le s \le 2 \\ \gamma(s-2) & 2 \le s \le 3 \end{cases}$$

3. (10 points) Let X be a topological space and let β be a path from x_0 to x_1. Show that the map

$$\Phi_\beta \colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1) \qquad [\gamma] \mapsto [\bar{\beta} * \gamma * \beta]$$

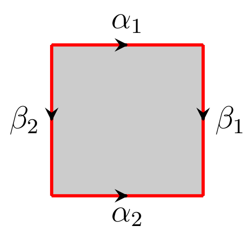
is an isomorphism of groups. In particular, the isomorphism class of the fundamental group $\pi(X, x_0)$ of a path connected space does not depend on the choice of the base point $x_0 \in X$. Hint: for any path γ in X, there are homotopies

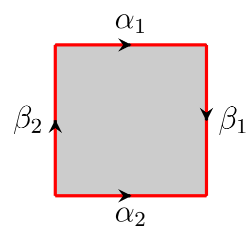
$$\gamma * \bar{\gamma} \simeq c_{\gamma(0)} \qquad \bar{\gamma} * \gamma \simeq c_{\gamma(1)} \qquad c_{\gamma(0)} * \gamma \simeq \gamma, \qquad \gamma * c_{\gamma(1)} \simeq \gamma$$

where c_x for $x \in X$ denotes the constant path at x. Make use of these (we proved one of these in class; no need to prove the others).

4. (10 points) Let (X, x_0), (Y, y_0) be pointed topological spaces. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the Cartesian product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ of the fundamental groups of (X, x_0) and (Y, y_0).

5. (10 points) We recall that the torus T and the Klein bottle K can both be described as quotient spaces of the square by identifying the α_1 edge with the α_2 edge, and the β_1 edge with the β_2 edge as shown in the following pictures.

$T =$ 

$K =$ 

The edges of the square can be interpreted as paths in the square, which project to a based *loops* $p \circ \alpha_1 = p \circ \alpha_2$ and $p \circ \beta_1 = p \circ \beta_2$ in the quotient space T resp. K. Here the base point x_0 of the quotient space is the projection of each of the four vertices of the square. Let $a := [p \circ \alpha_i]$ and $b := [p \circ \beta_i]$ be the elements in the fundamental group of T (resp. K) represented by these loops. Prove the following identities:

$$aba^{-1}b^{-1} = 1 \in \pi_1(T, x_0) \qquad aba^{-1}b = 1 \in \pi_1(K, x_0).$$

Observe that $T = \Sigma(aba^{-1}b^{-1})$ and $K = \Sigma(aba^{-1}b)$. Further question to think about: does this generalize to a statement for all quotients of a polygon described by a word?

Remarks that might be useful for problems #4 and #5. Let $f \colon X \to Y$ be a continuous map. If $\gamma \colon I \to X$ is a path in X, then f can be used to produce a path $f \circ \gamma \colon I \to Y$ in Y. This is useful in # 4 by using a projection map from $X \times Y$ to one of the factors to build from a loop in $X \times Y$ a loop in that factor. In # 5 it can be applied to the projection map p from the square $[0, 1] \times [0, 1]$ to the torus (or the Klein bottle) in order to construct paths in T (or K) from paths in $[0, 1] \times [0, 1]$.

It is helpful to know (and easy to prove) that this construction is compatible with homotopy and concatenation in the following sense. Let $f \colon X \to Y$ be continuous map and let $\gamma, \delta \colon I \to X$ be paths in X.

- **Compatibility with homotopy.** If γ, δ have the same starting/end point and are homotopic (relative endpoints), then the paths $f \circ \gamma$, $f \circ \delta$ have the same starting/end point and are homotopic (relative endpoints).
- **Compatibility with concatenation.** If $\gamma(1) = \delta(0)$, which assures that the concatenation $\gamma * \delta$ exist, then also the concatenation $(f \circ \gamma) * (f \circ \delta)$ exists and

$$(f \circ \gamma) * (f \circ \delta) = f \circ (\gamma * \delta).$$