Homework Assignment # 5, due Oct. 4

1. (10 points) A subset $U \subset \mathbb{R}^n$ is called *starlike* if there is some point $x_0 \in U$ such that for any point $x \in U$ the straight line segment connecting x and x_0 is contained in U.

(a) Show that $\pi_1(U, x_0)$ is trivial.

(b) Show that $\pi_1(U, x)$ for any point $x \in U$ is trivial.

2. (10 points) Let $f\colon S^1 \to S^1$ be defined by $f(z) = z^n$ for some $n \in \mathbb{Z}$. Calculate the induced homomorphism

$$f_*\colon \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1).$$

Clarification: We've proved that the fundamental group $\pi_1(S^1, 1)$ is isomorphic to $\mathbb{Z}$. In particular, any endomorphism of $\pi_1(S^1, 1)$ is given by multiplication by some integer $k \in \mathbb{Z}$. "Calculating f_*" means determining that integer k for the endomorphism $f_* \in \mathrm{End}(\pi_1(S^1, 1))$.

3. (10 points) A subspace $A \subset X$ of a topological space X is called a *retract of* X if there is a continuous map $r\colon X \to A$ whose restriction to A is the identity.

(a) Show that S^1 is not a retract of D^2. Hint: Show that the assumption that there is a continuous map $r\colon D^2 \to S^1$ which restricts to the identity on S^1 leads to a contradiction by contemplating the induced map r_* of fundamental groups.

(b) Brouwer's Fixed Point Theorem states that every continuous map $f\colon D^n \to D^n$ has a fixed point, i.e., a point x with $f(x) = x$. Prove this for $n = 2$. Hint: show that if f has no fixed point, then a retraction map $r\colon D^2 \to S^1$ can be constructed out of f.

4. (10 points) Let $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$. Let $v \in \mathbb{R}^n_+$ be a point of the boundary of $\mathbb{R}^n_+$, i.e. $v = (0, v_2, \dots, v_n)$, and let $V \subset \mathbb{R}^n_+$ be an open neighborhood of v. Let $w \in \mathbb{R}^n_+$ be a point in the interior of $\mathbb{R}^n_+$, i.e. $w = (w_1, \dots, w_n)$, with $w_1 > 0$ and let $W \subset \mathbb{R}^n_+$ be an open neighborhood of w. It can be shown that there is no homeomorphism $\varphi\colon V \to W$ with $\varphi(v) = w$, but we don't have the necessary tools (homology groups) to prove this for general n. We do have the tools to prove the following special cases:

(a) Prove this statement for $n = 1$.

(b) Prove this statement for $n = 2$.

Hint: Assume that there is a homeomorphism $\varphi\colon V \xrightarrow{\approx} W$ with $\varphi(v) = w$. The open neighborhood $V \ni v$ contains the semi-ball $B_\epsilon(v) \cap \mathbb{R}^n_+$ for small enough $\epsilon > 0$. This is a smaller open neighborhood of v and by restricting the homeomorphism φ to this neighborhood, we can assume with loss of generality that V is a semi-ball around v. Then φ restricts to a homeomorphism $V \setminus \{v\} \approx W \setminus \{w\}$. Derive a contradiction for $n = 1$ by considering whether the spaces $V \setminus \{v\}$, $W \setminus \{w\}$ are connected. For $n = 2$ show that the fundamental group one of these spaces is trivial, while the fundamental group of the other is not.

Remark. A *manifold of dimension n with boundary* is a topological space M (Hausdorff and second countable) which is locally homeomorphic to the half-space

$$\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\},$$

i.e., for every $x \in M$ there is some open neighborhood U and a homeomorphism

$$M \underset{\text{open}}{\supset} U \xrightarrow[\approx]{\varphi} V \underset{\text{open}}{\subset} \mathbb{R}^n_+$$

The statement above shows that there are two kinds of points in M: those points $x \in M$ which under such a homeomorphism $\varphi \colon U \xrightarrow{\approx} V$ map to a boundary point of $\mathbb{R}^n_+$ (this subspace of M is called the *boundary of* M, denoted ∂M), and those that map to interior points of $\mathbb{R}^n_+$ (this is the *interior of* M, often denoted $\mathring{M}$).

5. Let G_1 and G_2 be groups. Show that the free product $G_1 * G_2$ is the coproduct of G_1 and G_2 in the category of groups.