Homework Assignment # 6, due Oct. 11

1. (10 points) Show that the following five topological spaces are all homotopy equivalent:

   1. the circle S^1,
   2. the open cylinder $S^1 \times \mathbb{R}$,
   3. the annulus $A = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$,
   4. the solid torus $S^1 \times D^2$,
   5. the Möbius strip

Hint: show that each of the spaces (2)-(5) contains a subspace S homeomorphic to the circle S^1 which is a deformation retract of the bigger space it is contained in.

2. (10 points) Identify the pushout of the diagram of topological spaces

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \\ D^n & \end{tikzcd},$$

where both maps are the standard inclusion maps. In other words, show that the pushout of this diagram is homeomorphic to a well-known topological space X. Hint: It suffices to construct maps $i_1, i_2 \colon D^n \to X$ to your candidate space X, and to show that the resulting diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d, "i_1"] \\ D^n \arrow[r, "i_2"] & X \end{tikzcd},$$

is a pushout diagram (i.e., it is commutative and satisfies the universal property of a pushout diagram).

3. (10 points) Use the Seifert van Kampen Theorem to show $\pi_1(S^n, x_0) = \{1\}$ for $n \geq 2$. Hint: Use without proof the fact that the stereographic projection gives a homeomorphism

$$f \colon S^n \setminus \{N\} \xrightarrow{\approx} \mathbb{R}^n,$$

where $N := (0, \ldots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$ is the "north pole" of the n-sphere. Explicitly, f maps a point $x \in S^n \setminus N$ to the unique intersection point of the straight line through the points N and x with the subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ (draw a picture for $n = 2$). Make sure to verify that the hypotheses of the Seifert van Kampen Theorem are satisfied in the situation you apply it to.

4. (10 points) Let X be the subspace of $\mathbb{R}^3$ given by the union of the 2-sphere S^2 and the segment S of the x-axis given by $S = \{(t, 0, 0) \in \mathbb{R}^3 \mid -1 \leq t \leq 1\}$. Calculate the fundamental group of X. Hint: use the Seifert van Kampen Theorem.

5. (10 points) Let G be a group. The *abelianization of* G, is the abelian group G^{ab} obtained as the quotient of G modulo the commutator subgroup $[G, G]$, the normal subgroup generated by all commutators $[g, h] := ghg^{-1}h^{-1}$ for $g, h \in G$.

(a) Show that the abelianization of the free group $\langle S \rangle$ generated by a set $S = \{s_1, \dots, s_k\}$ is the free abelian group

$$\mathbb{Z}[S] := \mathbb{Z}s_1 \oplus \cdots \oplus \mathbb{Z}s_k$$

whose elements are the linear combinations $\sum_{i=1}^k n_i s_i$ of the elements of S with coefficients $n_i \in \mathbb{Z}$ (the group structure is given by the evident sum of such linear combinations). More precisely, show that an isomorphism

$$\Psi \colon \langle S \rangle^{\mathrm{ab}} \longrightarrow \mathbb{Z}[S]$$

is given by sending a word W in the letters s_i, s_i^{-1} to the linear combination $\sum_{i=1}^k n_i s_i$, where

$$n_i = \#\{\text{occurrences of } s_i \text{ in } W\} \quad - \quad \#\{\text{occurrences of } s_i^{-1} \text{ in } W\}.$$

(b) Let $R_1, \dots, R_k$ be elements of the free group $\langle S \rangle$, and let $\langle S \mid R_1, \dots, R_k \rangle$ be the quotient group of $\langle S \rangle$ modulo the normal subgroup generated by the elements $R_1, \dots, R_k$. Show that the there is an isomorphism

$$\langle S \mid R_1, \dots, R_k \rangle^{\mathrm{ab}} \cong \mathbb{Z}[S]/(\Psi(R_1), \dots, \Psi(R_k)),$$

where $(\Psi(R_1), \dots, \Psi(R_k)) \subset \mathbb{Z}[S]$ is the subgroup generated by $\Psi(R_1), \dots, \Psi(R_k)$.

(c) Show that $\pi_1(\Sigma_g)^{\mathrm{ab}} \cong \mathbb{Z}^{2g}$. Hint: recall that

$$\pi_1(\Sigma_g) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle,$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator.

(d) Show that $\pi_1(\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{k})^{\mathrm{ab}} \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2$. Hint: recall that

$$\pi_1(\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{k}) \cong \langle a_1, \dots, a_k \mid a_1 a_1 a_2 a_2 \dots a_k a_k \rangle.$$

It will be convenient to use the basis of $\mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_k$ provided by $a_1, \dots, a_{k-1}, c$, where $c = a_1 + \cdots + a_k$.

Remark. In general it is very difficult to determine whether two groups G, G' are isomorphic. By contrast, this is easy to determine for finitely generated *abelian* groups, since by the *Fundamental Theorem of finitely generated abelian groups* such a group G is isomorphic to the direct product of the infinite cyclic group $\mathbb{Z}$ and finite cyclic groups $\mathbb{Z}/q = \mathbb{Z}/q\mathbb{Z}$ whose order q is a prime power. Moreover, two finitely generated abelian groups are isomorphic if and only if their direct sum decomposition contains the same number of summands of order q for any prime power q and $q = \infty$. Hence the simplest way to show that two groups (e.g., the fundamental groups of topological spaces X, X') are *not* isomorphic, is to show that their abelianizations are not isomorphic.