Homework Assignment # 7, due Nov. 1

1. (10 points) Let M, N be path-connected manifolds of dimension $n \geq 3$. The goal of this problem is to compute the fundamental group of their connected sum M # N in terms of the fundamental groups of M and N. We recall that for the construction of the connected sum we picked points $x_0 \in M$, $y_0 \in N$ and maps $\phi: B_2^n \to M$, $\psi: B_2^n \to N$ which are are homeomorphisms onto their image with $\phi(0) = x_0$, $\psi(0) = y_0$; here $B_2^n = \{v \in \mathbb{R}^n \mid ||v|| < 2\} \subset \mathbb{R}^n$ is the open ball of radius 2. Then we defined

$$M \# N := (M \setminus \phi(B_1^n) \amalg N \setminus \psi(B_1^n)) / \sim$$

where the equivalence relation is given by identifying for $v \in S^{n-1}$ the point $\phi(v) \in M \setminus \phi(B_1^n)$ with the point $\psi(v) \in N \setminus \psi(B_1^n)$.

For the problem at hand, as well as for defining the connected sum for smooth manifolds, a modification of the definition of the connected sum is convenient. Let

$$\alpha \colon S^{n-1} \times (-1,1) \xrightarrow{\approx} B_2^n \setminus \{0\} \qquad \text{be given by} \qquad (v,t) \mapsto (1-t)v.$$

This is a homeomorphism with inverse given by $B_2^n \setminus \{0\} \ni w \mapsto (w/||w||, 1-||w||)$. Let

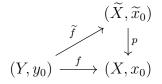
$$M \# N := (M \setminus \{x_0\} \amalg N \setminus \{y_0\}) / \sim_{\mathbb{R}}$$

where the equivalence relation identifies $\phi(\alpha(v, -t))$ and $\psi(\alpha(v, t))$ for $(v, t) \in S^{n-1} \times (-1, 1)$ (warning: $M \widetilde{\#} N$ is just an ad hoc notation).

- (a) Show that $M \widetilde{\#} N$ is homeomorphic to M # N. Hint: it might be helpful to draw a picture of $M \widetilde{\#} N$, indicating the image of $\phi(B_2^n \setminus \{0\}) = \psi(B_2^n \setminus \{0\})$.
- (b) How are the fundamental groups of M and $M \setminus \{x_0\}$ related? Hint: Use the Seifert van Kampen Theorem.
- (c) Express the fundamental group of M # N in terms of the fundamental groups of $M \setminus \{x_0\}$ and $N \setminus \{y_0\}$.
- 2. (10 points)
- (a) Show that the projection map $p: S^n \to \mathbb{RP}^n$ is a double covering.
- (b) Calculate the fundamental group of \mathbb{RP}^n for $n \ge 2$. Hint: use part (a).

3. (10 points) Let $p: (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Let Y be a path-connected and let $f: (Y, y_0) \to (X, x_0)$ be a map such that the image $f_*\pi_1(Y, y_0)$ is contained in the image

 $p_*\pi_1(\widetilde{X},\widetilde{x}_0)$. We proved in class that then there exists a unique (not necessarily continuous) map \widetilde{f} making the diagram

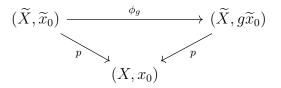


commutative. We constructed $\tilde{f}(y)$ by picking a path $\gamma: I \to Y$ from y_0 to y, composed with the map $f: Y \to X$ to obtain the path $f\gamma: I \to X$, and defined $\tilde{f}(y) := \tilde{f\gamma}(1)$, where $\tilde{f\gamma}: I \to \tilde{X}$ is the unique lift of $f\gamma$ with starting point \tilde{x}_0 .

Show that \tilde{f} is continuous under the additional assumption that Y is locally path-connected. Hint: It suffices to show that \tilde{f} is continuous in some open neighborhood V of every point $y \in Y$. Show that the assumption that Y is locally path-connected can be used to choose for every point $y \in Y$ a path-connected neighborhood V such that f(V) is contained in a evenly covered open subset $U \subset X$. To analyze $\tilde{f}(y')$ for $y' \in V$, use the concatenation $\gamma * \delta$ of a path γ from y_0 to y and $\delta \colon I \to V$ from y to y'.

4. (10 points) Let $p: (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the universal covering of a path-connected and locally path-connected space X.

(a) It follows from the General Lifting Criterion that for $g \in G := \pi_1(X, x_0)$ there is a unique map $\phi_g : (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, g\widetilde{x}_0)$ making the diagram



commutative. Here $g\tilde{x}_0 := \tilde{\gamma}(1)$ is the endpoint of a lift $\tilde{\gamma} \colon I \to \tilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ of any based loop γ in (X, x_0) which represents $g \in \pi_1(X, x_0)$ (we have shown that $\tilde{\gamma}(1)$ depends only on $[\gamma] \in \pi_1(X, x_0)$, not on the particular loop γ). Show that the map

$$G \times \widetilde{X} \longrightarrow \widetilde{X}$$
 $(g, \widetilde{x}) \mapsto \phi_g(\widetilde{x})$

is an action map.

- (b) Show that the action is free, i.e., for every $\tilde{x} \in \tilde{X}$, the only element of $g \in G$ with $g\tilde{x} = \tilde{x}$ is the identity element.
- (c) Show that the action is transitive on the fiber $p^{-1}(x)$ for all $x \in X$, i.e., for $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is some $g \in G$ such that $g\tilde{x} = \tilde{x}'$.

5. (10 points) Let (X, x_0) be a pointed space which is path-connected, locally path-connected, and semilocally simply connected. The goal of this assignment is to show that there is a bijection Ψ between

{based coverings $p: (E, e_0) \to (X, x_0 \text{ with } E \text{ path-connected})/\text{isomorphism}$

and

{subgroups of
$$\pi_1(X, x_0)$$
}

It is given by sending a covering p to the subgroup $p_*\pi_1(E, e_0) \subset \pi_1(X, x_0)$.

- (a) Show that Ψ is injective. Hint: use the general lifting criterion to show that any two path-connected based coverings $p: (E, e_0) \to (X, x_0)$ and $p': (E', e'_0) \to (X, x_0)$ are isomorphic.
- (b) Let $p: \widetilde{X} \to X$ be the universal covering of X, on which the fundamental group $G = \pi_1(X, x_0)$ acts freely by covering maps; this action is transitive on all fibers $p^{-1}(x)$ for $x \in X$. Let H be a subgroup of G and let $H \setminus \widetilde{X}$ be the orbit space of action of the subgroup H and let $p^H: (H \setminus \widetilde{X}, [\widetilde{x}_0]) \to (X, x_0), [\widetilde{x}] \mapsto p(\widetilde{x})$ be the projection map. Here $[\widetilde{x}] = H\widetilde{x}$ denotes the orbit through the point \widetilde{x} . Show that p^H is a covering and that $p_*^H \pi_1(H \setminus \widetilde{X}, [\widetilde{x}_0]) \subset G$ is the subgroup H.