Homework Assignment # 9, due Nov. 15

1. (10 points) For $i = 0, 1, 2, 3$, let $\phi_i\colon (-1, +1) \to \mathbb{R}$ be the following map

$$\phi_0(x) = x \qquad \phi_1(x) = \begin{cases} x & x \le 0 \\ 2x & x \ge 0 \end{cases} \qquad \phi_2(x) = x^3 \qquad \phi_3(x) = \tan\frac{\pi x}{2}$$

All of these maps are homeomorphisms between $M := (-1, +1)$ and an open subset of $\mathbb{R}$, allowing us to interpret (M, ϕ_i) as a *chart* for M. Let $\mathcal{A}_i$ be the atlas of M consisting of the single chart (M, ϕ_i). The *standard smooth structure* on M is the smooth structure determined by the smooth atlas $\mathcal{A}_0$.

(a) For which $i = 1, 2, 3$ is $\mathcal{A}_i$ a *smooth atlas* of M?

(b) Let S be the subset of $\{1, 2, 3\}$ such that $\mathcal{A}_i$ is a smooth atlas of M. For which $i \in S$ does the smooth atlas $\mathcal{A}_i$ determine the *standard* smooth structure on M?

(c) For $i \in S \cup \{0\}$, let M_i be the smooth manifold given by the topological manifold M equipped with the smooth structure determined by $\mathcal{A}_i$. For which $i, j \in S$ are the manifolds M_i, M_j diffeomorphic? Hint: If M is a smooth n manifold, and (U, ϕ) is a chart belonging to the maximal smooth atlas of M, then ϕ is a diffeomorphism from U to $\phi(U) \subset \mathbb{R}^n$.

2. (10 points) Let M be a smooth n-manifold. For a point $p \in M$ let

$$d^M\colon T_p^{\text{geo}}M = \{\gamma\colon (-\epsilon, \epsilon) \to M \mid \gamma(0) = p,\ \gamma \text{ smooth}\}/\sim \;\longrightarrow\; T_p^{\text{alg}}M = \operatorname{Der}(C_p^\infty(M), \mathbb{R})$$

be the map that sends $[\gamma]$ to the derivation d_γ. More explicitly, if f is (the germ of) a function $f\colon M \to \mathbb{R}$ then $d_\gamma f \in \mathbb{R}$ is defined by

$$d_\gamma f := \lim_{t\to 0} = \lim_{t\to 0} \frac{f(\gamma(t)) - f(p)}{t} = (f \circ \gamma)'(0);$$

i.e., d_γ is the derivative of functions at p in the direction of the path γ.

(a) Show that the geometric and the algebraic definition of the differential of a smooth map $F\colon M \to N$ are compatible in the sense that for $p \in M$ the following diagram is commutative:

$$\begin{tikzcd} T_p^{\text{geo}}M \arrow[r, "F_*^{\text{geo}}"] \arrow[d, "d^M"] & T_{F(p)}^{\text{geo}}N \arrow[d, "d^N"] \\ T_p^{\text{alg}}M \arrow[r, "F_*^{\text{alg}}"] & T_{F(p)}^{\text{alg}}N \end{tikzcd}$$

(b) Show that d^M is a bijection for $M = \mathbb{R}^n$. Hint: Show that the map d^M for $M = \mathbb{R}^n$ factors in the form

$$T_p^{\mathrm{geo}}\mathbb{R}^n \xrightarrow{\Phi} \mathbb{R}^n \xrightarrow{\Psi} T_p^{\mathrm{alg}}\mathbb{R}^n,$$

where Φ, Ψ are the maps we showed in class are bijections (which motivated our definition of the geometric/algebraic tangent space).

(c) Show that the map d^M is a bijection for general M. Hint: use a chart for M and part (a) to reduce to the case $M = \mathbb{R}^n$.

3. (10 points) Let M, N be smooth manifolds, and let $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ be the projection maps. Show that for any $(x, y) \in M \times N$ the map

$$\alpha \colon T_{(x,y)}(M \times N) \longrightarrow T_x M \oplus T_y N$$

defined by

$$\alpha(v) = ((\pi_1)_*(v), (\pi_2)_*(v))$$

is an isomorphism. Hint: To prove this, it is unnecessary to "unpack" the definition of the tangent space of manifolds by using either the geometric or algebraic definition. Rather, only the functorial properties of the tangent space, i.e., the chain rule, is needed, applied to suitable projection/inclusion maps. Remark: Using this isomorphism, we will routinely identify $T_x M$ and $T_y N$ with subspaces of $T_{(x,y)}(M \times N)$.

4. (10 points) Let $M_{n\times k}(\mathbb{R})$ be the vector space of $n \times k$-matrices. For $A \in M_{n\times k}(\mathbb{R})$ let $A^t \in M_{k\times n}(\mathbb{R})$ be the transpose of A, and let $\mathsf{Sym}(\mathbb{R}^k) = \{B \in M_{k\times k}(\mathbb{R}) \mid B^t = B\}$ be the vector space of *symmetric $k \times k$-matrices.*

(a) Show that the map $\Phi \colon M_{n\times k}(\mathbb{R}) \to \mathsf{Sym}(\mathbb{R}^k)$, $A \mapsto A^t A$ is smooth, and that its differential

$$\Phi_* \colon T_A M_{n\times k}(\mathbb{R}) = M_{n\times k}(\mathbb{R}) \longrightarrow T_{\Phi(A)}\mathsf{Sym}(\mathbb{R}^k) = \mathsf{Sym}(\mathbb{R}^k)$$

is given by $\Phi_*(C) = C^t A + A^t C$. Hint: Use the geometric description of tangent spaces. More explicitly, the tangent space $T_A M_{n\times k}(\mathbb{R})$ can be identified with $M_{n\times k}(\mathbb{R})$ by sending a matrix $C \in M_{n\times k}(\mathbb{R})$ to the path $\gamma(t) := A + tC$.

(b) Show that the identity matrix is a regular value of the map Φ. This implies in particular that the level set Φ^{-1}(identity matrix) is a smooth manifold. We recall that we showed in class that Φ^{-1}(identity matrix) is the Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$. Hint: to show that $\Phi_* \colon T_A M_{n\times n}(\mathbb{R}) \to T_e\mathsf{Sym}(\mathbb{R}^k)$ is surjective for $e =$ identity matrix, compute $\Phi_*(C)$ for $C = AB$ for $B \in \mathsf{Sym}(\mathbb{R}^k)$.

(c) What is the dimension of $V_k(\mathbb{R}^n)$?

We remark that identifying $M_{n\times k}(\mathbb{R})$ in the usual way with the vector space $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n)$ of linear maps $f\colon \mathbb{R}^k \to \mathbb{R}^n$, a matrix belongs to $V_k(\mathbb{R}^n)$ if and only if the corresponding linear map f is an *isometry*, that is, if f preserves the length of vectors in the sense that $||f(v)|| = ||v||$, or equivalently, if f preserves the scalar product in the sense that

$$\langle f(v), f(w)\rangle = \langle v, w\rangle \qquad \text{for all } v, w \in \mathbb{R}^k.$$

The manifold $V_k(\mathbb{R}^n)$ is called the *Stiefel manifold*. We observe that $V_n(\mathbb{R}^n)$ is the orthogonal group $O(n)$ of isometries $\mathbb{R}^n \to R^n$.

5. (10 points) Recall that the special linear group $SL_n(\mathbb{R})$ and the orthogonal group $O(n)$ are both submanifolds of the vector space $M_{n\times n}(\mathbb{R})$ of $n \times n$ matrices. In particular, the tangent spaces $T_A SL_n(\mathbb{R})$ for $A \in SL_n(\mathbb{R})$ and $T_A O(n)$ for $A \in O(n)$ are subspaces of the tangent space $T_A M_{n\times n}(\mathbb{R})$, which can be identified with $M_{n\times n}(\mathbb{R})$, since $M_{n\times n}(\mathbb{R})$ is a vector space.

(a) Show that $T_e SL_n(\mathbb{R}) = \{C \in M_{n\times n} \mid \mathrm{tr}(C) = 0\}$, where e is the identity matrix, and $\mathrm{tr}(C)$ denotes the trace of the matrix C. Hint for parts (a) and (b): $SL_n(\mathbb{R})$ and $O(n)$ can be both be described as level sets $F^{-1}(c)$ of a regular value c for a suitable smooth map F.

(b) Show that $T_e O(n) = \{C \in M_{n\times n} \mid C^t = -C\}$.

(c) Let $G \subset M_{n\times n}(\mathbb{R})$ be either the group $SL_n(\mathbb{R})$ or the group $O(n)$. For $A \in G$ let $L_A\colon G \to G$ be the map given by left multiplication by A, i.e., $B \mapsto AB$. Show that the differential

$$(L_A)_*\colon T_B G \longrightarrow T_{AB} G \qquad \text{is given by} \qquad C \mapsto AC,$$

where we identify all of these tangent spaces as subspaces of $M_{n\times n}(\mathbb{R})$. Hint: Compute first the differential of the map $\mathbb{M}_{n\times n}(\mathbb{R}) \to \mathbb{M}_{n\times n}(\mathbb{R})$, $B \mapsto AB$, and then compare with $(L_A)_*$.

(d) Use parts (a)–(c) to determine the tangent space $T_A G \subset M_{n\times n}(\mathbb{R})$ for $A \in G$ and $G = SL_n(\mathbb{R})$, as well as $G = O(n)$.