Index theory

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1 Elliptic differential operators and their index

1.1 Differential operators

Before defining differential operators, let us first present two examples of differential operators.

Example 1.1. (Examples of differential operators in $\mathbb{R}^n$).

(i) Let V be a smooth vector field in $\mathbb{R}^n$. There are equivalent ways to define/think of a vector field:

- as a smooth function $V\colon \mathbb{R}^n \to \mathbb{R}^n$, or
- as a derivation $V\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ of the algebra of smooth functions on $\mathbb{R}^n$, i.e., a linear map satisfying the product rule $V(fg) = V(f)g + fV(g)$ for $f, g \in C^\infty(\mathbb{R}^n)$.

Explicitly, if $V_1, \dots, V_n \in C^\infty(\mathbb{R}^n)$ are the component functions of a smooth function $V\colon \mathbb{R}^n \to \mathbb{R}^n$, then the corresponding derivation

$$V\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

maps a smooth function $f \in C^\infty(\mathbb{R}^n)$ to the function $Vf \in C^\infty(\mathbb{R}^n)$ defined by

$$Vf := \sum_{j=1}^n V_j \frac{\partial f}{\partial x_j}.$$

(ii) The *Laplace operator* $\Delta\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is defined by $\Delta := -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. Warning: the definition of the Laplace operator is not consistent in the literature. The definition above is the convention preferred by geometers, while analysts define Δ without the minus sign.

Definition 1.2. A linear map $D\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a *differential operator of order* $\leq k$ if it can be written in the form

$$Df = \sum_{|\alpha| \leq k} A_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f \tag{1.3}$$

for $f \in C^\infty(\mathbb{R}^n)$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi index with $\alpha_j \in \mathbb{N}$, $|\alpha| := \alpha_1 + \dots + \alpha_n|$, and $A_\alpha \in C^\infty(\mathbb{R}^n)$. The operator has *order* k if some function A_α with $|\alpha| = k$ is non-zero.

In particular, $D\colon C^\infty(\mathbb{R}^n) \to \mathbb{C}^\infty(\mathbb{R}^n)$ is a differential operator of order 0 if it given by multiplication by a smooth function $A_0 \in C^\infty(\mathbb{R}^n)$. A vector field $V\colon C^\infty(\mathbb{R}^n) \to \mathbb{C}^\infty(\mathbb{R}^n)$ is a first order differential operator, and the Laplace operator Δ is a second order differential operator.

More generally, if E, F are vector spaces (finite dimensional, real or complex), let $C^\infty(\mathbb{R}^n; E)$ be the space of smooth maps $f\colon \mathbb{R}^n \to E$ and similar for $C^\infty(\mathbb{R}^n; F)$. A linear map

$$D\colon C^\infty(\mathbb{R}^n; E) \longrightarrow C^\infty(\mathbb{R}^n; E)$$

is a differential operator of order $\leq k$ if $(Df)(x)$ is given by the equation (1.3) where now A_α is a smooth function on $\mathbb{R}^n$ with values in the space $\mathrm{Hom}(E, F)$ of linear maps from E to F.

This definition can be further generalized from the Euclidean space $\mathbb{R}^n$ to n-dimensional smooth manifolds X as follows. Let E, F be smooth vector bundles on X (finite dimensional, real or complex), and let $\Gamma(X; E)$ resp. $\Gamma(X; F)$ be the vector space of smooth sections. A linear map

$$D\colon \Gamma(X; E) \longrightarrow \Gamma(X; E)$$

is a differential operator of order $\leq k$ if locally Df has the form (1.3) (using a smooth chart $X \supset U \xrightarrow{\approx} V \subset \mathbb{R}^n$ and local trivializations of $E_{|U}$ and $F_{|U}$ the section spaces $\Gamma(U; E)$, $\Gamma(U; F)$ can be identified with vector valued functions on V).

1.2 Digression: the adjoint of a differential operator

Let $C_c^\infty(\mathbb{R}^n)$ be the smooth functions on $\mathbb{R}^n$ with compact support (we recall that the *support* of a function or section f is the closure of the subset of the domain of f where f is non-zero). There is an inner product

$$\langle\ ,\ \rangle\colon C_c^\infty(\mathbb{R}^n)\times C_c^\infty(\mathbb{R}^n)\longrightarrow\mathbb{R}$$

defined by

$$\langle f,g\rangle := \int_{\mathbb{R}^n} f(x)g(x)dx.$$

We recall that *inner product* means that $\langle f,g\rangle$ is multlinear (i.e., it is linear in each slot), it is symmetric (i.e., $\langle f,g\rangle=\langle g,f\rangle$), and positive definite (i.e., $\langle f,f\rangle\geq 0$ for all f and $\langle f,f\rangle=0$ if and only if $f=0$). All of these properties are immediate from the definition, except possibly the statement that $\langle f,f\rangle=0$ implies $f=0$. To argue this, suppose that $f(x)\neq 0$ for some point x. The continuity of f implies that $f(y)\geq\epsilon$ for all y in some ball around x, which forces $\langle f,f\rangle=\int_{\mathbb{R}^n} f(x)^2dx$ to be strictly positive; this provides the desired contradiction.

Remark 1.4. The inner product space $C_c^\infty(\mathbb{R}^n)$ of smooth compactly supported functions is not *complete*, that is, not every Cauchy sequence converges. In other words, $C_c^\infty(\mathbb{R}^n)$ is not a Hilbert space, and so we need to be careful not to use well-known facts for Hilbert spaces that rely on the completeness assumption. Of course, we can *complete* the inner product space $C_c^\infty(\mathbb{R}^n)$ to obtain a real Hilbert space (whose elements are equivalence classes of Cauchy sequences in the inner product space). This Hilbert space is denoted $L^2(\mathbb{R}^n)$, the *Hilbert space of square integrable functions on* $\mathbb{R}^n$.

Similarly, on the space $C_c^\infty(\mathbb{R}^n;\mathbb{C})$ of $\mathbb{C}$-valued smooth functions on $\mathbb{R}^n$ with compact support, there is a hermitian inner product

$$\langle\ ,\ \rangle\colon C_c^\infty(\mathbb{R}^n;\mathbb{C})\times C_c^\infty(\mathbb{R}^n;\mathbb{C})\longrightarrow\mathbb{C}$$

defined by

$$\langle f,g\rangle := \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx.$$

We recall that *hermitian inner product* means that $\langle f,g\rangle$ is is $\mathbb{C}$-linear in the first slot, and $\mathbb{C}$-antilinear in the second slot (i.e., $\langle f,zg\rangle=\bar{z}\langle f,g\rangle$ for $z\in\mathbb{C}$), it is conjugate-symmetric (i.e., $\langle f,g\rangle=\overline{\langle g,f\rangle}$), and positive definite (i.e., $\langle f,f\rangle\geq 0$ for all f and $\langle f,f\rangle=0$ if and only if $f=0$).

More generally, if E is a complex vector space equipped with Hermitian inner product $\langle\ ,\ \rangle_E$, this allows us to define a Hermitian inner product on $C_c^\infty(\mathbb{R}^n;\mathbb{E})$ the space of E-valued functions with compact support on $\mathbb{R}^n$ by

$$\langle f,g\rangle := \int_{\mathbb{R}^n}\langle f(x),g(x)\rangle_E dx. \tag{1.5}$$

Remark 1.6. If E is a real vector space equipped with an inner product $\langle\ ,\ \rangle_E$, then the formula above defines an inner product on the real vector space $C_c^\infty(\mathbb{R}^n; E)$ of smooth functions $f\colon \mathbb{R}^n \to E$ with compact support. Moreover, the complexification the real vector space $C_c^\infty(\mathbb{R}^n; E)$ can be identified with $C^\infty(\mathbb{R}^n; E \otimes_{\mathbb{R}} \mathbb{C})$. The inner product on E induces a hermitian inner product on $E \otimes_{\mathbb{R}} \mathbb{C}$, leading to a hermitian inner product on $C^\infty(\mathbb{R}^n; E \otimes_{\mathbb{R}} \mathbb{C}) = C_c^\infty(\mathbb{R}^n; E) \otimes_{\mathbb{R}} \mathbb{C}$, which in turn is given by extending the inner product on $C^\infty(\mathbb{R}^n; E)$ to a hermitian inner product on $C_c^\infty(\mathbb{R}^n; E) \otimes_{\mathbb{R}} \mathbb{C}$. In other words, the inner product on $C_c^\infty(\mathbb{R}^n; E)$ and the hermitian inner product on $C_c^\infty(\mathbb{R}^n; E \otimes_{\mathbb{R}} \mathbb{C})$ are closely related, and so we might drop the adjective "hermitian" in front of "inner product" when it is clear from the context whether our vector spaces are real or complex.

Let E, F be complex vector spaces equipped with Hermitian inner products and let $T\colon E \to F$ be a $\mathbb{C}$-linear operator. Then a $\mathbb{C}$-linear operator $T^*\colon F \to E$ is called the *adjoint of* T if

$$\langle Tv, w\rangle_F = \langle v, T^* w\rangle_F \qquad \text{for all } v \in E,\ w \in F.$$

Example 1.7.

1. The operator $D = \frac{\partial}{\partial x}\colon C_c^\infty(\mathbb{R}; \mathbb{C}) \to C_c^\infty(\mathbb{R}; \mathbb{C})$ has adjoint $D^* = -D$, since for every $f, g \in C_c^\infty(\mathbb{R}; \mathbb{C})$

$$\langle Df, g\rangle = \int_{\mathbb{R}} f'(x)\overline{g(x)}dx = -\int_{\mathbb{R}} f(x)\overline{g'(x)}dx = -\langle f, Dg\rangle.$$

Here the second equality is given by integration by parts.

2. Let E, F be hermitian inner product spaces, and let

$$D\colon C_c^\infty(\mathbb{R}; E) \to C_c^\infty(\mathbb{R}; F)$$

be the first order differential operator given by

$$(Df)(x) = A(x)f'(x),$$

where $A\colon \mathbb{R} \to \operatorname{Hom}(E, F)$ is a smooth function. To determine the adjoint of D, let us differentiate the complex valued function $\langle Af, g\rangle_F = \langle A(x)f(x), g(x)\rangle_F$, where $\langle\ ,\ \rangle_F$ is the hermitian inner product on F:

$$\begin{aligned}\frac{\partial}{\partial x}\langle Af, g\rangle_F =&\langle A'f, g\rangle_F + \langle Af', g\rangle_F + \langle Af, g'\rangle_F\\ =&\langle Df, g\rangle_F + \langle f, (A')^* g\rangle_F + \langle f, A^* g'\rangle_F\\ =&\langle Df, g\rangle_F + \langle f, A^* g' + (A')^* g\rangle_F\end{aligned}$$

Due to our assumption that the functions f, g are compactly supported, we can integrate these functions over $\mathbb{R}$ and obtain

$$0 = \langle Df, g\rangle + \langle f, A^* g' + (A')^* g\rangle.$$

This shows that D^*, the adjoint of D is given by $D^* = -A^*\frac{\partial}{\partial x} - (A')^*$. We note that although the operator D did not involve a zero order term, the operator D^* does involve the zero order term $-(A')^*$. While the formula for D^* is somewhat unpleasant, the formula for the principal symbol couldn't be nicer:

$$\sigma^{D^*}_{dx}(x) = i(-A^*) = (iA)^* = (\sigma^D_{dx}(x))^*$$

In other words, the factor of i in the definition of the principal symbol is helpful since it cancels the effect to the annoying minus sign coming from integration by parts.

The last statement holds much more generally.

Proposition 1.8. *For any differential operator $D\colon C^\infty_c(\mathbb{R}^n;E) \to C^\infty_c(\mathbb{R}^n;F)$ of order k, there is an adjoint differential operator $D^*\colon C^\infty_c(\mathbb{R}^n;F) \to C^\infty_c(\mathbb{R}^n;E)$ of order k. Moreover, the principal symbol of D^* is given by*

$$\sigma^{D^*}_\xi(x) = (\sigma^D_\xi(x))^* \in \mathrm{Hom}(F,E) \qquad \textit{for } x \in \mathbb{R}^n,\ \xi \in T^*_x\mathbb{R}^n.$$

The proof involves a straightforward calculation using Stokes' Theorem, which is completely analogous, but lengthier than our calculation above.

Next we would like to generalize our construction of the inner product on $C^\infty_c(\mathbb{R}^n)$ above to an inner product on the vector space $C^\infty_c(X)$ of compactly supported smooth functions $f\colon X \to \mathbb{R}$ on an n-manifold X. It is tempting to define $\langle f,g\rangle := \int_X f(x)g(x)$ for $f,g \in C^\infty(X)$, but this is nonsense since compactly supported smooth functions *cannot be integrated* over a manifold X without additional geometric data. For example, if $f\colon X \to \mathbb{R}$ is the constant function $f \equiv 1$ on a compact manifold X, then our experience with integrating functions over compact subsets of $\mathbb{R}^n$ would lead us to expect that integral is the volume of X. It is pretty clear that it doesn't make sense to talk about the *volume* of a manifold without additional geometric structure on X.

If X is an oriented manifold, and $\omega \in \Omega^n_c(X) = \Gamma_c(X;\Lambda^n T^*X)$ is a compactly supported differential form on X of degree n, then the integral

$$\int_X \omega \in \mathbb{R}$$

is well-defined. An n-form ω is a *volume form* if for every $x \in X$ and every oriented basis $\{e_1,\dots,e_n\}$ of T_xX

$$\omega_x(e_1,\dots,e_n) > 0$$

Here $\omega_x \in \Lambda^n T^*_x X$ denotes the evaluation of the section $\omega \in \Gamma(X;\Lambda^n_x T^*X)$ at the point $x \in X$. Volume forms exist on any oriented manifold. They can be constructed locally using the local trivializations of TX, and then combined to a volume form on all of X via partitions of unity.

Example 1.9. Let g be a Riemannian metric on X, i.e., every tangent space T_xX is equipped with an inner product $g_x\colon T_xX \times T_xX \to \mathbb{R}$ which depends smoothly on X (technically, this is expressed by saying that g is a section of the vector bundle $\mathsf{Sym}^2(TX,\mathbb{R})$ whose fiber $\mathsf{Sym}^2(TX,\mathbb{R})_x$ at a point $x \in X$ is the vector space of symmetric bilinear maps $T_xX \times T_xX \to \mathbb{R}$). Then an orientation on X determines a volume form $\mathrm{vol}_g \in \Omega^n(X)$ which is determined by the requirement that

$$\mathrm{vol}_g(e_1,\dots,e_n) = 1$$

for any oriented orthonormal basis $\{e_1,\dots,e_n\}$ of T_xX. The volume form vol_g is called *Riemannian volume form.*

Summarizing this discussion one could say that a Riemannian metric on X is a measuring device for the length of tangent vectors, and the angle between two tangent vectors belonging to the same tangent space. If X is oriented, a volume form $\mathrm{vol} \in \Omega^n(X)$ is a measuring device for volumes of compact codimension 0 pieces of X: if $K \subset X$ is a compact codimension 0 submanifold of X, then we interpret $\int_K \mathrm{vol}$ as the volume of K. So the construction of the Riemannian volume form vol_g associated to a Riemannian metric g means geometrically that knowing how to measure lengths of tangent vectors of X allows us to measure volumes of compact codimension 0 submanifolds of X.

Let X be an oriented n-manifold and E a complex vector bundle over X equipped with a volume form vol (e.g., the volume form vol_g associated to a Riemannian metric g on X). A *bundle metric* $\langle\ ,\ \rangle_E$ on E consists of hermitian inner products on the fibers E_x for all $x \in X$ which depend smoothly on x (in the same sense as explained for Riemannian metrics above). Then there is a hermitian inner product on the space $\Gamma_c(X;E)$ of compactly supported smooth sections of E defined by

$$\langle f,g\rangle := \int_X \langle f(x),g(x)\rangle_E \,\mathrm{vol} \tag{1.10}$$

Similarly, if E is a real vector bundle, a bundle metric on E consists of inner products on the fibers E_x (which are now real vector spaces), depending smoothly on x. For example, a bundle metric on the tangent bundle TX is the same thing as a Riemannian metric on X. Then the real vector space $\Gamma_c(X;E)$ can be equipped with an inner product (real valued) defined by formula (1.10) (now the function $x \mapsto \langle f(x),g(x)\rangle_E$ is real-valued and hence $\langle f,g\rangle$ is a real number).

The statement of Proposition 1.8 concerning the existence of adjoints of differential operators holds in this more general setting.

Proposition 1.11. *Let X be an oriented n-manifold equipped with a volume form* vol. *Let E, F be real or complex vector bundles over X equipped with bundle metrics, and let*

$$D\colon \Gamma_c(X;E) \longrightarrow \Gamma_c(X;F)$$

be a differential operator of order k. Then there is a differential operator

$$D^*\colon \Gamma_c(X;F) \longrightarrow \Gamma_c(X;E)$$

of order k which is adjoint to D. The principal symbol of D^* is given by

$$\sigma_\xi^{D^*}(x) = (\sigma_\xi^D(x))^* \in \operatorname{Hom}(F_x, E_x) \qquad \textit{for } x \in \mathbb{R}^n,\ \xi \in T_x^*\mathbb{R}^n.$$

Remark 1.12. The assumption that X is oriented in not necessary for the above result. We used the orientation to define the inner product on $\Gamma(X;E)$ using a bundle metric on E and a volume form $\mathrm{vol} \in \Omega^n(X)$. More generally, the volume form vol could be replaced by a *density*, a section of a real line bundle $|\Lambda^n T^*X|$ satisfying a positivity condition. The line bundle $|\Lambda^n T^*X|$ is built from $\Lambda^n T^*X$, but it differs from $\Lambda^n T^*X$ in the following ways:

- Sections of $|\Lambda^n T^*X|$ with compact support can be integrated over X to obtain a real number *without* requiring an orientation on X;
- The line bundle $|\Lambda^n T^*X|$ is always trivializable, unlike $\Lambda^n T^*X$ which is trivializable if and only if X is orientable.

1.3 Elliptic operators built from the de Rham differential

We recall from Lemma **??** that the principal symbol of the de Rham differential

$$\Omega^k(X) = \Gamma(X, \Lambda^k T^*X) \xrightarrow{d} \Omega^{k+1}(X) = \Gamma(X, \Lambda^{k+1}T^*X)$$

evaluated at a cotangent vector $\xi \in T_x^*X$ is the homomorphism $\sigma_\xi^d(x)\colon \Lambda^k T_x^*X \to \Lambda^{k+1}T_x^*X$ given by $\omega \mapsto i\xi \wedge \omega$. If X is a Riemannian n-manifold, then the inner product on the tangent space T_xX given by the Riemannian metric induces an inner product on the dual space T_x^*X. This in turn leads to an inner product on the exterior power space $\Lambda^k T_x^*X$ for any k. These inner products depends smoothly on $x \in X$; in other words, the Riemannian metric gives us bundle metrics on TX, T^*X and $\Lambda^k T * X$. In particular, if X is oriented, the Riemannian metric determines a volume form vol_g, thus giving us an inner product on the space of compactly supported k-forms $\Omega_c^k(X) = \Gamma_c(X; \Lambda^k T_x^*X)$ for any k.

Remark 1.13. The orientation on X is not necessary here, since a riemannian metric g on X always determines a *riemannian density* $|\mathrm{vol}_g|$, a section of the density line bundle $|\Lambda^n T^*X|$ mentioned in Remark 1.12. As discussed there, this leads to an inner product on $\Gamma_c(X;E)$ for any vector bundle E with bundle metric.

Then we know from Proposition **??** that there is a first order differential operator

$$d^*\colon \Omega^{k+1}(X) \longrightarrow \Omega^k(X)$$

which is adjoint to the de Rham differential, and that its principal symbol $\sigma_\xi^{d^*}(x)$ is the adjoint of the linear map

$$\Lambda^k T_x^* X \longrightarrow \Lambda^{k+1} T_x^* X \qquad \text{given by} \qquad \omega \mapsto i\xi \wedge \omega$$

with respect to the inner product on domain and codomain induced by the inner product on $T_x X$ provided by the Riemannian metric. A calculation (exercise!) gives the following result.

Lemma 1.14. *The principal symbol of d^* at $\xi \in T_x^* X$ is given by $\sigma_\xi^{d^*}(x) = -i\iota_v$, where*

- *$v \in T_x X$ is the tangent vector corresponding to $\xi \in T_x^* X$ via the isomorphism*

$$T_x X \xrightarrow{\cong} T_x^* X$$

  which sends a vector $v \in T_x X$ to the linear map $(w \mapsto \langle v, w\rangle \in \operatorname{Hom}(T_x X, \mathbb{R}) = T_x^* X$.

- *ι_u is the derivation on the exterior algebra $\Lambda^* T_x^* X := \bigoplus_{k=0}^n \Lambda^k T_x^* X$ determined by $\iota_u(\omega) = \omega(u)$ for $\omega \in T_x^* X = \operatorname{Hom}(T_x X, \mathbb{R})$.*

Let $D\colon \Omega^*(X) = \bigoplus_{k=0}^n \Omega^k(X) \longrightarrow \Omega^*(X)$ be the first order differential operator given by $D = d + d^*$. This operator is sometimes called the *de Rham operator* (to be carefully distinguished from the de Rham differential d).

Lemma 1.15. *The de Rham operator $D = d + d^*$ is an elliptic operator.*

Proof. We need to show that for any non-zero cotangent vector $\xi \in T_x^* X$ the principal symbol

$$\sigma_\xi^{d+d^*}(x)\colon \Lambda^* T_x^* X \longrightarrow \Lambda^* T_x^* X$$

is invertible. Applying the principal symbol to $\omega \in \Lambda^* T_x^* X$ we obtain

$$(\sigma_\xi^{d+d^*}(x))\omega = \sigma_\xi^{d+d^*}(x) + \sigma_\xi^{d+d^*}(x) = i\xi \wedge \omega - i\iota_v\omega = i(\xi \wedge \omega - \iota_v\omega).$$

To show that $\sigma_\xi^{d+d^*}(x)$ is invertible it suffices to show that its square is invertible. Writing $\xi \wedge \_\_$ for the map given by wedging with ξ, we calculate

$$\begin{aligned}
(\sigma_\xi^{d+d^*}(x))^2\omega &= -(\xi \wedge __ - \iota_v)(\xi \wedge \omega - \iota_v\omega) \\
&= -\xi \wedge \xi \wedge \omega + \iota_v(\xi \wedge \omega) + \xi \wedge \iota_v\omega - \iota_v(\iota_v\omega) \\
&= \iota_v(\xi \wedge \omega) + \xi \wedge \iota_v\omega \\
&= \iota_v(\xi) \wedge \omega - \xi \wedge \iota_v\omega + \xi \wedge \iota_v\omega \\
&= \iota_v(\xi) \wedge \omega = \xi(v)\omega = ||\xi||^2\omega
\end{aligned}$$

In other words, the linear map $(\sigma_\xi^{d+d^*}(x))^2$ is simply given by multiplication by $||\xi||^2$. In particular, this is an isomorphism for $\xi \neq 0$ as claimed. $\square$

It will be useful to also consider the *Laplace-Beltrami operator*

$$\Delta := D^2 = (d + d^*)^2 = dd^* + d^*d \colon \Omega^*(X) \to \Omega^*(X).$$

We note that unlike the de Rahm operator D the Laplace operator Δ maps k-form to k-form. In other words, Δ is the direct sum of the operators $\Delta_k \colon \Omega^k(X) \to \Omega^k(X)$ obtained by restricting Δ to k-forms. The following lemma implies that compositions of elliptic operators are elliptic; in particular $\Delta = D^2$ is elliptic since D is. The operator $\Delta_0 \colon C^\infty(X) \to C^\infty(X)$ is the usual Laplace operator acting on the functions on X.

Lemma 1.16. *Let $D_1 \colon \Gamma(X; E^1) \to \Gamma(X; E^2)$ and $D_2 \colon \Gamma(X; E^2) \to \Gamma(X; E^3)$ be differential operators of order k_1 resp. k_2. Then their composition $D_2 \circ D_1$ is a differential operator of order $k_1 + k_2$ whose principal symbol $\sigma_\xi^{D_2 \circ D_1}(x) \colon E_x^1 \to E_x^3$ for $\xi \in T_x^*X$ is given by the composition*

$$E_x^1 \xrightarrow{\sigma_\xi^{D_1}(x)} E_x^2 \xrightarrow{\sigma_\xi^{D_2}(x)} E_x^3$$

To calculate the index of the de Rham operator D and the Laplace operators Δ_k it will be useful to express the dimension of the cokernel of an elliptic operator D in terms of its adjoint D^*. So let X be a compact oriented n-manifold, let E, F be vector bundles over X equipped with bundle metrics, and let $D \colon \Gamma(X, E) \to \Gamma(X, F)$ be a differential operator and D^* its adjoint. We note that

$$\begin{aligned} \ker(D^*) =& \{g \in \Gamma(X; F) \mid \langle f, D^* g\rangle = 0 \text{ for all } f \in \Gamma(X; E)\} \\ =& \{g \in \Gamma(X; F) \mid \langle Df, g\rangle = 0 \text{ for all } f \in \Gamma(X; E)\} \\ =& \operatorname{im}(D)^\perp. \end{aligned}$$

We might be tempted to conclude that there is an orthogonal direct sum decomposition

$$\Gamma(X; F) = \operatorname{im}(D) \oplus \operatorname{im}(D)^\perp = \operatorname{im}(D)^\perp \oplus \ker(D^*).$$

However, the first equality might not hold if the image of D is not equal to $(\operatorname{im}(D)^\perp)^\perp$; in general $\operatorname{im}(D)$ is just a subspace of $(\operatorname{im}(D)^\perp)^\perp$.

Theorem 1.17. (**see e.g. Theorem 5.5 in [LM]**)**.** *Let X be a compact Riemannian manifold, let E, F be vector bundles over X equipped with bundle metrics, and let*

$$D \colon \Gamma(X; E) \longrightarrow \Gamma(X; F)$$

be an elliptic differential operator. Then there is an orthogonal direct sum decomposition

$$\Gamma(X; F) = \operatorname{im}(D) \oplus \ker(D^*).$$

As mentioned in Remarks 1.12 and 1.13 the assumption that X is oriented in not needed here.

Corollary 1.18. *If D is elliptic, then* $\dim \operatorname{coker} D = \dim \ker D^*$ *and hence*

$$\operatorname{index} D = \dim \ker D - \dim \ker D^*.$$

In particular, if D is self-adjoint, i.e., $D^* = D$, then $\operatorname{index} D = 0$.

This implies the disappointing statement that the indices of the de Rham operator D, the Laplace operator Δ and all its summands Δ_k are zero since these operators are all self-adjoint. Before constructing examples of elliptic operators with non-zero indices, we would like to point out another consequence of Theorem 1.17.

Corollary 1.19. (The Hodge Decomposition Theorem) *Let X be a compact riemannian n-manifold. Then there is an orthogonal direct sum decomposition*

$$\Omega^*(X) = \mathcal{H}^* \oplus \operatorname{im} d \oplus \operatorname{im} d^*,$$

where $\mathcal{H}^* := \ker \Delta \subset \Omega^*(X)$ *is the space of* harmonic forms.

The proof of this statement will use the following useful alternative ways to describe harmonic forms.

Lemma 1.20. *Let $D = d + d^*$ be the de Rham operator and $\Delta = D^2$ be the Laplace operator. Then*

$$\ker \Delta = \ker D = \ker d \cap \ker d^*.$$

Proof. It is clear that $\ker D \subseteq \ker D^2 = \ker \Delta$. To prove equality, let $\omega \in \ker \Delta$. Then

$$||D\omega||^2 = \langle D\omega, D\omega \rangle = \langle \omega, D^2\omega \rangle = \langle \omega, \Delta\omega \rangle = 0$$

and hence $\omega \in \ker D$.

Concerning the second equality, it is clear that $\ker d \cap \ker d^* \subseteq \ker(d + d^*) = \ker D$. To prove the converse inclusion, let $\omega \in \ker D = \ker(d + d^*)$. Then $d\omega = -d^*\omega$ and hence

$$||d\omega||^2 = \langle d\omega, d\omega \rangle = -\langle d^*\omega, d\omega \rangle = -\langle \omega, d^2\omega \rangle = 0.$$

This implies $d\omega = 0$, i.e., $\omega \in \ker d$. Similarly,

$$||d^*\omega||^2 = \langle d^*\omega, d^*\omega \rangle = -\langle d\omega, d^*\omega \rangle = -\langle \omega, (d^*)^2\omega \rangle = 0,$$

and hence $d^* \in \ker d^*$. $\square$

Proof of Corollary 1.19. Applying Theorem 1.17 to the self-adjoint Beltrami-Laplace operator $\Delta\colon \Omega^*(X) \to \Omega^*(X)$ gives an orthogonal direct sum decomposition

$$\Omega^*(X) = \operatorname{im}\Delta \oplus \ker\Delta = \operatorname{im}\Delta \oplus \mathcal{H}^*. \tag{1.21}$$

It remains to show that there is an orthogonal direct sum decomposition $\operatorname{im}\Delta = \operatorname{im} d \oplus \operatorname{im} d^*$. To argue that $\operatorname{im} d$ is orthogonal to $\operatorname{im} d^*$ we note that

$$\langle d\omega, d^*\eta\rangle = \langle d^2\omega, \eta\rangle = 0.$$

It is clear that the image of $\Delta = (d + d^*)^2$ is contained in $\operatorname{im} d \oplus \operatorname{im} d^*$. To prove the converse inclusion, we need to show that $d\omega$ and $d^*\eta$ are contained in $\operatorname{im}\Delta$ for any $\omega, \eta \in \Omega^*(X)$. Using the orthogonal direct sum decomposition (1.21) it suffices to show that $d\omega$ and $d^*\eta$ belong to $(\ker\Delta)^\perp$. To check this, let $\zeta \in \ker\Delta$. Then

$$\langle d\omega, \zeta\rangle = \langle \omega, d^*\zeta\rangle = 0 \qquad \text{and} \qquad \langle d^*\eta, \zeta\rangle = \langle \eta, d\zeta\rangle = 0$$

since $\zeta \in \ker\Delta = \ker d \cap \ker d^*$. □

Restricting the decomposition (**??**) to the summand $\Omega^k(X) \subset \Omega^*(X)$ we obtain an orthogonal direct sum decomposition

$$\Omega^k(X) = \mathcal{H}^k(X) \oplus \operatorname{im}(d\colon \Omega^{k-1}(X) \to \Omega^k(X)) \oplus \operatorname{im}({}^*\colon \Omega^{k+1}(X) \to \Omega^k(X)),$$

where $\mathcal{H}^k(X) := \mathcal{H}^*(X) \cap \Omega^k(X)$ is the space of harmonic forms of degree k. This has interesting consequences for the *de Rham cohomology*

$$H^k_{\mathrm{dR}}(X) := \frac{\ker d\colon \Omega^k(X) \to \Omega^{k+1}(X)}{\operatorname{im} d\colon \Omega^{k-1}(X) \to \Omega^k(X)}.$$

Clearly the summands $\mathcal{H}^k(X)$ and $\operatorname{im} d$ are in the kernel of the de Rham differential $d\colon \Omega^k(X) \to \Omega^{k-1}(X)$. To see that d restricted to $\operatorname{im} d^*$ is injective, let $d^*\eta \in \operatorname{im} d^*$. Then

$$\langle dd^*\eta, \eta\rangle = \langle d^*\eta, d^*\eta\rangle = ||d^*\eta||^2,$$

which implies that if $d^*\eta$ is in the kernel of d, then $d^*\eta = 0$ as desired. This shows that

$$\ker(d\colon \Omega^k(X) \to \Omega^{k-1}(X)) = \mathcal{H}^k \oplus \operatorname{im} d$$

and hence

$$H^k_{\mathrm{dR}}(X) = \frac{\mathcal{H}^k \oplus \operatorname{im} d}{\operatorname{im} d} \cong \mathcal{H}^k.$$

This statement is known as the Hodge Theorem. For future reference we state it explicitly.

Theorem 1.22. Hodge Theorem. *Let X be a compact riemannian manifold. Then the map*

$$\mathcal{H}^k(X) \longrightarrow H^k_{\mathrm{dR}}(X) \qquad \textit{given by} \qquad \omega \mapsto [\omega]$$

is a vector space isomorphism. Here $[\omega]$ is the de Rham cohomology class of the harmonic form ω.

The ellipticity of the Beltrami-Laplace operator Δ implies that the space of harmonic forms $\mathcal{H}^* = \ker \Delta$ is finite dimensional. Hence the Hodge Theorem has the following consequence.

Corollary 1.23. *The de Rham cohomology groups $H^k_{\mathrm{dR}}(X)$ of a compact manifold X are finite dimensional.*

It is interesting to contrast the two sides of the isomorphism of the Hodge Theorem. The advantage of the de Rham cohomology $H^k_{\mathrm{dR}}(X)$ is that its construction does not require a Riemannian metric on X, unlike the the space of harmonic forms $\mathcal{H}^*(X) = \ker \Delta$. The advantage of harmonic forms is that each de Rham cohomology class has a *unique harmonic cocycle representative.*

1.4 The Euler characteristic operator and the signature operator

For an n-manifold X, let

$$\Omega^{\mathrm{ev}}(X) := \bigoplus_{k \text{ even}} \Omega^k(X) \qquad \text{and} \qquad \Omega^{\mathrm{odd}}(X) := \bigoplus_{k \text{ odd}} \Omega^k(X).$$

We observe that the de Rham operator $D = d + d^* \colon \Omega^k(X) \to \Omega^k(X)$ maps even forms to odd forms and vice versa. In other words, D restricts to operators

$$D^+ \colon \Omega^{\mathrm{ev}}(X) \to \Omega^{\mathrm{odd}}(X) \qquad \text{and} \qquad D^- \colon \Omega^{\mathrm{odd}}(X) \to \Omega^{\mathrm{ev}}(X)$$

and with respect to the orthogonal direct sum decomposition

$$\Omega^*(X) = \Omega^{\mathrm{ev}}(X) \oplus \Omega^{\mathrm{odd}}(X) \tag{1.24}$$

the operator D has the form

$$\Omega^{\mathrm{ev}}(X) \oplus \Omega^{\mathrm{odd}}(X) \xrightarrow{\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}} \Omega^{\mathrm{ev}}(X) \oplus \Omega^{\mathrm{odd}}(X).$$

In particular, the ellipticity of D implies that D^+ and D^- are elliptic, and the self-adjointness of D implies that D^- is the adjoint of D^+, since

$$\langle D^+\omega, \eta\rangle = \langle D\omega, \eta\rangle = \langle \omega, D\eta\rangle = \langle \omega, D^-\eta\rangle \qquad \text{for } \omega \in \Omega^{\mathrm{ev}}(X) \text{ and } \eta \in \Omega^{\mathrm{odd}}(X).$$

It follows that

$$\begin{aligned}\operatorname{index} D^+ &= \dim\ker D^+ - \dim\ker D^- \\ &= \sum_{k \text{ even}} \dim \mathcal{H}^k(X) - \sum_{k \text{ odd}} \dim \mathcal{H}^k(X) \\ &= \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(X)\end{aligned}$$

In conjunction with the Hodge isomorphism $\mathcal{H}^k(X) \cong H^k_{\mathrm{dR}}(X)$, this implies the following result.

Proposition 1.25. *For a compact n-manifold X the index of the operator*

$$d + d^* \colon \Omega^{\mathrm{ev}}(X) \to \Omega^{\mathrm{odd}}(X)$$

is the Euler characteristic of X, defined by $\chi(X) := \sum_{k=0}^{n} (-1)^k \dim H^k_{\mathrm{dR}}(X)$.

This statement is the reason that the elliptic operator $d+d^* \colon \Omega^{\mathrm{ev}}(X) \to \Omega^{\mathrm{odd}}(X)$ is called the *Euler characteristic operator*.

Let us step back and look at the abstract features of this example.

Definition 1.26. A $\mathbb{Z}/2$-*graded vector space* or *super vector space* is a vector space V equipped with a $\mathbb{Z}/2$-grading, i.e., direct sum decomposition $V = V^+ \oplus V^-$. Alternatively, a $\mathbb{Z}/2$-grading can be described as given by an involution $\epsilon \colon V \to V$. An involution ϵ determines a decomposition $V = V^+ \oplus V^-$, where $V^\pm$ are the ± 1 eigenspaces of ϵ. Conversely, a decomposition $V = V^+ \oplus V^-$ determines an involution $\epsilon \colon V \to V$ by $\epsilon(v) = v$ for $v \in V^+$ and $\epsilon(v) = -v$ for $v \in V^-$.

If V is finite dimensional, its *super dimension* $\operatorname{sdim} V \in \mathbb{Z}$ is defined to by

$$\operatorname{sdim} V := \dim V^+ - \dim V^-.$$

A $\mathbb{Z}/2$-*graded (or super) inner product space* is an inner product space V equipped with a $\mathbb{Z}/2$-grading $V = V^+ \oplus V^-$ such that this is an *orthogonal* direct sum decomposition. Equivalently, in terms of a grading involution ϵ, the required compatibility with the inner product is that ϵ is an isometry.

Exercise: prove the last statement.

Using this terminology, $\Omega^*(X)$ is a $\mathbb{Z}/2$-graded inner product space with $\mathbb{Z}/2$-grading given by the decomposition (1.24) into even/odd forms.

Definition 1.27. Let V a graded vector space and $D \colon V \to W$ a linear operator. The operator D is *even* if

- D maps V^+ to V^+ and V^- to V^-, or equivalently if
- D commutes with the grading involution ϵ

and D is *odd* if

- D maps V^+ to V^- and V^- to V^+, or equivalently, if
- D anti-commutes with the grading involution ϵ.

More explicitly, with respect to the decomposition $V = V^+ \oplus V^-$, the operator D is given by a 2×2-matrix. The operator D is even (resp. odd) if the off-diagonal (resp. diagonal) entries of this matrix vanish.

Proposition 1.28. *Let X be a compact riemannian manifold, E a vector bundle over X equipped with a bundle metric, and let*

$$D\colon \Gamma(X;E) \longrightarrow \Gamma(X;E)$$

be an elliptic differential operator which is self-adjoint (i.e., $D^* = D$) or skew-adjoint (i.e., $D^* = -D$). Assume that $\Gamma(X;E)$ is equipped with a $\mathbb{Z}/2$-grading such that D is an odd operator. Then $D^+\colon \Gamma(X;E)^+ \to \Gamma(X;E)^-$ is an elliptic operator and

$$\mathrm{index} D^+ = \operatorname{sdim} \ker D,$$

where the $\mathbb{Z}/2$-grading on $\ker D$ is given by restricting the grading involution ϵ to $\ker D$ (the assumption that D is odd, i.e., anti-commutes with ϵ implies that ϵ maps $\ker D$ to itself).

Proof. We note that if D is self-adjoint, then $(D^+)^* = D^-$; if D is skew-adjoint, then $(D^+)^* = -D^-$. In either case, $\dim \operatorname{coker} D^+ = \dim \ker (D^+)^* = \dim \ker D^-$, and hence

$$\mathrm{index} D^+ = \dim \ker D^+ - \dim \ker D^- = \dim (\ker D)^+ - \dim (\ker D)^- = \operatorname{sdim} \ker D.$$

□

1.4.1 Hodge star and the signature operator

For an n-manifold X the dimension of $\Lambda^k T_x^* X$ is $\binom{n}{k}$. In particular, $\Lambda^k T_x^* X$ and $\Lambda^{n-k} T_x^* X$ have the same dimension. In fact the wedge product

$$\Lambda^k T_x^* X \times \Lambda^{n-k} T_x^* X \xrightarrow{\wedge} \Lambda^n T_x^* X$$

provides a pairing which is non-degenerate in the sense that the map

$$\begin{aligned} \Lambda^{n-k} T_x^* X &\longrightarrow \mathrm{Hom}(\Lambda^k T_x^* X, \Lambda^n T_x^* X) \\ \omega &\longmapsto (\beta \mapsto \beta \wedge \omega) \end{aligned}$$

is an isomorphism. If X is an oriented Riemannian n-manifold with volume form vol, then $\mathrm{vol}_x \in \Lambda^n T_x^* X$ is a distinguished non-zero element and the vector space $\Lambda^k T_x^* X$ has an inner product determined by the Riemannian metric. Hence for any element $\alpha \in \Lambda^k T_x^* X$ determines an element

$$(\beta \mapsto \langle \alpha, \beta \rangle \, \mathrm{vol}_x) \in \mathrm{Hom}(\Lambda^k T_x^* X, \Lambda^n T_x^* X).$$

Hence the isomorphism above determines a unique element $\star\alpha \in \Lambda^{n-k} T_x^* X$. More explicitly, $\star\alpha$ is uniquely determined by

$$\beta \wedge \star\alpha = \langle \alpha, \beta \rangle \, \mathrm{vol}_x \qquad \text{for all } \beta \in \Lambda^k T_x^* X.$$

The linear isomorphism $\star\colon \Lambda^k T_x^* X \to \Lambda^{n-k} T_x^* X$ depends smoothly on x and hence these maps fit together to give a vector bundle isomorphism

$$\star\colon \Lambda^k T^* X \xrightarrow{\cong} \Lambda^{n-k} T^* X.$$

This in turn induces an isomorphism on the associated section spaces:

$$\star\colon \Gamma(X; \Lambda^k T^* X) = \Omega^k(X) \xrightarrow{\cong} \Gamma(X; \Lambda^{n-k} T^* X) = \Omega^{n-k}(X)$$

The last isomorphism are referred to as *Hodge star operator*.

Lemma 1.29. *The Hodge star operator $\star\colon \Omega^*(X) \to \Omega^*(X)$ acting on the differential forms of an n-manifold X has the following properties.*

1. $\star \star \alpha = (-1)^{k(n-k)} \alpha$ *for $\alpha \in \Omega^k(X)$.*

2. *The adjoint d^* of the de Rham operator can be expressed in terms of the Hodge star operator; namely*

$$d^* \alpha = (-1)^{nk+n+1} \star d \star \alpha \qquad \text{for } \alpha \in \Omega^k(X)$$

3. *The Hodge star operator anti-commutes with the de Rham operator $D = d + d^*$, i.e., $D\star = -\star D$.*

Part (3) shows that for $\alpha \in \mathcal{H}^* = \ker D$ the element $\star\alpha$ is again harmonic, since

$$D \star \alpha = -\star D\alpha = 0.$$

This implies:

Corollary 1.30. *The star operator $\star\colon \Omega^k(X) \to \Omega^{n-k}(X)$ restricts to an isomorphism $\mathcal{H}^k(X) \cong \mathcal{H}^{n-k}(X)$. In particular, by the Hodge Theorem 1.22, it follows that* $\dim H^k_{\mathrm{dR}}(X) = \dim H^{n-k}_{\mathrm{dR}}(X)$*, which is one form of Poincare duality.*

Let X be an oriented compact manifold of dimension $n = 2\ell$. Then the map

$$\mathcal{H}^\ell \times \mathcal{H}^\ell \longrightarrow \mathbb{R} \qquad \text{defined by} \qquad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

is a non-degenerate bilinear form. Via the Hodge isomorphism $\mathcal{H}^\ell(X) \cong H^\ell_{\mathrm{dR}}(X)$ it corresponds to the bilinear form

$$H^\ell_{\mathrm{dR}}(X) \times H^\ell_{\mathrm{dR}}(X) \longrightarrow \mathbb{R} \qquad \text{defined by} \qquad (a, b) \mapsto \langle a \cup b, [X] \rangle,$$

where $a \cup b \in H^n_{\mathrm{dR}}(X)$ is the cup product of the cohomology classes a, b, and $\langle a \cup b, [X] \rangle \in \mathbb{R}$ is the evaluation of $a \cup b$ on the fundamental class $[X] \in H_n(X)$.

Due to $\alpha \wedge \beta = (-1)^\ell \beta \wedge \alpha$, this form is symmetric if ℓ is even and skew-symmetric if ℓ is odd. In the former case, i.e., if the dimension of X is divisible by 4, then the vector space $\mathcal{H}^\ell(X)$ can be decomposed in the form

$$\mathcal{H}^\ell(X) = \mathcal{H}^\ell_+(X) \oplus \mathcal{H}^\ell_-(X),$$

where the form is positive definite on $\mathcal{H}^\ell_+(X)$ and negative definite on $\mathcal{H}^\ell_-(X)$. The *signature of* X is defined by

$$\mathrm{sign}(X) := \dim \mathcal{H}^\ell_+(X) - \dim \mathcal{H}^\ell_-(X).$$

We note that for ℓ even $\star\colon \mathcal{H}^\ell(X) \to \mathcal{H}^\ell(X)$ is an involution. If $\alpha \in \mathcal{H}^\ell(X)$ belongs to the $+1$-eigenspace, i.e., $\star\alpha = \alpha$, then

$$\int_X \alpha \wedge \alpha = \int_X \langle \alpha, \star\alpha \rangle \,\mathrm{vol} = \int_X \langle \alpha, \alpha \rangle \,\mathrm{vol} = \int_X ||\alpha||^2 \,\mathrm{vol} \geq 0.$$

In other words, we can take $\mathcal{H}^\ell_\pm(X)$ to be the ± 1-eigenspace of $\star$. Summarizing the discussion so far we conclude that for a manifold X of dimension $n = 2\ell$ with ℓ even

$$\mathrm{sign}(X) = \mathrm{sdim}\, \mathcal{H}^\ell(X), \tag{1.31}$$

where $\mathcal{H}^\ell(X)$ is $\mathbb{Z}/2$-graded with grading involution $\star$.

Unfortunately, the Hodge star operator is not an involution on all of $\Omega^*(X)$, and hence we cannot use $\star$ as grading involution giving us an new $\mathbb{Z}/2$-grading on $\Omega^*(X)$.

Lemma 1.32. *Let X be an oriented riemannian manifold of dimension $n = 2\ell = 4m$. Let $\tau\colon \Omega^*(X) \to \Omega^*(X)$ be defined by $\tau\alpha := (-1)^{\binom{k}{2}+m} \star \alpha$ for $\alpha \in \Omega^k(X)$. Then*

1. *τ is an involution,*
2. *τ agrees with $\star$ on $\Omega^\ell(X)$, and*
3. *τ anti-commutes with the de Rham operator $D = d + d^*$.*

Proof: exercise.

The involution τ gives $\Omega^*(X)$ a $\mathbb{Z}/2$-grading; in particular, we have direct sum decomposition of $\Omega^*(X)$ as

$$\Omega^*(X) = \Omega^*_+(X) \oplus \Omega^*_-(X), \tag{1.33}$$

where $\Omega^*_\pm(X)$ is the ±1-eigenspace of τ. Then the fact that τ anti-commutes with D implies that applying D to $\alpha \in \Omega^*_\pm(X)$ produces an element $D\alpha \in \Omega^*_\mp(X)$, since

$$\tau(D\alpha) = -D\tau\alpha = \begin{cases} -D\alpha & \text{if } \alpha \in \Omega^*_+(X) \\ D\alpha & \text{if } \alpha \in \Omega^*_-(X) \end{cases}$$

In other words, with respect to the decomposition (1.33) the de Rham operator D has the form

$$\Omega^*_+(X) \oplus \Omega^*_-(X) \xrightarrow{\begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}} \Omega^*_+(X) \oplus \Omega^*_-(X).$$

The operator $D_+ \colon \Omega^*_+(X) \to \Omega^*_-(X)$, given by the restriction of the de Rham operator D to $\Omega^*_+(X) \subset \Omega^*(X)$ is called the *signature operator*. This terminology is motivated by the following result.

Proposition 1.34. *Let X be a compact oriented riemannian manifold of dimension $n = 2\ell$ with ℓ even. Then the index of the signature operator D_+ is equal to the signature of the manifold X.*

Proof. By the Proposition 1.28 the index of D_+ is the super dimension of $\ker D = \mathcal{H}^*(X)$, graded by the involution τ (by construction, τ is an involution on $\Omega^*(X)$, but it restricts to an involution on $\ker D$ since τ anti-commutes with D by part (3) of the previous lemma). We observe that the involution τ on

$$\mathcal{H}^*(X) = \mathcal{H}^0(X) \oplus \mathcal{H}^1(X) \oplus \cdots \oplus \mathcal{H}^n(X)$$

does not restrict to an involution of each piece $\mathcal{H}^k(X)$, since applying τ to $\alpha \in \mathcal{H}^k(X)$ yields an element $\tau\alpha \in H^{n-k}(X)$, except for $k = \ell = n/2$. However, if we decompose $\mathcal{H}^*(X)$ in the form

$$\mathcal{H}^*(X) = (\mathcal{H}^0(X) \oplus \mathcal{H}^n(X)) \oplus \cdots \oplus (\mathcal{H}^{\ell-1}(X) \oplus \mathcal{H}^{\ell+1}(X)) \oplus \mathcal{H}^\ell(X),$$

then τ does preserve each of the block summands $\mathcal{H}^k(X) \oplus \mathcal{H}^{n-k}(X)$ for $k = 0, \dots, \ell - 1$ and the summand $\mathcal{H}^\ell(X)$. It follows that the super dimension of $\mathcal{H}^*(X)$ is the sum of the super dimensions of the pieces:

$$\operatorname{sdim} \mathcal{H}^*(X) = \sum_{k=0}^{\ell-1} \operatorname{sdim}(\mathcal{H}^k(X) \oplus \mathcal{H}^{n-k}(X)) \ + \ \operatorname{sdim} \mathcal{H}^\ell(X)$$

We observe that for $k = 1, \dots, \ell - 1$ the map

$$\mathcal{H}^k(X) \longrightarrow \mathcal{H}^k(X) \oplus \mathcal{H}^{n-k}(X) \qquad \text{given by} \qquad \alpha \mapsto \alpha \pm \tau\alpha$$

in an isomorphism onto the ± 1-eigenspace of the involution τ acting on $\mathcal{H}^k(X) \oplus \mathcal{H}^{n-k}(X)$. In particular, the dimension of the $+1$-eigenspace is *equal* to the dimension of the -1-eigenspace and hence the super dimension of $\mathcal{H}^k(X) \oplus \mathcal{H}^{n-k}(X)$ (the difference between these dimensions) is zero. It follows that

$$\operatorname{sdim} \mathcal{H}^*(X) = \operatorname{sdim} \mathcal{H}^\ell(X) = \operatorname{sign}(X)$$

as claimed. □

2 Dirac operators and their index

Dirac operators and twisted Dirac operators are a very important class of first order elliptic operators defined on spin manifolds (see Definition 2.3 below). The reasons are:

- Many elliptic operators that show up in a geometric context are (twisted) Dirac operators; for example, the Euler characteristic operator or the signature operator are examples of twisted Dirac operators (at least if the manifold is spin, or with a suitably generalized notion of "Dirac operator).
- The index of an elliptic operator D on a compact manifold X depends only on its principal symbol σ^D, more precisely on the element $[\sigma^D]$ in the K-theory $K(T^*X, T^*X_0)$ of the pair (T^*X, T^*X_0) consisting of the total space of the cotangent bundle T^*X and its subspace $T^*X_0 \subset T^*X$ consisting of all non-zero cotangent vectors $\xi \in T^*X$ (this will be dealt with in detail later; this is an essential part of the "K-theory proof of the Index Theorem" that we will follow). It turns out that for a spin manifold X every class of $K(T^*X, T^*X_0)$ is given by the symbol of a twisted Dirac operator.

2.1 Spin structures

The notion of "spin manifold" has its origin in the fact that the group $SO(n)$ of orientation preserving isometries $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is not simply connected for $n \geq 2$; rather, its fundamental is

$$\pi_1 SO(n) = \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}/2 & n \geq 3 \end{cases}$$

In particular, for $n \geq 2$ there is a non-trivial double covering of $SO(n)$, unique up to isomorphism, which is usually denoted $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. In particular, $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the universal covering for $n \geq 3$. This is extended to $n = 1$ by noting that $\mathrm{SO}(1)$ is the trivial group, and defining $\mathrm{Spin}(1) := \mathbb{Z}/2$.

Exercise 2.1. Show that there is a unique group structure on $\mathrm{Spin}(n)$ such that the projection map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a homomorphism. Hint: Show that for any connected Lie group G, there is a group structure on its universal covering $\widetilde{G}$ by using the usual description of the universal covering of a space as homotopy classes relative endpoints of paths starting at the base point.

Remark 2.2. The spin groups show up in physics in the following way. A classical mechanical system with rotational symmetry, e.g., a planet orbiting around a sun, has a phase space on which the rotation group $\mathrm{SO}(3)$ acts. This might lead one to suspect that in a quantum system with rotational symmetry, e.g., the electron orbiting the proton in a hydrogen atom, the symmetry group $\mathrm{SO}(3)$ should act on the mathematical object describing the states of this quantum mechanical system, which is given by a Hilbert space and operators on that Hilbert space. The intriguing fact is that in general $\mathrm{SO}(3)$ does not act on the relevant Hilbert space, but only its double covering group $\mathrm{Spin}(3) = \mathrm{SU}(2)$.

Let X be an oriented riemannian n-manifold. Then its *oriented frame bundle* is the smooth fiber bundle $p\colon \mathrm{SO}(X) \to X$ is given by

$$\mathrm{SO}(X) := \{(x, f) \mid x \in X,\ f\colon \mathbb{R}^n \to T_x \text{ orientation preserving isometry}\}$$

and $p(x, f) = x$. Notice that if $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \to T_x$ is an orientation preserving isometry, then $\{f(e_1), \dots, f(e_n)\}$ is an oriented orthonormal basis of T_xX, which is the more traditional way to think of "frames". The advantage of thinking in terms of isometries $f\colon \mathbb{R}^n \to T_xX$ is that there is an evident right action:

$$\mathrm{SO}(X) \times \mathrm{SO}(n) \longrightarrow \mathrm{SO}(X) \qquad \text{given by} \qquad (x, f), g \mapsto (x, f \circ g).$$

This action is free, and its orbits are the fibers of $p\colon \mathrm{SO}(X) \to X$; in other words, the oriented frame bundle is a principal $\mathrm{SO}(n)$-bundle.

Definition 2.3. Let X be an oriented riemannian n-manifold. A *spin structure* on X is a double covering

$$\pi\colon \mathrm{Spin}(X) \longrightarrow \mathrm{SO}(X)$$

with the property that for each fiber $\mathrm{SO}(X)_x$ the restriction

$$\pi\colon \mathrm{Spin}(X)_x := \pi^{-1}(SO(X)_x) \longrightarrow \mathrm{SO}(X)_x$$

is a non-trivial double covering of $\mathrm{SO}(X)_x \cong \mathrm{SO}(n)$ for $n \geq 2$.

2.2 Principal bundles and associated vector bundles

Definition of principal bundle

Examples of principal bundles: frame bundle of a vector bundle, oriented frame bundle, orthogonal frame bundle, $\mathrm{SO}(X)$, $\mathrm{Spin}(X)$, etc

Classification of principal bundles

Homotopy theoretic interpretation of spin structure

Associated vector bundle construction

Examples of associated vector bundles (giving the sense that it is compatible with "linear algebra" constructions).

Generalizing the argument that the group structure on $\mathrm{SO}(n)$ induces a group structure on the total space of the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, it can be shown that the right $\mathrm{SO}(n)$-action on $\mathrm{SO}(X)$ can be lifted to an action of $\mathrm{Spin}(n)$ on $\mathrm{Spin}(X)$ in the sense that the diagram

$$\begin{tikzcd}
\mathrm{Spin}(X) \times \mathrm{Spin}(n) \arrow[r] \arrow[d] & \mathrm{Spin}(X) \arrow[d] \\
\mathrm{SO}(X) \times \mathrm{SO}(n) \arrow[r] & \mathrm{SO}(X)
\end{tikzcd}$$

is commutative. The action of $\mathrm{Spin}(n)$ on $\mathrm{Spin}(X)$ is again free and transitive on the fibers of the projection map $\mathrm{Spin}(X) \to X$; in other words, $\mathrm{Spin}(X) \to X$ is a principal $\mathrm{Spin}(n)$-bundle. This motivates the notation $\mathrm{Spin}(X)$ for the total space of the double covering $\mathrm{Spin}(X) \to \mathrm{SO}(X)$.

Remark 2.4. The homotopy theoretic take on orientations and spin structures via maps to $BO(n)$, $BSO(n)$, $BSpin(n)$.

A spin structure on a manifold X enables us to construct new vector bundles on X using the principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(X) \to X$ via the following construction.

The associated vector bundle construction. Let G be a Lie group, $G \times V \to V$ a representation of G, and $\pi\colon P \to X$ a principal G-bundle over X. Then

$$P \times_G V := \{(p, v) \mid p \in P,\ v \in V\}/\sim \longrightarrow X \qquad [p, v] \mapsto \pi(p)$$

is a vector bundle over X of rank $\dim V$. Here the equivalence relation $\sim$ on $P \times V$ is defined by $(pg, v) \sim (p, gv)$ for all $p \in P$, $g \in G$, $v \in V$ (notice the formal similarity with the definition of the tensor product $M \otimes_A N$ of a right A-module M and a left A-module N). To check that for $x \in X$ the fiber $(P \times_G V)_x = \pi^{-1}(x)$ is isomorphic to the vector space V, we pick an element $p \in P_x$ and note that the map

$$V \longrightarrow (P \times_G V)_x \qquad v \mapsto [p, v]$$

is a vector space isomorphism.

Example 2.5. (Examples of associated vector bundles)

1. Let X be an oriented riemannian manifold and let $\mathrm{SO}(X) \to X$ be the oriented orthonormal frame bundle, principal $\mathrm{SO}(n)$-bundle. Consider $\mathbb{R}^n$, $(\mathbb{R}^n)^*$ and $\Lambda^k(\mathbb{R}^n)^*$ as representations of $\mathrm{SO}(n)$ given by the standard action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$. Then
$$\mathrm{SO}(X) \times_{\mathrm{SO}(n)} \mathbb{R}^n \cong TX$$
$$\mathrm{SO}(X) \times_{\mathrm{SO}(n)} (\mathbb{R}^n)^* \cong T^*X$$
$$\mathrm{SO}(X) \times_{\mathrm{SO}(n)} \Lambda^k(\mathbb{R}^n)^* \cong \Lambda^k T^*X.$$

2. Consider $\mathbb{R}^n$ as a representation of $\mathrm{Spin}(n)$ via the double covering map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Then the associated bundle $\mathrm{Spin}(n) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \cong \mathrm{SO}(n) \times_{\mathrm{SO}(n)} \mathbb{R}^n \cong TX$. More generally, if for any $\mathrm{SO}(n)$-representation V there is an isomorphism of vector bundles
$$\mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} V \cong \mathrm{SO}(X) \times_{\mathrm{SO}(n)} V.$$
In other words, we don't get any "new" vector bundles associated to the principal bundle $\mathrm{Spin}(X) \to X$ as long as we use representations of $\mathrm{Spin}(n)$ which factor through $\mathrm{SO}(n)$.

3. There is a complex representation Δ of $\mathrm{Spin}(n)$ called the *complex spinor representation* where the non-trivial element $-1 \in \pi^{-1}(1) \subset \mathrm{Spin}(n)$ acts by multiplication by -1. In particular, the representation does not factor through $\mathrm{SO}(n)$. The representation Δ is irreducible for n odd; for n even it splits as a sum $\Delta = \Delta_+ \oplus \Delta_-$ of two non-isomorphic irreducible representations $\Delta_\pm$. If X is a riemannian spin n-manifold, the associated vector bundle
$$S := \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \Delta$$
is called the *spinor bundle*. If n is even, this bundle has a direct sum decomposition $S = S^+ \oplus S^-$, where $S^\pm := \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \Delta^\pm$.

The spinor bundle on a spin manifold X is a much more subtle vector bundle then the usual vector bundles that we built from the tangent bundle by using direct sums, dualizing, tensoring or forming symmetric/exterior powers of bundles. Ultimately, the spinor bundle **is** constructed from the tangent bundle TX, but via a detour through principal bundles:

1. We associate to the tangent bundle TX the oriented orthonormal frame bundle $\mathrm{SO}(X) \to X$, a principal $\mathrm{SO}(n)$ bundle.

2. A spin structure on X allows us to pass to the principal bundle $\mathrm{Spin}(X) \to X$ for the double covering group $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.

3. The spinor bundle is the vector bundle $S = \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \Delta$ associated to the spinor representation Δ.

The crucial property of the spinor bundle relevant for the construction of the Dirac operator is that there is a vector bundle map

$$T^*X \otimes S \xrightarrow{c} S \tag{2.6}$$

called *Clifford multiplication* since (as we will see later this semester), it is constructed using Clifford algebras. With respect to the decomposition $S = S^+ \oplus S^-$ for n even, Clifford multiplication by a cotangent vector $\xi \in T^*_x X$ is an *odd* endomorphism of the fiber $S_x = S^+_x \oplus S^-_x$, i.e., it maps elements in S^+_x to S^-_x and vice versa.

2.3 Connections

The Dirac operator on a spin manifold X will be a first order differential operator

$$D\colon \Gamma(X;S) \longrightarrow \Gamma(X;S)$$

acting on the sections of the spinor bundle $S = \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \Delta$, where Δ is the spinor representation of $\mathrm{Spin}(n)$. Besides the Clifford multiplication (2.6) the other essential ingredient in the construction of the Dirac operator is a connection on the spinor bundle.

We review the notion of a connection on a vector bundle, first motivating it by discussing "directional derivatives" of a smooth function $h \in C^\infty(X)$. The differential of f is the 1-form $dh \in \Omega^1(X) = \Gamma(X;T^*X)$. Given a tangent vector $v \in T_xX$, the differential ds can be evaluated on v to obtain a real number $ds(v) \in \mathbb{R}$. Geometrically, $ds(v)$ is the derivative of s at x in the direction of the tangent vector v. If V is a vector field on X, i.e., a section of the tangent bundle TX, then we can evaluate ds on V to obtain the function

$$d_Vs := ds(V) \in C^\infty(X),$$

which we think of as the *derivative of s in the direction of the vector field V*. More generally, if $s\colon X \to E$ is a smooth map with values in a (finite dimensional) vector space E, then $ds \in \Omega^1(X;E) := \Gamma(X;T^*X \otimes E)$ (where we abuse notation by using the symbol E also to refer to the trivial vector bundle over X with fiber E), and hence $d_Vs \in C^\infty(X;E)$. We observe the following algebraic properties of d_Vs with respect to the V-slot and the s-slot:

(i) for fixed $s \in C^\infty(X;E)$, the map $\Gamma(X;TX) \longrightarrow C^\infty(X;E)$ is a map of $C^\infty(X)$-modules, i.e.

$$\begin{aligned} d_{V_1+V_2}s =& d_{V_1}s + d_{V_2}s && \text{for } V_1,V_2 \in \Gamma(X;TX),\ s \in C^\infty(X;E) \\ d_{fV}s =& f d_V f && \text{for } V \in \Gamma(X;TX),\ f \in C^\infty(X),\ s \in C^\infty(X;E). \end{aligned}$$

(ii) for fixed vector field $V \in \Gamma(X;TX)$ the map $C^\infty(X;E) \longrightarrow C^\infty(X;E)$ has the properties

$$\begin{aligned} d_V(s_1+s_2) =& d_Vs_1 + d_Vs_2 && \text{for } V \in \Gamma(X;TX),\ s_1,s_2 \in C^\infty(X;E) \\ d_Vfs =& (d_Vf)s + fd_Vs && \text{for } V \in \Gamma(X;TX),\ f \in C^\infty(X),\ s \in C^\infty(X;E). \end{aligned}$$

Thinking of the section $s \in \Gamma(X; E)$ of a vector bundle E as a generalization of a vector valued smooth function, one would like to talk about the derivative of s in the direction of a vector field V. Unlike for vector valued functions, there is no distinguished way to make sense of the derivative of a section s in the direction of a vector field V.

Definition 2.7. Let E be vector bundle over the manifold X. A *covariant derivative* on E is a map

$$\nabla \colon \Gamma(TX) \times \Gamma(E) \longrightarrow \Gamma(E) \qquad (V, s) \mapsto \nabla_V s$$

which shares the algebraic properties of differentiation of vector-valued functions, i.e.,

$$\begin{aligned}
&\nabla_{V_1+V_2} s = \nabla_{V_1} s + \nabla_{V_2} s \\
&\nabla_{fV} s = fs \\
&\nabla_V (s_1 + s_2) = \nabla_V s_1 + \nabla_V s_2 \\
&\nabla_V fs = (d_V f)s + f\nabla_V s
\end{aligned}$$

for $V, V_1, V_2 \in \Gamma(TX)$, $s, s_1, s_2 \in \Gamma(E)$, $f \in C^\infty(X)$.

We note that for fixed $s \in \Gamma(E)$, the first two properties say that the map

$$\nabla s \colon \Gamma(TX) \to \Gamma(E) \qquad \text{given by} \qquad V \mapsto \nabla_V s$$

is a map of $C^\infty(X)$-modules. Equivalently, this means it is induced by a map of vector bundles $TX \to E$, which in turn is a section of the vector bundle $\mathrm{Hom}(TX, E) \cong T^*X \otimes E$. Abusing notation, we call this section of $T^*X \otimes E$ again ∇s. The first two properties of the covariant derivative can then by recast in terms of the map

$$\nabla \colon \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E) \qquad \text{given by} \qquad s \mapsto \nabla s,$$

namely,

1. ∇ is a linear map of vector spaces (over $\mathbb{R}$ resp. $\mathbb{C}$ depending on whether E is a real or a complex vector bundle);

2. ∇ satisfies the *Leibnitz rule* $\nabla(fs) = df \otimes s + f\nabla s$.

Definition 2.8. A connection on a vector bundle E is a map $\nabla \colon \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E)$ satisfying these two properties.

So a covariant derivative and a connection are just two different way to think about the same mathematical concept, and we will always pass back and forth between these two. Any smooth vector bundle E has a connection; they can be constructed local trivializations, and combining these via partitions of unity. Much more precisely, the space of connections on E is a *torsor* for $\Gamma(T^*X \otimes \mathrm{End}(E))$, i.e., this abelian group acts freely and transitively on the space of connections.

If the vector bundle E has additional structures, then we can require connections to be compatible with these structures. Here are some examples.

1. Suppose E is equipped with a bundle metric $\langle\ ,\ \rangle$. Then the point-wise inner product $\langle s_1, s_2\rangle$ of two sections $s_1, s_2 \in \Gamma(E)$ is a smooth function on X. We can require that a connection ∇ is compatible with the bundle metric in the sense that we have the "product rule"
$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$$
for all sections s_1, s_2. Such a connection is called a *metric connection*.

2. The tangent bundle of a manifold has additional structure provided by the Lie bracket $[V, W]$ of vector fields $V, W \in \Gamma(TX)$. We can require that a connection ∇ on TX is compatible with this structure in the sense that
$$[V, W] = \nabla_V W - \nabla_W V.$$
Such a connection ∇ is called *torsion free*.

3. If X is a Riemannian manifold, there is a unique connection on the tangent bundle TX which is metric and torsion free. This is the *Levi-Civita connection*.

If E, F are vector bundles with connections ∇^E, ∇^F, then all the bundles "built from E and F by linear algebra", e.g., $E\oplus F$, $E\otimes F$, E^*, $\mathrm{Hom}(E, F) \cong E^*\otimes F$, $\Lambda^k E$ have connections ∇ built from ∇^E and ∇^F. For example if $s \in \Gamma(E)$ and $t \in \Gamma(F)$, then $(s, t) \in \Gamma(E \oplus F)$ and $s \otimes t \in \Gamma(E \otimes F)$. Then
$$\begin{aligned}\nabla(s, t) :=& (\nabla^E s, \nabla^F t) \in \Gamma(T^*X \otimes (E \otimes F)) = \Gamma(T^*X \otimes E \ \oplus\ T^*X \otimes F)\\ \nabla(s \otimes t) :=& (\nabla^E s) \otimes t + s \otimes \nabla^F t \in \Gamma(T^*X \otimes E \otimes F).\end{aligned}$$

As discussed above, the spinor bundle S on a riemannian spin manifold X is "built from the tangent bundle". Hence it might be expected that the Levi-Civita connection on TX induces a connection ∇^S on the spinor bundle. This is in fact true, but alas the construction of that connection is harder and proceeds by carrying along the connection in each of the steps in the construction of the spinor bundle. In particular, it requires talking about connections on *principal bundles*. We will do that later this semester.

Definition 2.9. Let X be a riemannian spin manifold with spinor bundle S. Then the *Dirac operator* on X is the first order differential operator given by the composition
$$\Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*X \otimes S) \xrightarrow{c} \Gamma(S),$$
where ∇^S is the connection on the spinor bundle induced by the Levi-Civita connection on TX, and c is Clifford multiplication. If E is a vector bundle with connection over X, then

the connection on E and the connection on S combine to define a connection ∇ on the tensor product $S \otimes E$. The composition

$$\Gamma(S \otimes E) \xrightarrow{\nabla} \Gamma(T^*X \otimes S \otimes E) \xrightarrow{c \otimes \mathrm{id}_E} \Gamma(S \otimes E),$$

is a first order differential operator called *twisted Dirac operator* or *Dirac operator twisted by E*.

Later this semester when we construct the group $\mathrm{Spin}(n)$ and the spinor representation Δ using Clifford algebras, we will also construct the Clifford multiplication map $c\colon T^*X \otimes S \to S$ and calculate the principal symbol of the Dirac operator. In particular, it will turn out that the Dirac operator and twisted Dirac operators are elliptic operators.

3 The Index Theorem for Dirac operators

In this section we state the Index Theorem for (twisted) Dirac operators, which expresses the index of the twisted Dirac operator D_E on a closed even-dimensional riemannian spin manifold X in terms of topological invariants of X and E. In fact, we give two versions, a *K-theory version*, where the topological invariant is expressed in terms of the K-theory of X, and a *cohomological version* where the topological invariant is expressed in terms of the cohomology ring $H^*(X;\mathbb{Q})$ and characteristic classes of TX and E (which are elements of $H^*(X;\mathbb{Q})$).

Theorem 3.1. Index Theorem for Dirac operators, K-theory version *Let X be a closed riemannian spin n-manifold, n even. Let E be a complex vector bundle over X equipped with a connection, and let D_E be the Dirac operator on X twisted by E. Then*

$$\mathrm{index}(D_E^+) = p_!([E]) \in K(\mathrm{pt}) = \mathbb{Z},$$

where $[E] \in K(X)$ is the K-theory class represented by E, and $p_!\colon K(X) \to K(\mathrm{pt})$ is the pushforward map in K-theory induced by the projection map $p\colon X \to \mathrm{pt}$.

Theorem 3.2. Index Theorem for Dirac operators, cohomology version *With the same hypothesis as above,*

$$\mathrm{index}(D_E^+) = \langle \widehat{A}(TX)\,\mathrm{ch}(E), [X]\rangle,$$

where

- $\widehat{A}(TX) \in H^*(X;\mathbb{Q})$ *is the* A-roof class *of the tangent bundle, which is a polynomial (with rational coefficients) in the Pontryagin classes $p_i(TX) \in H^{4i}(X;\mathbb{Z})$;*

- $\mathrm{ch}(E) \in H^*(X;\mathbb{Q})$ *is the* Chern character *of the complex vector bundle E, which is a polynomial (with rational coefficients) in the Chern classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$;*
- $\langle \widehat{A}(TX)\,\mathrm{ch}(E), [X]\rangle \in \mathbb{Q}$ *is the evaluation of $\widehat{A}(TX)\,\mathrm{ch}(E) \in H^*(X;\mathbb{Q})$ (the cup product of $\widehat{A}(TX)$ and $\mathrm{ch}(E)$) on the fundamental class $[X] \in H_n(X;\mathbb{Z})$ (which ignores all but the degree n part of $\widehat{A}(TX)\,\mathrm{ch}(E)$).*

Our next goal is the construction of the K-theory umkehr map $p_!\colon K(X) \to K(\mathrm{pt})$ used in the K-theory formulation of the index theorem. In section **??** we will discuss K-theory and the umkehr map $p_!$. While the construction of the umkehr map in K-theory is not particularly involved, it is hard to motivate, and hence to understand conceptually. For motivational reasons, we prefer to discuss umkehr maps first in ordinary cohomology. For a closed oriented n-manifold X, there is a map

$$p_!\colon H^*(X) \longrightarrow H^{*-n}(\mathrm{pt}) \qquad \text{given by} \qquad a \mapsto \begin{cases} \langle a, [X]\rangle \in \mathbb{Z} = H^0(\mathrm{pt}) & \text{for } \deg(a) = n \\ 0 & \text{otherwise} \end{cases}$$

where $\langle a, [X]\rangle$ is the evaluation of a on the fundamental class $[X] \in H_n(X;\mathbb{Z})$. This is quite straightforward, but it requires talking about the fundamental class, which is a *homology class*. The same strategy works in principle for K-theory, but it would require to talk not only about K-cohomology (the usual thing one has in mind when talking about K-theory), but K-homology and the K-homology fundamental class. There is a beautiful analytic description of K-homology, including a construction of the K-theory fundamental class of spin manifold in terms of its Dirac operator. Alas, that's a long story...

Fortunately, there is one setting in which the description of the pushforward map is straightforward, namely in de Rham cohomology. The map

$$p_!\colon H^*_{\mathrm{dR}}(X) \longrightarrow H^{*-n}(\mathrm{pt}) \qquad \text{is given by} \qquad [\alpha] \mapsto \begin{cases} \int_X \alpha \in \mathbb{R} = H^0_{\mathrm{dR}}(\mathrm{pt}) & \deg(\alpha) = n \\ 0 & \deg(\alpha) \neq n \end{cases}$$

So our strategy will be to start with the umkehr map in de Rham cohomology, and then to show that $p_!$ can be written as a composition of maps all of which involving just de Rham cohomology of auxiliary spaces. Those maps can then be generalized to other cohomology theories, and their composition will serve then as the *definition* of $p_!$ there. So the next section is purely motivational; the impatient reader is welcome to go straight to the definition of $p_!$ in K-theory in section **??**.

3.1 The umkehr map in de Rham cohomology

The construction of the pushforward map in K-theory (or in any other generalized cohomology theory) is based on the Thom isomorphism. The Thom isomorphism in K-theory in turn is based on the construction of the spinor representation Λ of $\mathrm{Spin}(n)$ via Clifford algebras, which we will do later this semester. The representation Λ is also crucial for the construction of the spinor bundle $S = \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \Delta$, and hence for the construction of the Dirac operator. In fact, I think of the Dirac operator as the geometric/analytic incarnation of the Thom class in K-theory, since the principal symbol of the Dirac operator represents the K-theory Thom class of the cotangent bundle.

While the construction of the pushforward map is pretty straightforward once the Thom isomorphism has been established (see **??**), I feel that construction is not well-motivated. By contrast, the pushforward map in de Rham cohomology has a great geometric interpretation as "integration over the manifold". For that reason, it seems better to first define the pushforward via integration, and then show that this integration map can equivalently be formulated via the Thom isomorphism in de Rham cohomology.

Let X be an oriented n-manifold (without boundary), and for $k \in \mathbb{Z}$ let $\Omega^k_c(X)$ be the vector space of k-forms on X with compact support (by definition, $\Omega^k_c(X)$ is the trivial vector space for $k < 0$ or $k > n$). Integration gives a linear map

$$\Omega^n_c(X) \longrightarrow \mathbb{R} \qquad \text{given by} \qquad \alpha \mapsto \int_X \alpha.$$

If $\alpha = d\beta$ for some form $\beta \in \Omega^{n-1}_c(X)$, then $\int_X \alpha = 0$ by Stokes' Theorem. This can be interpreted cohomologically as follows. For $k \in \mathbb{Z}$, let $H^k_{\mathrm{dR},c}(X)$ be the *de Rham cohomology with compact support of degree* k, defined to be the real vector space

$$H^k_{\mathrm{dR},c}(X) := \frac{\{\text{closed } k\text{-forms with compact support}\}}{\{\text{exact } k\text{-forms with compact support}\}} = \frac{\ker d\colon \Omega^k_c(X) \to \Omega^{k+1}_c(X)}{\operatorname{im} d\colon \Omega^{k-1}_c(X) \to \Omega^k_c(X)}.$$

As usual, let $H^*_{\mathrm{dR},c}(X)$ be the $\mathbb{Z}$-graded vector space obtained as the direct sum

$$H^*_{\mathrm{dR},c}(X) := \bigoplus_{k\in\mathbb{Z}} H^k_{\mathrm{dR},c}(X)$$

Definition 3.3. For an oriented n-manifold X, let

$$p_!\colon H^n_{\mathrm{dR},c}(X) \longrightarrow \mathbb{R} \qquad \text{be the linear map defined by} \qquad p_!([\alpha]) := \int_X \alpha$$

Abusing notation, we will also write

$$p_!\colon H^*_{\mathrm{dR},c}(X) \longrightarrow H^{*-n}_{\mathrm{dR}}(\mathrm{pt})$$

for the linear map of $\mathbb{Z}$-graded vector spaces which in degree n is the map $p_!$ above, with $\mathbb{R}$ identified with $H^0_{\mathrm{dR}}(\mathrm{pt})$, the degree 0 de Rham cohomology of the point, and which is trivial in degrees $k \neq n$. This map is called the *integration map* or *umkehr map* associated to the projection map $p\colon X \to \mathrm{pt}$.

To motivate the terminology, we note that a smooth map $f\colon X \to Y$ induces maps $f^*\colon \Omega^*(Y) \to \Omega^*(X)$ on differential forms and $f^*\colon H^*_{\mathrm{dR}}(Y) \to H^*_{\mathrm{dR}}(X)$ on de Rham cohomology. The map $p_!$ goes in the *opposite direction* of the induced map p^*, which is the origin of the terminology *umkehr map* (*umkehr* is the German word for *reversion*).

We want to generalize this integration map from a single oriented manifold to a family of oriented manifolds. Later for us the parameter space will be the n-manifold X, and so from the outset we look at a family E of oriented k-manifolds parametrized by X. In other words,

$$\pi\colon E \longrightarrow X$$

be a smooth fiber bundle with k-dimensional fibers such that the *vertical tangent bundle* VE (also known as *tangent bundle along the fibers*) is oriented. This is a vector bundle over the total space E whose fiber at a point $e \in E$ is given by

$$V_eE = \ker \pi_*\colon T_eE \to T_{\pi(e)}X = T_eE_x,$$

where π_* is the differential of π, and $E_x = \pi^{-1}(x) \subset E$ is is fiber over a point $x \in X$. In particular, an orientation on VE restricts to an orientation on the tangent bundle on each fiber $TE_x = VE_{|E_x}$, thus giving an orientation on each fiber. However, orientability of each fiber does not guarantee orientability of the vertical tangent bundle.

For example, a diffeomorphism $g\colon F \to F$ on an k-manifold F yields a smooth fiber bundle

$$\pi\colon (F \times \mathbb{R})/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1 \qquad [z,t] \mapsto [t] \qquad \text{for } z \in F,\, t \in \mathbb{R},$$

where $n \in \mathbb{Z}$ acts on $\mathbb{R}$ by $t \mapsto t+n$, and on $F \times \mathbb{R}$ by $(z,t) \mapsto (g^n(z), t+n)$. Hence each fiber is diffeomorphic to F and hence orientable if F is, but the tangent bundle along the fibers turns out to be orientable if and only if the diffeomorphism g is orientation preserving.

Given a fiber bundle $\pi\colon E \to X$ with k-dimensional fibers and oriented vertical tangent bundle, we want to construct an integration map

$$\pi_!\colon \Omega^*_{cv}(E) \longrightarrow \Omega^{*-k}(X),$$

where the subscript cv stands for *compact vertical support*, meaning that $\Omega^*_{cv}(E)$ consists of all forms $\alpha \in \Omega^*(E)$ such that $\mathrm{supp}(\alpha) \cap E_x$, the intersection of the support of α and the fiber E_x is compact for all $x \in X$. In particular, if X is compact, this amounts to the requirement that α has compact support.

Let $\alpha \in \Omega^n_{cv}(E)$. To describe $\pi_!\alpha \in \Omega^{n-k}_c(X)$ we need to specify $(\pi_!\alpha)_x(w_1, \dots, w_{n-k}) \in \mathbb{R}$ for any point $x \in X$ and tangent vectors $w_1, \dots, w_{n-k} \in T_xX$. Let

$$\alpha_{w_1,\dots,w_{n-k}} \in \Omega^k_c(E_x)$$

be the differential form determined by

$$(\alpha_{w_1,\dots,w_{n-k}})_x(v_1 \dots, v_k) = \alpha_x(v_1 \dots, v_k, \widetilde{w}_1, \dots, \widetilde{w}_{n-k}),$$

for $e \in E_x$ and $v_1, \ldots, v_n \in T_e E_x$. Here the $\widetilde{w}_i \in T_e E$ are *lifts* of $w_i \in T_x X$ in the sense that $\pi_*(\widetilde{w}_i) = w_i$ for $i = 1, \ldots, n-k$. We note that this is independent of the choice of the lifts $\widetilde{w}_i$, since if $\widetilde{w}'_i$ is another choice of lift, then $\widetilde{w}'_i = \widetilde{w}_i + u_i$ for vertical tangent vectors u_i. If the vertical tangent vectors $v_1, \ldots, v_k$ form a basis for the vertical tangent space $V_e E = T_e E_x$, then u_i is a linear combination of the v_j's and hence

$$\begin{aligned}
&\alpha_e(v_1 \ldots, v_n, \widetilde{w}_1, \ldots, \widetilde{w}'_i, \ldots, \widetilde{w}_{n-k}) - \alpha_e(v_1 \ldots, v_n, \widetilde{w}_1, \ldots, \widetilde{w}_i, \ldots, \widetilde{w}_{n-k}) \\
=&\alpha_e(v_1 \ldots, v_n, \widetilde{w}_1, \ldots, u'_i, \ldots, \widetilde{w}_{n-k}) \\
=&0.
\end{aligned}$$

If the v_i are not linearly independent, then $\alpha_e(v_1 \ldots, v_k, \widetilde{w}_1, \ldots, \widetilde{w}_i, \ldots, \widetilde{w}_{n-k})$ vanishes in any case. We define $p_! \alpha \in \Omega^{n-k}_c(Y)$ by

$$(p_! \alpha)_x(w_1, \ldots, w_{n-k}) := \int_{E_x} \alpha_{w_1, \ldots, w_{n-k}} \in \mathbb{R}. \tag{3.4}$$

This is a multilinear and alternating function of the tangent vectors $w_1, \ldots, w_{n-k} \in T_x X$, and hence $p_! \alpha \in \Omega^{n-k}_c(X)$.

Proposition 3.5. *The umkehr map $\pi_! \colon \Omega^*_{cv}(E) \longrightarrow \Omega^{*-k}(X)$ has the following properties.*

1. *$\pi_!$ is chain map, i.e., it commutes with the de Rham differential and hence induces an Umkehr map of de Rham cohomology with compact support*

$$p_! \colon H^*_{\mathrm{dR},c}(E) \longrightarrow H^{*-k}_{\mathrm{dR},c}(X)$$

2. *$\pi_!(\alpha \wedge \pi^* \beta) = \pi_! \alpha \wedge \beta$ for $\alpha \in \Omega^*_{cv}(E)$, $\beta \in \Omega^*(X)$.*

3. *The Umkehr map is compatible with compositions in the sense that if $\pi \colon E \to X$ and $p \colon X \to Y$ are fiber bundles with oriented vertical tangent bundles, then $(p \circ \pi)_! = p_! \circ \pi_!$.*

Let V be a real oriented vector space of dimension k. Then it is not hard to construct a compactly supported n-form $\omega \in \Omega^n_c(V)$ with $\int_V \omega = 1$. More generally, if $\pi \colon V \to X$ is an oriented vector bundle with n-dimensional fibers, one can construct an n-form with compact vertical support $\omega \in \Omega^n_{cv}(V)$ such that

- $d\omega = 0$.
- $\int_{V_x} \omega = 1$ for any $x \in X$ ($V_x = \pi^{-1}(x)$ is the fiber over x); in particular, $\pi_! \omega = 1 \in \Omega^0(X) = C^\infty(X)$.

Such a form ω is called a *Thom form*. Using a bundle metric on V and a metric connection, there are explicit formulas for such forms.

Now assume that X is compact and choose an embedding $X \hookrightarrow \mathbb{R}^{n+k}$ of the n-manifold X into the Euclidean space of sufficiently large dimension ($k = n$ will do by Whitney's embedding theorem). Let $V \to X$ be the normal bundle of X, a vector bundle of rank k. Regarding X as a subspace of V (namely, the zero section), by the tubular neighborhood theorem, the embedding $X \hookrightarrow \mathbb{R}^{n+k}$ extends to an embedding $i\colon V \hookrightarrow \mathbb{R}^{n+k}$. Consider the following commutative diagram of smooth maps

$$\begin{tikzcd} V \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathbb{R}^{n+k} \arrow[d, "q"] \\ X \arrow[r, "p"] & \mathrm{pt} \end{tikzcd}$$

We claim that this (really dumb) commutative diagram induces a commutative diagram of de Rham complexes

$$\begin{tikzcd} \Omega_c^{*+k}(V) \arrow[r, "i_!"] \arrow[d, "\pi_!"'] & \Omega_c^{*+k}(\mathbb{R}^{n+k}) \arrow[d, "q_!"] \\ \Omega^*(X) \arrow[r, "p_!"] & \Omega^{*-n}(\mathrm{pt}) \end{tikzcd}$$

Here $i_!$ is the map given by extending a form α with compact support on $V \subset \mathbb{R}^{n+k}$ to a form on all of $\mathbb{R}^{n+k}$ which vanishes outside the support of α. The commutativity of this diagram is easy to prove (but not tautological as the commutativity of the previous diagram). To prove it, let $\beta \in \Omega_c^{n+k}(V)$ (for all other degrees, the target $\Omega_c^{*-(n+k)}(\mathrm{pt})$ is trivial). Then

$$p_!\pi_!\beta = (p \circ \pi)_! = \int_V \beta = \int_{\mathbb{R}^{n+k}} i_!\beta = q_!i_!\beta.$$

Let ω be a Thom form for the normal bundle $V \to X$. Our assumption that X is compact implies that ω has not only compact vertical support, but indeed compact support, i.e., $\omega \in \Omega_c^k(V)$. By the definition of the Thom class we have $\pi_!\omega = 1 \in \Omega^0(X)$, and by property (2) of Proposition 3.5 it follows that

$$\pi_!(\omega \wedge \pi^*\alpha) = (\pi_!\omega) \wedge \alpha = \alpha.$$

Hence the commutativity of the previous diagram applied to $\beta = \omega \wedge \pi^*\alpha$ for $\alpha \in \Omega^*(X)$ implies the commutativity of the diagram

$$\begin{tikzcd} \Omega_c^{*+k}(V) \arrow[r, "i_!"] & \Omega_c^{*+k}(\mathbb{R}^{n+k}) \arrow[d, "q_!"] \\ \Omega^*(X) \arrow[r, "p_!"] \arrow[u, "\omega\wedge\pi^*(\ )"] & \Omega^{*-n}(\mathrm{pt}) \end{tikzcd}$$

All of these maps are chain maps, and so we obtain a commutative diagram of de Rham cohomology groups

$$\begin{tikzcd} H^{*+k}_{\mathrm{dR},c}(V) \arrow[r, "i_!"] & H^{*+k}_{\mathrm{dR},c}(\mathbb{R}^{n+k}) \arrow[d, "q_!"] \\ H^*_{\mathrm{dR}}(X) \arrow[u, "{[\omega]\cup\pi^*(\ )}"] \arrow[r, "p_!"] & H^{*-n}_{\mathrm{dR}}(\mathrm{pt}) \end{tikzcd}$$

We note that integration map $q_!$ is an isomorphism. Its inverse can be described explicitly by

$$H^{*-n}_{\mathrm{dR}}(\mathrm{pt}) \longrightarrow H^{*+k}_{\mathrm{dR},x}(\mathbb{R}^{n+k}) \qquad \text{given by} \qquad [\beta] \mapsto [\omega_{n+k} \wedge q^*(\beta)] = [\omega_{n+k}] \cup q^*([\beta]).$$

Here $\omega_{n+k} \in \Omega^{n+k}_c(\mathbb{R}^{n+k})$ is a differential form with $\int_{R^{n+k}} \omega_{n+k} = 1$; this determines its de Rham cohomology class $[\omega_{n+k}] \in H^{n+k}_{\mathrm{dR},c}(\mathbb{R}^{n+k}) \cong \mathbb{R}$. In other words, ω_{n+k} is a Thom form for the trivial vector bundle $q\colon \mathbb{R}^{n+k} \to \mathrm{pt}$.

We write $\cup$ for the cup-product for de Rham cohomology which is induced by the wedge product of forms.

Summarizing the discussion we state the following result.

Lemma 3.6. *Let X be closed oriented n-manifold. The umkehr map $p_!$ in de Rham cohomology can be written as the composition*

$$H^*_{\mathrm{dR}}(X) \xrightarrow{[\omega]\cup\pi^*(\ )} H^{*+k}_{\mathrm{dR},c}(V) \xrightarrow{i_!} H^{*+k}_{\mathrm{dR},c}(\mathbb{R}^{n+k}) \xleftarrow[\cong]{[\omega_{n+k}]\cup q^*(\ )} H^{*-n}_{\mathrm{dR}}(\mathrm{pt}) \tag{3.7}$$

3.2 The umkehr map for a generalized cohomology theory

The goal of this section is to mimic the description of the umkehr map in de Rham cohomology given by the composition (3.7) in a generalized cohomology theory. In the next section, we will specialize the generalized cohomology theory to be K-theory. There are essentially two ways to describe the umkehr map, one via generalized cohomology with compact support, the other via generalized cohomology of Thom spaces. The latter is the more common point of view in algebraic topology, but working with compactly supported cohomology, as we did with de Rham cohomology in the previous section, is well adapted to describe Thom classes in K-theory which we wil do in the next section.

Definition 3.8. A *generalized cohomology theory E* is a contravariant functor that associates to a pair (X, A) of topological spaces a graded abelian group

$$E^*(X, A) = \bigoplus_{k\in\mathbb{Z}} E^k(X, A)$$

and to each continuous map $f\colon (X,A) \to (Y,B)$ a homomorphism

$$f^*\colon E^*(Y,B) \longrightarrow E^*(X,A)$$

of graded abelian groups (which has degree 0, i.e., f^* maps $E^k(Y,B)$ to $E^k(X,A)$). It is required to have the following properties:

homotopy invariance: If two maps $f,g\colon (X,A) \to (Y,B)$ are homotopic, then $f^* = g^*$.

long exact sequence of a pair: For any pair (X,A), the sequence

$$\dots \longrightarrow E^k(X,A) \xrightarrow{j^*} E^k(X) \xrightarrow{i^*} E^k(A) \xrightarrow{\delta} E^{k+1}(X,A) \xrightarrow{j^*} \dots$$

Here $E^k(X)$ is shorthand for $E^k(X,\emptyset)$, and $i\colon A \to X$ and $j\colon (X,\emptyset) \to (X,A)$ are the evident inclusion maps. The map δ is a natural transformation from the functor given by $(X,A) \mapsto E^k(A)$ to the functor given by $(X,A) \mapsto E^{k+1}(X,A)$.

Excision Let (X,A) be a pair of spaces and let $U \subset A$ be a subspace whose closure $\bar{U}$ is contained in A. Then the inclusion map $i\colon (X \setminus U, A \setminus U) \to (X,A)$ induces an isomorphism

$$i^*\colon E^*(X,A) \xrightarrow{\cong} E^*(X \setminus U, A \setminus U).$$

The group $E^k(X,A)$ is called the *E-cohomology group of (X,A) in degree k*. The graded group $E^*(\mathrm{pt})$ is called the *coefficient group* of the generalized cohomology theory.

Strictly speaking, the natural transformation δ is a *datum* and hence it is more precise to say that the pair (E^*,δ) consisting of the functor E^* and the natural transformation δ is a generalized cohomology theory.

The basic example of a generalized cohomology theory is the singular cohomology theory $H^*(X,A;R)$ with coefficients in an abelian group R. The coefficient group for this cohomology group is $H^k(\mathrm{pt};R) = 0$ for $k \neq 0$ and $H^0(\mathrm{pt};R) = R$. Conversely, if E^* is a generalized cohomology theory with trivial coefficient groups $E^k(\mathrm{pt})$ for $k \neq 0$, then there is a natural isomorphism

$$E^*(X,A) \cong H^*(X,A;R) \qquad \text{with } R = E^0(\mathrm{pt}).$$

Another example of a generalized cohomology theory is K-theory, which we will discuss in the next section.

For a pointed space X, the *reduced E-cohomology* is defined to be

$$\widetilde{E}^*(X) := \ker(E^*(X) \to E^*(\mathrm{pt})),$$

where the map is the induced by the inclusion $\mathrm{pt} \to X$ of the basepoint. There is an obvious splitting

$$E^k(X) \cong \widetilde{E}^k(X) \oplus E^k(\mathrm{pt}).$$

There are number of useful consequences of the axioms:

1. If $X = U \cup V$ is the union of two open subspaces U, V, then there is a long exact sequence called the *Mayer-Vietoris sequence*

$$\xrightarrow{\delta} E^k(X) \xrightarrow{(j_U^*, j_V^*)} E^k(U) \oplus E^k(V) \xrightarrow{i_U^* - i_V^*} E^k(U \cap V) \xrightarrow{\delta} E^{k-1}(X) \dots$$

where the maps are induced by the evident inclusion maps in the commutative diagram

$$\begin{tikzcd} U \cap V \arrow[r, "i_U"'] \arrow[d, "i_V"] & U \arrow[d, "j_U"] \\ V \arrow[r, "j_V"] & X \end{tikzcd}$$

2. The long exact sequence of the pair (D^k, S^{k-1}), excision, and the homeomorphism $D^k/S^{k-1} \approx S^k$ results in the isomorphism $E^*(\mathrm{pt}) \cong \widetilde{E}^{*+k}(S^k)$. More generally, we have an isomorphism $E^*(X) \cong E^{*+k}(X_+ \wedge S^n)$, called *suspension isomorphism.*

With the goal of mimicking the composition (3.7) in de Rham cohomology for a generalized cohomology theory E^*, we will construct for a closed n-manifold X, embedded in $\mathbb{R}^{n+k}$ with normal bundle $\pi\colon V \to X$ the following homomorphisms:

$$E^*(X) \xrightarrow{U_V \cup \pi^*(\ )} E_c^{*+k}(V) \xrightarrow{i_!} E_c^{*+k}(\mathbb{R}^{n+k}) \xleftarrow[\cong]{U_{n+k} \cup q^*(\ )} E^{*-n}(\mathrm{pt}) \tag{3.9}$$

The construction of these maps involves four ingredients:

1. A cup product in E-cohomology (see definition 3.10);
2. The definition of E-cohomology groups with compact support, indicated by the subscript c (see definition 3.11).
3. The construction of the map $i_!$ induced by the inclusion $i\colon V \hookrightarrow \mathbb{R}^{n+k}$ of the total space of the normal bundle of X as a tubular neighborhood of X in $\mathbb{R}^{n+k}$ (see definition 3.12).
4. The isomorphism $E^*(\mathrm{pt}) \cong E_c^{*+n+k}(\mathbb{R}^{n+k})$, which is a form of the suspension isomorphism (see diagram (3.16)); the class $U_{n+k} \in E_c^{n+k}(\mathbb{R}^{n+k})$ corresponds to the unit $1 \in E^0(\mathrm{pt})$.
5. The class $U_V \in E_c^k(V)$, called an *E-Thom class* or *E-orientation* (see definition 3.23).

Definition 3.10. A *multiplicative cohomology theory* is a cohomology theory E^* equipped with a natural cup-product pairing

$$\cup\colon E^*(X, A) \otimes E^*(X, B) \longrightarrow E^*(X, A \cup B)$$

which is associative, graded commutative and has a unit $1 \in E^0(X)$.

The basic example of a multiplicative cohomology theory is $H^*(X, A; R)$, singular cohomology with coefficients in a commutative ring R, equipped with the ordinary cup-product. In the next section we will use the tensor product of vector bundles and vector bundle maps to produce a cup-product on K-theory (at least on K^0).

Definition 3.11. (Compactly supported E-cohomology). Let E be a generalized cohomology theory. Then the *E-cohomology* of a topological space X is defined by

$$E_c^*(X) := \varinjlim_{K \subset X \text{ compact}} E^*(X, X \setminus K).$$

We note that if $K \subset K'$ are compact subsets of X, then we have inclusion maps $X \setminus K' \to X \setminus K$, hence a map of pairs $(X, X \setminus K') \to (X, X \setminus K)$, which in cohomology induces a map

$$E^*(X, X \setminus K) \longrightarrow E^*(X, X \setminus K'),$$

allowing us to take the direct limit above. More generally, if $A \subset X$ is a subspace of X, the *relative E-cohomology with compact support* of the pair (X, A) is defined by

$$E_c^*(X, A) := \varinjlim_{K \subset X \text{ compact}} E^*(X, A \cup (X \setminus K)).$$

Definition 3.12. (Construction of $i_!\colon E_c^*(V) \to E_c^*(\mathbb{R}^{n+k})$.) A compact subset $K \subset V \subset \mathbb{R}^{n+k}$ leads to an inclusion map $(V, V \setminus K) \to (\mathbb{R}^{n+k}, \mathbb{R}^{n+k} \setminus K)$. The induced map in E-cohomology

$$E^*(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} \setminus K) \xrightarrow{\cong} E^*(V, V \setminus K)$$

is an isomorphism by excision. Hence the inverse of this map gives a homomorphism

$$i_!\colon E_c^*(V) = \varinjlim_{K \subset V \text{ compact}} E^*(V, V \setminus K) \longrightarrow \varinjlim_{L \subset \mathbb{R}^{n+k} \text{ compact}} E^*(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} \setminus L) = E_c^*(\mathbb{R}^{n+k})$$

Example 3.13. (Compactly supported E-cohomology of $\mathbb{R}^k$). For calculating $E_c^*(\mathbb{R}^k)$ we note that by Heine-Borel every compact subset $K \subset \mathbb{R}^k$ is contained in some closed k-ball D_ϵ^k of radius $\epsilon > 0$ around the origin. Hence

$$E_c^*(\mathbb{R}^k) = \varinjlim_{K \subset \mathbb{R}^k \text{ compact}} E^*(\mathbb{R}^k, \mathbb{R}^k \setminus K) = \varinjlim_{\epsilon} E^*(\mathbb{R}^k, \mathbb{R}^k \setminus D_\epsilon^k) \tag{3.14}$$

We claim that for any $\epsilon > 0$ the inclusion map $(\mathbb{R}^k, \mathbb{R}^k \setminus D_\epsilon^k) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$ induces an isomorphism in cohomology

$$E^*(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \xrightarrow{\cong} E^*(\mathbb{R}^k, \mathbb{R}^k \setminus D_\epsilon^k).$$

To see this, we note that the inclusion map $\mathbb{R}^k \setminus \{0\} \hookrightarrow \mathbb{R}^k \setminus D^k_\epsilon$ is a homotopy equivalence, and hence induces an isomorphism in cohomology. The map of pairs induces the following commutative diagram whose rows are the long exact cohomology sequences of these pairs.

$$\begin{tikzcd}
E^*(\mathbb{R}^k) \arrow[r] \arrow[d, equal] & E^*(\mathbb{R}^k \setminus \{0\}) \arrow[r] \arrow[d, "\cong"] & E^{*+1}(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \arrow[r] \arrow[d] & E^{*+1}(\mathbb{R}^k) \arrow[r] \arrow[d, equal] & E^{*+1}(\mathbb{R}^k \setminus \{0\}) \arrow[d, "\cong"] \\
E^*(\mathbb{R}^k) \arrow[r] & E^*(\mathbb{R}^k \setminus D^k_\epsilon) \arrow[r] & E^{*+1}(\mathbb{R}^k, \mathbb{R}^k \setminus D^k_\epsilon) \arrow[r] & E^{*+1}(\mathbb{R}^k) \arrow[r] & E^{*+1}(\mathbb{R}^k \setminus D^k_\epsilon)
\end{tikzcd}$$

By the 5-Lemma the middle vertical map is an isomorphism as claimed. Since the connecting maps in the direct limit of the cohomology groups $E^*(\mathbb{R}^k, \mathbb{R}^k \setminus D^k_\epsilon)$ are compatible with these isomorphisms, this yields the isomorphism

$$\varinjlim_{\epsilon} E^*(\mathbb{R}^k, \mathbb{R}^k \setminus D^k_\epsilon) \cong E^*(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}).$$

There are further isomorphisms

$$E^*(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \cong E^*(\mathbb{R}^k, \mathbb{R}^k \setminus B^k_1) \cong E^*(D^k, \partial D^k) \cong \widetilde{E}^*(D^k/\partial D^k) \cong \widetilde{E}^*(S^k);$$

the first one follows with the same argument as above, the second one is excision, and the third one holds since the inclusion $\partial D^k = S^{k-1} \hookrightarrow D^k$ of the boundary of the closed k-disk is a cofibration. Composing all these isomorphisms, we obtain an isomorphism

$$E^*_c(\mathbb{R}^k) \cong \widetilde{E}^*(S^k) \tag{3.15}$$

The composition of this isomorphism and the suspension isomorphism then yields the desired isomorphism

$$E^*(\mathrm{pt}) \cong E^{*+k}(S^k) \cong E^{*+k}_c(\mathbb{R}^k).$$

Let $U_k \in E^{*+k}_c(\mathbb{R}^k)$ be the class corresponding to the unit $1 \in E^0(\mathrm{pt})$ of the multiplicative cohomology theory E. The graded groups above are modules over the coefficient ring $E^*(\mathrm{pt})$ and the isomorphisms above are isomorphisms of modules over $E^*(\mathrm{pt})$. It follows that the above isomorphism

$$E^*(\mathrm{pt}) \xrightarrow{\cong} E^{*+k}_c(\mathbb{R}^k) \qquad \text{is given by} \qquad a \mapsto U_k \cup q^*(a) \tag{3.16}$$

There is a useful variant of compactly supported cohomology for fiber bundles.

Definition 3.17. Let $\pi\colon X \to Y$ be a fiber bundle. A subset $K \subset X$ is *vertically compact* if for each $y \in Y$ the intersection $K \cap X_y$ with the fiber $X_y = \pi^{-1}(y)$ is compact. Then

$$E^*_{cv}(X) := \varinjlim_{K \subset X \text{ vertically compact}} E^*(X, X \setminus K)$$

is the *E-cohomology of X with compact vertical support.*

Example 3.18. Let $\pi\colon V \to X$ be a vector bundle of dimension k, equipped with a bundle metric. Given a function $\epsilon\colon X \to (0,\infty)$, the ϵ-disk bundle

$$D_\epsilon(V) = \{(x,v) \mid x \in X,\ v \in V_x,\ ||v|| \le \epsilon(x)\}$$

is vertically compact (but not compact, unless X is). In fact, if $K \subset V$ is any vertically compact subset, then K is contained in some disk bundle $D_\epsilon(V)$ and hence the direct limit over all vertically compact subsets can be replaced by the limit of the groups $E^*(X, X \setminus D_\epsilon(V))$, where the limit is taken over all functions $\epsilon\colon X \to (0,\infty)$. Analogous to the Example 3.13 we note that $E^*(V, V \setminus D_\epsilon(V))$ is isomorphic to $E^*(V, V_0)$, where V_0 is the complement of the zerosection of V. This gives the first of the following isomorphisms

$$E^*_{cv}(V) \cong E^*(V, V_0) \cong E^*(V, V_0 \setminus B_1(V)) \cong E^*(D(V), S(V)) \cong \widetilde{E}^*(D(V)/S(V))$$

Here

$$D(V) = \{(x,v) \mid x \in X,\ v \in V_x,\ ||v|| \le 1\} \subset V$$

is the *unit disk bundle*, and

$$S(V) = \{(x,v) \mid x \in X,\ v \in V_x,\ ||v|| = 1\} \subset V$$

is the unit *sphere bundle*. The other isomorphisms are again analogous to the isomorphisms discussed in example 3.13. The quotient space $D(V)/S(V)$ is called the *Thom space* of V; a common notation is $X^V := D(V)/S(V)$. Summarizing, there is an isomorphism

$$E^*_{cv}(V) \cong E^*(X^V). \tag{3.19}$$

If V is the trivial vector bundle $X \times \mathbb{R}^k \to X$, then the Thom space X^V can be identified with the suspension $S^k \wedge X_+$ where X_+ denotes the pointed topological space obtained by adding a disjoint basepoint to X. Specializing the above isomorphisms we obtain

$$E^*_{cv}(\mathbb{R}^k \times X) \cong \widetilde{E}^*(S^k \wedge X_+). \tag{3.20}$$

Specializing further to $X = \mathrm{pt}$, we obtain the isomorphism (3.15).

The homomorphism $i_!\colon E^*_c(V) \to E^*_c(\mathbb{R}^{n+k})$ was constructed in Definition 3.12. Here $V \to X$ is the normal bundle of the compact n-manifold X in $\mathbb{R}^{n+k}$. Due the compactness of X, vertically compact subsets of V are in fact compact, and hence $E^*_{cv}(V) = E^*_c(V)$. After identifying the domain of $i_!$ with $E^*(X^V)$ by equation (3.19) and the codomain with $\widetilde{E}^*(S^{n+k})$ by equation (3.15), it is a natural question whether there is a map from S^{n+k} to the Thom space X^V which induces $i_!$. The answer is yes, and the *collapse map* c which corresponds to $i_!$ is described in the following lemma. We will not use this result, but it seems useful to state this result, since algebraic topologists typically describe the umkehr map in terms of the Thom space X^V and the collapse map. The collapse map is also important for other construction, e.g., it is the essential ingredient in the *Pontryagin-Thom construction.*

Lemma 3.21. *Let $V \to X$ be the normal bundle of a compact n-manifold embedded in $\mathbb{R}^{n+k}$, and let $i\colon V \to \mathbb{R}^{n+k}$ be the embedding of V as tubular neighborhood of X in $\mathbb{R}^{n+k}$. Let $c\colon S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to X^V$ be the* collapse map *defined by*

$$c(x) := \begin{cases} * & \text{if } x \notin i(V) \text{ or } x = \infty \\ v & \text{if } x = i(v) \end{cases} \tag{3.22}$$

Then the following diagram commutes:

$$\begin{tikzcd} E_c^*(V) \arrow[r, "i_!"] \arrow[d, "\cong"'] & E_c^*(\mathbb{R}^{n+k}) \arrow[d, "\cong"] \\ \widetilde{E}^*(X^V) \arrow[r, "c^*"] & \widetilde{E}^*(S^{n+k}) \end{tikzcd}$$

Generalizing the argument leading to the isomorphism (3.16), we obtain the following commutative diagram of isomorphisms.

$$\begin{tikzcd} E_c^*(X) \arrow[r, "p_1^*U_k \cup p_2^*(\ )"] \arrow[d, equal] & E_{cv}^{*+k}(\mathbb{R}^k \times X) \arrow[d, "\cong"] \\ E_c^*(X) \arrow[r, "\text{suspension isomorphism}", "\cong"'] & E_c^*(S^k \wedge X_+) \end{tikzcd}$$

Our goal is to generalize this diagram to the more general situation where the trivial vector bundle $\mathbb{R}^k \times X \to X$ is replaced by a general vector bundle $\pi\colon V \to X$ of dimension k. Looking to generalize the top horizontal map, the projection map p_2 to the base X is simply replaced by the projection map $\pi\colon V \to X$. The problem is that for a non-trivial bundle V, there is no projection map p_1 to the fiber $\mathbb{R}^k$. We note that the class $p_1^*U_k \in E_{cv}^k(\mathbb{R}^k \times X)$ restricts on each fiber $\mathbb{R}^k$ to $U_k \in E_c^k(\mathbb{R}^k)$. This motivates the following definition.

Definition 3.23. Let $\pi\colon V \to X$ be a k-dimensional vector bundle over X. An *E-orientation* or *E-theory Thom class* is an element $U_V \in E_{cv}^k(V)$ with the property that its restriction to each fiber V_x corresponds to the suspension class $U_k \in E_c^k(\mathbb{R}^k)$ via the isomorphism $E_c^k(V_x) \cong E_c^k(\mathbb{R}^k)$ induced by some vector space isomorphism $V_x \cong \mathbb{R}^k$.

Theorem 3.24. Thom Isomorphism Theorem. *Let $V \to X$ be a vector bundle of rank k over a topological space X and let $U_V \in E_{cv}^k(V)$ be an E-oriention on V. Then the maps*

$$E^*(X) \xrightarrow{U_V \cup \pi^*(\ )} E_{cv}^*(V) \qquad \text{and} \qquad E_c^*(X) \xrightarrow{U_V \cup \pi^*(\ )} E_c^*(V)$$

*are isomorphisms of $\mathbb{Z}$-graded abelian groups, called the E-*theory Thom isomorphisms*.*

Example 3.25. 1. Every vector bundle has a canonical E-orientation if E is ordinary cohomology with $\mathbb{Z}/2$-coefficients.

2. Let $V \to X$ be a vector bundle and let E ordinary cohomology with $\mathbb{Z}$-coefficients. Then there is a natural bijection between the set of orientations on V and the set of E-orientations on V.

Remark 3.26. Via the isomorphisms $E^*_{cv}(V) \cong E^*(V, V_0) \cong \widetilde{E}^*(X^V)$ the Thom class U_V can also be seen as an element in $E^k(V, V_0)$ or $\widetilde{E}^k(X^V)$, which is more common in algebraic topology. The Thom isomorphism is then seen as the isomorphism

$$E^*(X) \cong E^{*+k}(V, V_0) \qquad \text{or} \qquad E^*(X) \cong \widetilde{E}^{*+k}(X^V).$$

Both of these isomorphisms are also described as an appropriate cup product with the Thom class.

Now we have established all the ingredients for the construction of the umkehr map in E-theory for a closed n-manifold X as the composition (**??**). For future reference, we state this as the definition of $p_!$.

Definition 3.27. Let X be a closed n-manifold embedded in $\mathbb{R}^{n+k}$ with normal bundle V. Assume that V has an E-orientation $U_V \in E^k_{cv}(V)$. Then the *umkehr map*

$$p_! \colon E^*(X) \to E^{*-n}(\mathrm{pt})$$

is the composition

$$E^*(X) \xrightarrow{U_V \cup \pi^*(\ )} E_c^{*+k}(V) \xrightarrow{i_!} E_c^{*+k}(\mathbb{R}^{n+k}) \xleftarrow[\cong]{U_{n+k} \cup q^*(\ )} E^{*-n}(\mathrm{pt}) \tag{3.28}$$

Here the first map is the Thom isomorphism in E-cohomology given by the cup-product with the Thom class U_V (Theorem 3.24, the second is the *extend by zero map* $i_!$ (Definition 3.12), and the third the suspension isomorphism (given by the cup-product with U_{n+k}, see (3.16)).

As mentioned, all of the maps of the composition above can be described alternatively in terms of E-theory rather than E-theory with compact support, which is the more common description in algebraic topology. The following commutative diagram relates both points of view.

$$\begin{tikzcd}
E^*(X) \arrow[r, "U_V \cup \pi^*(\ )"] \arrow[d, "\cong"] & E_c^{*+k}(V) \arrow[r, "i_!"] \arrow[d, "\cong"] & E_c^{*+k}(\mathbb{R}^{n+k}) \arrow[d, "\cong"] & E^{*-n}(\mathrm{pt}) \arrow[l, "U_{n+k} \cup q^*(\ )"', "\cong"] \arrow[d, "\cong"] \\
E^*(X) \arrow[r, "\text{Thom isomorphism}", "\cong"'] & \widetilde{E}^{*+k}(X^V) \arrow[r, "c^*"] & \widetilde{E}^*(S^{n+k}) & E^{*-n}(\mathrm{pt}) \arrow[l, "\text{suspension isomorphism}"', "\cong"]
\end{tikzcd}, \tag{3.29}$$

where $c \colon S^{n+k} \to X^V$ is the collapse map (see equation (3.22)).

3.3 The umkehr map in K-theory

K-theory is a generalized cohomology theory; in particular, it associates to a pair (X, A), consisting of a topological space X and a subspace A, a $\mathbb{Z}$-graded abelian group

$$K^*(X, A) = \bigoplus_{n \in \mathbb{Z}} K^n(X, A).$$

This cohomology theory is 2-periodic in the sense that

$$K^{n+2}(X, A) = K^n(X, A). \tag{3.30}$$

There is a simple geometric description of $K^0(X)$ for compact X in terms of vector bundles over X, and more generally, for $K^0_c(X)$, $K^0_c(X, A)$, the compactly supported K-theory of X resp. (X, A). Fortunately, this is sufficient for our purposes, since we are interested in the umkehr map $p_!$ given by specializing the composition (3.28) to the generalized cohomology theory K^* in degree $* = 0$ and for $n = \dim X$ even.

$$K^*(X) \xrightarrow{U_V \cup \pi^*(\ )} K_c^{*+k}(V) \xrightarrow{i_!} K_c^{*+k}(\mathbb{R}^{n+k}) \xleftarrow[\cong]{U_{n+k} \cup q^*(\ )} K^{*-n}(\mathrm{pt})$$

We note that we choose $k = \dim V$ to be even. In that situation, thanks for the 2-periodicity of the K-theory groups the above simplifies to

$$K(X) \xrightarrow{U_V \cup \pi^*(\ )} K_c(V) \xrightarrow{i_!} K_c(\mathbb{R}^{n+k}) \xleftarrow[\cong]{U_{n+k} \cup q^*(\ )} K(\mathrm{pt})$$

where we write $K(\ )$ instead of $K^0(\ )$.

Let $\mathrm{Vect}(X)$ be the set of isomorphism classes of finite dimensional vector bundles over the compact space X. The direct sum of vector bundles gives $\mathrm{Vect}(X)$ the structure of an abelian semi-group. Let $K(X)$ be the abelian group obtained by the *group completion* of the semi-group $\mathrm{Vect}(X)$. This is the procedure used to construct to abelian group $\mathbb{Z}$ from the abelian semigroup $\mathbb{N}_0$ (the non-negative integers). Here is the formal definition.

Definition 3.31. For a compact topological space X, the set $K(X)$ consists of equivalence classes of pairs (E^+, E^-) of vector bundles, where the equivalence relation is defined by

$$(E^+, E^-) \ \sim \ (F^+, F^-)$$

if and only if there is some $V \in \mathrm{Vect}(X)$ such that

$$E^+ + F^- + V = F^+ + E^- + V$$

The set $K(X)$ is an abelian group with addition defined by $[(E^+, E^-)] + [(F^+, F^-)] := [(E^+ + F^+, E^- + F^-0)]$, unit element $[(V, V)]$ for any $V \in \mathrm{Vect}(X)$, and inverse given by

$-[(E^+, E^-)] = [(E^-, E^+)]$. In particular, a vector bundle $E \to X$ represents an element $[E] := [(E, 0)]$ in $K(X)$, where 0 denotes the 0-dimensional vector bundle. Its inverse is given by $-[E] = -[(E, 0)] = [(0, E)]$. Any element $[(E^+, E^-)] \in K(X)$ can then be written in the form

$$[(E^+, E^-)] = [(E^+ \oplus 0, 0 \oplus E^-)] = [(E^+ \oplus 0)] + [(0 \oplus E^-)] = [E^+] - [E^-],$$

i.e., as a difference of element in $K(X)$ represented by vector bundles.

Remark 3.32. Unlike $\mathbb{N}_0$, the abelian semi-group $\mathrm{Vect}(X)$ does not have the *cancellation property*, i.e., $E \oplus V = F \oplus V$ does not imply $E = F$. For example, let TS^n be the tangent bundle of the n-sphere S^n. It can be shown that TS^n is not isomorphic to the trivial vector bundle $\underline{\mathbb{R}}^n$ of dimension n, unless $n = 1$, 3 or 7 (this is easy to see for n even, since the Euler class $\chi^H(TS^n) \in H^n(S^n; \mathbb{Z})$ is non-trivial, see **??**). The normal bundle $\nu(S^n, \mathbb{R}^{n+1})$ of $S^n \hookrightarrow \mathbb{R}^{n+1}$ is isomorphic to the trivial bundle $\underline{\mathbb{R}}^1 = S^n \times \mathbb{R}$; an isomorphism from $\underline{\mathbb{R}}^1$ to the normal bundle is given by sending $(x, 1) \in S^n \times \mathbb{R}$ to the outward pointing unit normal vector at x. The direct sum of $TS^n \oplus \nu$ is isomorphic to the tangent bundle $T\mathbb{R}^{n+1}$ restricted to S^n, which is the trivial bundle $\underline{\mathbb{R}}^{n+1}$ of dimension $n+1$ on S^n. This implies that in $\mathrm{Vect}(S^n)$ we have

$$TS^n + \underline{\mathbb{R}}^1 = TS^n + \nu = T\mathbb{R}^{n+1}_{|S^n} = \underline{\mathbb{R}}^{n+1} = \underline{\mathbb{R}}^n + \underline{\mathbb{R}}^1,$$

but $TS^n \neq \underline{\mathbb{R}}^n$.

For abelian semi-groups that have the cancellation property, the definition of the equivalence relation can obviously simplified to $(E^+, E^-) \sim (F^+, F^-)$ if and only if $E^+ + F^- = F^+ + E^-$. For an abelian semi-group with cancellation property, the simplified relation is not an equivalence relation since it lacks transitivity.

When describing relating the umkehr map and the Thom isomorphism in de Rham cohomology, it was convenient for us to use relative de Rham cohomology $H^*_{\mathrm{dR}}(X, A)$ and de Rham cohomology with compact support $H^*_{\mathrm{dR},c}(X)$. Similarly, it will be useful to utilise relative K-theory $K(X, A)$ and K-theory with compact support $K_c(X)$. We recall that $H^*_{\mathrm{dR}}(X, A)$ and $H^*_{\mathrm{dR},c}(X)$ were defined in terms of the *support* of differential forms representing de Rham cohomology classes. Similarly, the definition of $K(X, A)$ and $K_c(X)$ will be based on a the notion of *support* of the geometric objects representing K-theory classes.

The objects used so far to represent elements in $K(X)$ are pairs (E^+, E^-) of vector bundles $E^\pm$. We know that such a pair represents $0 \in K(X)$ if there is an isomorphism between these vector bundles, but there is no way to assign to (E^+, E^-) a subset $\mathrm{supp}(E^+, E^-) \subset X$ in a functorial way such that $\mathrm{supp}(E^+, E_-) = \emptyset$ implies $[(E^+, E^-)] = 0 \in K(X)$.

The idea is to replace the simple-minded pairs (E^+, E^-) with slightly more sophisticated triples (E^+, E^-, α_+), where as before $E^\pm$ are finite dimensional vector bundles on X, and $\alpha_+ \colon E^+ \to E^-$ is a vector bundle morphism. The *support* of (E^+, E^-, α^+) is defined by

$$\mathrm{supp}(E^+, E^-, \alpha_+) := \text{closure of } \{x \in X \mid \alpha^+_x \colon E^+_x \to E^-_x \text{ is not an isomorphism}\}.$$

We observe that $\operatorname{supp}(E^+, E^-, \alpha_+) = \emptyset$ implies that α^+ is a vector bundle isomorphism, and hence $[(E^+, E^-)] = 0 \in K(X)$ as desired. Based on these triples, we give a new definition of K-theory.

Definition 3.33. (K-theory with compact support) Let X be a topological space (not required to be compact), and let $A \subset X$ be a subspace. Then the *K-theory with compact support* of X resp. (X, A) is the abelian group defined by

$$\begin{aligned} K_c(X) :=& \{(E^+, E^-, \alpha^+) \mid \operatorname{supp}(E^+, E^-, \alpha^+) \text{ is compact}\}/\sim \\ K_c(X, A) :=& \{(E^+, E^-, \alpha^+) \mid \operatorname{supp}(E^+, E^-, \alpha^+) \text{ is compact, } \operatorname{supp}(E^+, E^-, \alpha^+) \subset X \setminus A\} \sim \end{aligned}$$

Here E^+, E^- are finite dimensional vector bundles on X, and $\alpha^+ \colon E^+ \to E^-$ is a vector bundle morphism. The equivalence relation $\sim$ is generated by the following relations:

An isomorphism between (E^+, E^-, α^+) and (F^+, F^-, β^+) consists of vector bundle isomorphisms $f^\pm \colon E^\pm \to F^\pm$ such that the diagram

$$\begin{tikzcd} E^+, \arrow[r, "\alpha^+"] \arrow[d, "f^+", "\cong"'] & E^- \arrow[d, "f^-", "\cong"'] \\ F^+ \arrow[r, "\beta^+"] & F^- \end{tikzcd}$$

is commutative.

A homotopy between (E^+, E^-, α_0^+) and (E^+, E^-, α_1^+) is a path $\alpha_t^+ \colon E^+ \to E^-$ of vector bundle morphisms connecting α_0^+ and α_1^+ such that for all $t \in [0, 1]$ the relevant support condition for the triple (E^+, E^-, α_t^+) is satisfied.

Adding a trivial triple A triple (F^+, F^-, β^+) is called *trivial* if $\operatorname{supp}(F^+, F^-, \beta^+) = \emptyset$. The *sum* of (E^+, E^-, α_0^+) and (F^+, F^-, β^+) is defined by

$$(E^+, E^-, \alpha_0^+) + (F^+, F^-, \beta^+) := (E^+ \oplus F^+, E^- \oplus F^-, \alpha^+ \oplus \beta^+).$$

If (F^+, F^-, β^+) is trivial, this sum is declared to be equivalent to (E^+, E^-, α_0^+).

The sum of these triple gives $K_c(X)$ and $K_c(X, A)$ the structure of an abelian group.

Example 3.34. (The Bott class). Let $E^\pm$ be the trivial complex line bundle over $\mathbb{C}$. So an element of the total space is a pair $(v, w) \in \mathbb{C} \times \mathbb{C}$; our convention is the bundle projection map sends (v, w) to v, i.e., v is a point in the base space, and w is an element in the fiber. Let

$$\alpha^+ \colon E^+ \to E^- \qquad \text{be given by} \qquad (v, w) \mapsto (v, vw)$$

This bundle map is clearly an isomorphism for $v \neq 0 \in \mathbb{C}$; in other words, $\operatorname{supp}(E^+, E^-, \alpha^+) = \{0\} \subset \mathbb{C}$. In particular, this triple B represents an element $[B] \in K_c(\mathbb{C})$, called the *Bott class*. It turns out that $[B]$ is a generator of $K_c(\mathbb{C}) \cong \mathbb{Z}$.

The statement $K_c(\mathbb{C}) \cong \mathbb{Z}$ is a very special case of a much more general theorem known as Bott periodicity.

Theorem 3.35. (Bott periodicity). *For any space X there is an isomorphism*

$$K_c(X) \xrightarrow{\cong} K_c(X \times \mathbb{C})$$

which is given by sending an element $[E] \in K_c(X)$ represented by a triple $E = (E^+, E^-, \alpha^+)$ to the product $[p_2^* B \otimes p_1^* E]$. Here $p_1 \colon X \times \mathbb{C} \to X$, $p_2 \colon X \times \mathbb{C} \to \mathbb{C}$ are the projection maps onto the factors, and $p_1^* B$ resp. $p_2^* E$ are the triples over $X \times \mathbb{C}$ obtained by pulling back the vector bundles. The tensor product is constructed in Definition 3.38 below.

Addendum. There is a variant of the above definition using *Hermitian triples*, i.e., triples (E^+, E^-, α^+) as above, where the complex vector bundles $E^\pm$ are equipped with hermitian bundle metrics. The only modification necessary in the definition of the equivalence relation for these triples is that that *isomorphism of hermitian triples* involves bundle isomorphisms $f^\pm$ that are fiberwise isometries.

Exercise 3.36. (a) Show that the forgetful map from hermitian triples to general triples induces a bijection on equivalence classes.

(b) Show that for a hermitian triple (E^+, E^-, α^+) the inverse of $[E^+, E^-, \alpha^+] \in K_c(X, A)$ is given $[E^-, E^+, (\alpha^+)^*]$, where $(\alpha^+)^* \colon E^- \to E^+$ is the adjoint of α^+ with respect to the Hermitian bundle metrics on $E^\pm$. In more detail, $(\alpha^+)^*$ is a vector bundle map whose restriction $(\alpha^+)^*_x \colon E^-_x \to E^+_x$ to fibers over $x \in X$ is the adjoint to the linear map $\alpha^+_x \colon E^+_x \to E^-_x$ with respect to the hermitian inner product on $E^\pm_x$. This show a technical advantage of working with Hermitian triples: there is a canonical representative for the inverse of the class represented by a given Hermitian triple.

(c) Show that for compact X this new definition of $K(X)$ agrees with the classical one as the group completion of $\mathrm{Vect}(X)$.

There is a way to repackage Hermitian triples (E^+, E^-, α^+) that is very convenient for the construction of the tensor product of such triples. We recall that a $\mathbb{Z}/2$-*graded vector bundle* can be defined as

- a vector bundle E together with a decomposition $E = E^+ \oplus E^-$ as a sum of two complementary subbundles, or
- as a vector bundle E together with an involution $\tau \colon E \to E$ (i.e., a vector bundle map τ with $\tau^2 = \mathrm{id}_E$).

Definition 3.37. A *graded Hermitian triple* is a triple (E, τ, α), where E is a $\mathbb{Z}/2$-graded Hermitian vector bundle with grading involution τ (where E^+ and E^- are perpendicular with respect to the Hermitian metric), and $\alpha\colon E \to E$ is an odd self-adjoint vector bundle morphism.

Given a decomposition $E = E^+ \oplus E^-$, the corresponding grading involution $\tau\colon E \to E$ as a 2×2-matrix has the form $\begin{pmatrix} \mathrm{id}_{E_+} & 0 \\ 0 & -\mathrm{id}_{E^-} \end{pmatrix}$. Conversely, an involution $\tau\colon E \to E$ determines a vector bundle decomposition $E = E^+ \oplus E^-$, where the fiber $E^\pm_x$ is the ± 1-eigenspace of the involution $\tau_x\colon E_x \to E_x$.

Given a Hermitian triple (E^+, E^-, α^+), let (E, τ, α) be the triple consisting of the $\mathbb{Z}/2$-graded vector bundle $E = E^+ \oplus E^-$ with grading involution τ. The vector bundle map $\alpha\colon E \to E$ with respect to the decomposition $E = E^+ \oplus E^-$ is given by the 2×2-matrix

$$\alpha = \begin{pmatrix} 0 & (\alpha^+)^* \\ \alpha^+ & 0 \end{pmatrix},$$

where $(\alpha^+)^*\colon E^- \to E^+$ is the adjoint of α^+ with respect to the bundle metrics on $E^\pm$. We note that $\alpha\colon E \to E$ is an *odd* endomorphism of E (i.e., it sends vectors in $E^\pm$ to vectors in $E^\mp$, or, equivalently, it *anti-commutes* with the grading involution τ, i.e., $\tau\alpha = -\alpha\tau$). Moreover, α is *self-adjoint* with respect to the bundle metric on $E = E^+ \oplus E^-$ determined by the bundle metrics on $E^\pm$.

Conversely, if (E, τ) is a $\mathbb{Z}/2$-graded hermitian vector bundle (the bundle metric is required to be compatible with the grading in the sense that E^+_x is perpendicular to E^-_x), and $\alpha\colon E \to E$ is an odd, self-adjoint endomorphism, then (E^+, E^-, α^+) is a Hermitian triple, where $\alpha^+\colon E^+ \to E^-$ is the restriction of α to E_+ (which maps to E^- since α is odd). In other words, there is a natural bijection between the Hermitian triples (E^+, E^-, α^+) and (E, τ, α).

We will use this to move freely between these two descriptions, since there are pro's and con's for both points of view. The graded Hermitian triples (E, τ, α) are very convenient for the construction of a tensor product of such triples which will induce a product on K-theory.

Definition 3.38. (Tensor product of graded hermitian triples.) Let (E_1, τ_1, α_1) and (E_2, τ_2, α_2) be graded hermitian triples over the same space X. Their *tensor product* is the hermitian triple defined by

$$(E_1, \tau_1, \alpha_1) \otimes (E_2, \tau_2, \alpha_2) := (E_1 \otimes E_2, \tau_1 \otimes \tau_2, \mathrm{id}_{E_1} \otimes \alpha_2 + \alpha_1 \otimes \mathrm{id}_{E_2}). \tag{3.39}$$

A number of comments are in order.

- Hermitian inner products on vector spaces V, W induce a hermitian inner product on $V \otimes W$ defined by $\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle := \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle$. If these hermitian inner products on V, W are compatible with gradings on V resp. W, then so is the product on $V \otimes W$. If E_1, E_2 are vector bundles , these remarks apply to the fibers to show that hermitian bundle metrics on E_1, E_2 determine a bundle metric on $E_1 \otimes E_2$.

- It is easy to check that $\tau_1 \otimes \tau_2$ is again an involution. Suppose $v_i^\pm \in E_i^\pm$. Then

$$(\tau_1 \otimes \tau_2)(v_1^+ \otimes v_2^+) = \tau_1(v_1^+) \otimes \tau_2(v_2^+) = v_1^+ \otimes v_2^+$$
$$(\tau_1 \otimes \tau_2)(v_1^- \otimes v_2^-) = \tau_1(v_1^-) \otimes \tau_2(v_2^-) = (-v_1^- \otimes (-v_2^-) = v_1^- \otimes v_2^-$$

which shows that $(E_1^+ \otimes E_2^+) \oplus (E_1^- \otimes E_2^-)$ belongs to the positive eigenspace of $\tau_1 \otimes \tau_2$. Similarly, $(E_1^+ \otimes E_2^-) \oplus (E_1^- \otimes E_2^+)$ belongs to the negative eigenspace of $\tau_1 \otimes \tau_2$, which shows that this construction of the tensor product of $\mathbb{Z}/2$-graded vector spaces/bundles in terms of grading involutions agrees with the more definition in terms of a decomposition into E^+ and E^-.

- Some care has to be taken when dealing with tensor products of maps between graded vector spaces. Suppose V, V', W, W' are $\mathbb{Z}/2$-graded vector spaces and $f\colon V \to V'$ and $g\colon W \to W'$ are linear maps which are either odd or even (f and g don't need to have the same parity). Then their *tensor product*

$$f \otimes g\colon V \otimes W \longrightarrow V' \otimes W'$$

is defined by

$$(f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w)$$

for homogeneous elements $v \in V$, $w \in W$. Here $|v| \in \{0,1\}$ is the degree of v (i.e., $|v| = 0$ for $v \in V^+$ and $|v| = 1$ for $v \in V^-$), and $|g| \in \{0,1\}$ is the degree of g, i.e., $|g| = 0$ if g is even, and $|g| = 1$ if g is odd.

- An explicit calculation is needed to determine the support of the tensor product (3.39). We need to determine for which $x \in X$ the linear map

$$\mathrm{id}_{E_1} \otimes \alpha_2 + \alpha_1 \otimes \mathrm{id}_{E_2}\colon (E_1)_x \otimes (E_2)_x \longrightarrow (E_1)_x \otimes (E_2)_x$$

is an isomorphism, or – equivalently its square – is an isomorphism. Simplifying notation, let $\alpha := \alpha_1 \otimes 1 + 1 \otimes \alpha_2\colon (E_1)_x \otimes (E_2)_x \to (E_1)_x \otimes (E_2)_x$. Then

$$\begin{aligned}\alpha^2 =&(\alpha_1 \otimes 1)(\alpha_1 \otimes 1) + (\alpha_1 \otimes 1)(1 \otimes \alpha_2) + (1 \otimes \alpha_2)(\alpha_1 \otimes 1) + (1 \otimes \alpha_2)(1 \otimes \alpha_2)\\ =&\alpha_1^2 \otimes 1 + \alpha_1 \otimes \alpha_2 - \alpha_1 \otimes \alpha_2 + 1 \otimes \alpha_2^2\\ =&\alpha_1^2 \otimes 1 + 1 \otimes \alpha_2^2\end{aligned}$$

Here the minus sign in the equation $(1 \otimes \alpha_2)(\alpha_1 \otimes 1) = -\alpha_1 \otimes \alpha_2$ is due to the transposition of the two odd endomorphisms α_1, α_2. It is a consequence of the definition of the tensor product (**??**) of linear maps between graded vector spaces. We recall that α_i, $i = 1,2$ is self-adjoint, and hence its eigenvalues are *real*. It follows that the eigenvalues of α_i^2 are non-negative (denoted by writing $\alpha_i^2 \geq 0$, and α_i^2 is invertible if and only if all its eigenvalues are positive (denoted $\alpha_i^2 > 0$). Using that notation, the calculation above shows that $\alpha_1^2 \geq 0$ and $\alpha_2^2 \geq 0$ implies $\alpha^2 \geq 0$ and

$$\alpha^2 > 0 \qquad \Longleftrightarrow \qquad \alpha_1^2 > 0 \text{ or } \alpha_2^2 > 0$$

Equivalently, α is not invertible if and only if α_1 is not invertible and α_2 is not invertible. It follows that

$$\operatorname{supp}((E_1, \tau_1, \alpha_1) \otimes (E_2, \tau_2, \alpha_2)) = \operatorname{supp}(E_1, \tau_1, \alpha_1) \cap \operatorname{supp}(E_2, \tau_2, \alpha_2)$$

Our next goal is to construct a Thom class for K-theory. To motivate the upcoming construction, we recall that the principal symbol of the de Rham operator

$$d + d^* \colon \Omega^*(X) \longrightarrow \Omega^*(X) = \Gamma(X; \Lambda^* T^* X)$$

is given by

$$\sigma_\xi^{d+d^*}(x) = i(\xi \wedge \_ \; - \; \iota_\xi) \colon \Lambda^* T_x^* X \longrightarrow \Lambda^* T_x^* X \qquad \text{for } \xi \in T_x^* X. \tag{3.40}$$

Here $\xi \wedge \_$ is given by $\omega \mapsto \xi \wedge \omega$ and $\iota_\xi \colon \Lambda^* T_x^* X \to \Lambda^* T_x^* X$ is the graded derivation of degree -1 that is determined by $\iota_\xi(\omega) = \langle \xi, \omega \rangle$ for $\omega \in \Lambda^1 T_x^* X = T_x^* X$ where $\langle \xi, \omega \rangle \in \mathbb{R}$ is the inner product of $\xi, \omega \in T_x^* X$. Earlier, in Lemma 1.15, we proved that the de Rham operator is elliptic, i.e., we showed that the map (3.40) is an isomorphism for $\xi \neq 0$ (by showing that its square is simply multiplication by $||\xi||^2$).

This example serves as an inspiration for the following construction.

Definition 3.41. (K-Thom class for complex vector bundles). Let $\pi \colon V \to X$ be a complex vector bundle equipped with a hermitian metric. Let $U_{\mathbb{C}}^K(V)$ be the following hermitian triple over the total space of V:

$$\alpha \colon \Lambda^*(\pi^* V) \longrightarrow \Lambda^*(\pi^* V)$$

which at a point $(x, v) \in V$ is the linear map

$$\alpha_v \colon \Lambda^* V_x \longrightarrow \Lambda^* V_x \qquad \text{given by} \qquad \alpha_v := i(v \wedge \_ \; - \; \iota_v)$$

Here the vector spaces are complex, so the exterior algebra $\Lambda^* V_x$ has to be understood over the base field $\mathbb{C}$. A little care is needed to ensure that ι_v is complex linear: for $\omega \in \Lambda^1 V_x = V_x$, it is defined by $\iota_v(\omega) = \langle v, \omega \rangle$. Hence to ensure complex linearity in ω, we require our hermitian inner products to be complex linear in the second slot, and complex anti-linear in the first (due to the Koszul sign convention, I try hard to minimize unnecessary permutations of symbols).

Theorem 3.42. $[U^K_{\mathbb{C}}(V)] \in K_c(V)$ *is a Thom class.*

Remark 3.43. The factor of i in the definition above is convenient, but not essential. It is necessary if we wish to think in terms of hermitian triples: without the factor i the operator is *skew-adjoint* rather than *self-adjoint*. However, if we just extract the corresponding triple $\alpha^+ \colon E^+ \to E^-$ there is no requirement on α^+ to be self- or skew-adjoint, and in fact the triples (E^+, E^-, α^+) and $(E^+, E^-, i\alpha^+)$ represent the same K-theory class.

Exercise 3.44. Prove Theorem 3.42, i.e., show that the restriction of $U^K_{\mathbb{C}}(V)$ to each fiber V_x is a generator of $K_c(V_x)$. Hint:

(a) Show that for $\dim V = 1$ the class $[U^K_{\mathbb{C}}(V)] \in K_c(V)$ restricts to the Bott class $[B] \in K_c(V_x)$ on each fiber.

(b) Show that the Thom class $U^K_{\mathbb{C}}(V)$ is exponential in the sense that for complex vector bundles $V, W \to X$

$$[U^K_{\mathbb{C}}(V \oplus W)^{\mathbb{C}}] = [p_1^* U^K_{\mathbb{C}}(V) \otimes p_2^* U^K_{\mathbb{C}}(W)] \in K_c(V \oplus W),$$

where $p_1 \colon V \oplus W \to V$, $p_2 \colon V \oplus W \to W$ are the projection maps. In particular, if V is an n-dimensional bundle, then the restriction of the Thom class $U^K_{\mathbb{C}}(V)$ to the fiber $V_x = \mathbb{C}^n$ is the tensor product of n copies of the Bott class. This is the generator of $K_c(\mathbb{C}^n)$ by the Bott Periodicity Theorem.

The limitation of the Thom class $U^K_{\mathbb{C}}(V) \in K_c(V)$ is that it requires V to be a *complex vector bundle*. Since the Thom class for the normal bundle V of a compact manifold X in Euclidean space is the crucial ingredient for the construction of the umkehr map $p_! \colon K(X) \to K^{-n}(\mathrm{pt})$, this is an unwelcome restriction to the manifolds we can construct $p_!$ for. It turns out that a spin structure (or spin^c structure) on an even dimensional real vector bundle V determines a Thom class $U^K(V)$ (which does *not* agree with the Thom class $U^K_{\mathbb{C}}(V)$ if V happens to be a complex vector bundle whose underlying real vector bundle is equipped with a spin structure). The construction of $U^K(V)$ for a real vector bundle of dimension $2n$ for $n > 1$ will have to wait until we discuss Clifford algebras, but here is the construction for $n = 1$.

Definition 3.45. Let $\pi \colon V \to X$ be an oriented real vector bundle of dimension 2, equipped with a bundle metric. Note that equivalently, V can be viewed as a complex 1-dimensional vector bundle with hermitian metric. A spin structure on V can be interpreted as a *square root* of V, i.e., as a pair (L, β), where L is a complex line bundle and $\beta \colon L^{\otimes 2} \xrightarrow{\cong} V$ is a vector bundle isomorphism (exercise: prove this!). Abusing language, we write $V^{1/2}$ for the square root of V determined by its spin structure, and $V^{-1/2} := (V^{1/2})^*$ for its dual vector bundle. Then

$$\mathrm{Hom}(V_x^{-1/2}, V_x^{1/2}) \cong (V_x^{-1/2})^* \otimes V_x^{1/2} \cong V_x^{1/2} \otimes V_x^{1/2} \cong V_x,$$

and hence we can construct the vector bundle map

$$\alpha^+ : \pi^* V^{-1/2} \longrightarrow \pi^* V^{1/2} \qquad \text{given by} \qquad (v, w) \mapsto (v, v \cdot w)$$

for $v \in V_x$, $w \in V_x^{-1/2}$. Since the support of this triple is the zero section in V, this triple represents an element in $K_c(V)$. Restricted to each fiber of V it is isomorphic to the Bott triple, and hence it is a Thom class which we denote $U^K(V) \in K_c(V)$.

3.4 Chern classes

For general vector bundles it is difficult to decide whether two vector bundles over the same topological space X are isomorphic; for example the result that the tangent bundle TS^n of the n-sphere is isomorphic to the trivial bundle if and only if $n = 1$, 3, or 7 is a result that is relatively hard to prove. Fortunately, complex line bundles are easy to classify up to isomorphism due to the following two facts:

- Let $\mathbb{CP}^\infty$ the complex projective space, and let $\gamma \to \mathbb{CP}^\infty$ be the tautological line bundle. For a topological space X, let $[X, \mathbb{CP}^\infty]$ be the homotopy classes of maps $f : X \to \mathbb{CP}^\infty$. Then the map

  $$[X, \mathbb{CP}^\infty] \longrightarrow \{\text{complex line bundles over } X\}/\text{isomorphism} \qquad \text{given by} \qquad [f] \mapsto f^*\gamma$$

  is a bijection. This statement motivates calling $\mathbb{CP}^\infty$ the *classifying space for complex line bundles*, and γ the *universal complex line bundle*.

- Let $x \in H^2(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$ be the generator characterized by $\langle x, [\mathbb{CP}^1] \rangle = 1$. Here $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^1; \mathbb{Z})$ is the fundamental class of $\mathbb{CP}^1 \subset \mathbb{CP}^\infty$ with respect to the orientation given by the complex structure on $\mathbb{CP}^1$, and $\langle x, [\mathbb{CP}^1] \rangle \in \mathbb{Z}$ is the evaluation of x on $[\mathbb{CP}^1]$. Then the map

  $$[X, \mathbb{CP}^\infty] \longrightarrow H^2(X; \mathbb{Z}) \qquad \text{given by} \qquad [f] \mapsto -f^*x$$

  is also a bijection. So we could also refer to $\mathbb{CP}^\infty$ as the *classifying space for* 2*-dimensional cohomology classes* and to $x \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ as the *universal* 2*-dimensional cohomology class*.

The composition of these two bijections then gives a bijection

$$\{\text{complex line bundles over } X\}/\text{isomorphism} \;\longleftrightarrow\; H^2(X; \mathbb{Z}).$$

This is actually an isomorphism of abelian groups, with the obvious group structure on $H^2(X; \mathbb{Z})$, and the group structure on the left given by tensor products of line bundles. The cohomology class associated to a complex line bundle $L \to X$ is called the *first Chern class*, denoted $c_1(L) \in H^2(X; \mathbb{Z})$.

Remark 3.46. Why is there a minus sign in the definition of $c_1(L)$? For a closed surface Σ, we would like to have

$$\langle c_1(T\Sigma), [\Sigma]\rangle = \chi(\Sigma), \quad \text{the } \textit{Euler characteristic} \text{ of } \Sigma. \tag{3.47}$$

Let's check this for $\Sigma = S^2 = \mathbb{CP}^1$ with tangent bundle $TS^2 \cong (\gamma^*)^{\otimes 2}$ (prove this!).

$$c_1(TS^2) = c_1(\gamma^* \otimes 2\gamma^*) = 2c_1(\gamma^*) = -2c_1(\gamma),$$

and hence $\langle c_1(TS^2), [S^2]\rangle = -2\langle c_1(\gamma), [S^2]\rangle = 2 = \chi(S^2)$.

T For complex vector bundles $E \to X$, there are characteristic classes called *Chern classes* $c_i(E) \in H^{2i}(X;\mathbb{Z})$ for $i = 0, 1, \dots$, which only depend on the isomorphism class of E. They have the following properties:

vanishing $c_0(E) \in H^0(X;\mathbb{Z})$ is the unit in the cohomology ring $H^*(X;\mathbb{Z})$ (we denote the unit by 1), and $c_i(E) = 0$ for $i > \dim_{\mathbb{C}} E$ (the dimension of the fibers of E).

naturality If $f\colon Y \to X$ is a continuous map, then $c_i(f^*E) = f^*c_i(E) \in H^{2i}(Y;\mathbb{Z})$.

Exponential property Let $c(E) := 1 + c_1(E) + c_2(E) + \cdots \in H^*(X;\mathbb{Z})$ be the total Chern class of $E \to X$. If F is another complex vector bundle over X, then the total Chern class of the direct sum $E \oplus F$ is given by the formula

$$c(E \oplus F) = c(E)c(F) \in H^*(X;\mathbb{Z}),$$

where $c(E)c(F)$ is the cup product of these classes.

Normalization For the tautological line bundle $\gamma \to \mathbb{CP}^1$,

$$\langle c_1(\gamma), [\mathbb{CP}^1]\rangle = -1.$$

In other words, the first Chern class for complex line bundles agrees with our earlier construction at the beginning of this section.

Theorem 3.48. Axiomatic characterization of Chern classes. *There are cohomology classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ for complex vector bundles E. They are uniquely characterized by the four properties above.*

Exercise 3.49.

1. Show that for the trivial bundle $\underline{\mathbb{C}}^k$, its total Chern class is $c(\underline{\mathbb{C}}^k) = 1 \in H^*(X;\mathbb{Z})$. Hint: use naturality for the projection map $p\colon X \to \text{pt}$.
2. Show that $c(E \oplus \underline{\mathbb{C}}^k) = c(E)$.
3. Show that if L_1, L_2 are complex line bundle over X, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. Hint: Use naturality of c_1 for the diagonal map $\Delta\colon X \to X \times X$, $x \mapsto (x, x)$.

A little later, we will construct the Chern classes $c_i(E) \in H^{2i}(E;\mathbb{Z})$, but we won't show that the cohomology classes constructed we construct have all the properties listed above. There is an additional property of Chern classes: If X is a closed complex manifold of complex dimension n, then

$$\langle c_n(TX), [X]\rangle = \chi(X).$$

This is the generalization of our earlier statement (3.47) for $n = 1$

The Grassmann manifold $G_k(W)$ and its tangent bundle. Let W be a complex vector space of dimension n equipped with a Hermitian metric, and for $0 \le k \le n$, let $G_k(W)$ be the *Grassmann manifold* defined by

$$G_k(W) := \{V \subset W \mid V \text{ is a } k\text{-dimensional linear subspace of } W\}.$$

In particular, $G_1(\mathbb{C}^n) = \mathbb{CP}^{n-1}$ is the complex projective space of 1-dimensional subspaces of $\mathbb{C}^n$. For $V \in G_k(W)$, let $V^\perp \subset W$ be the orthogonal complement of V, and let

$$\phi_V\colon \operatorname{Hom}(V, V^\perp) \longrightarrow G_k(W) \qquad \text{be the map defined by} \qquad f \mapsto \operatorname{graph}(f).$$

Here $\operatorname{graph}(f) \subset V \times V^\perp = W$ is the graph of the linear map $f\colon V \to V^\perp$, which is a k-dimensional subspace of W. It can be shown that ϕ_V is a homeomorphism, and that the collection of these homeomorphisms provides a holomorphic atlas for $G_k(W)$, thus giving the Grassmann manifold $G_k(W)$ the structure of a complex manifold. In particular,

$$\dim_{\mathbb{C}} G_k(W) = \dim_{\mathbb{C}} \operatorname{Hom}(V, V^\perp) = \dim_{\mathbb{C}} V \dim_{\mathbb{C}} V^\perp = k(n-k).$$

Moreover, for $V \in G_k(W)$ the tangent space $T_V G_k(W)$ can be expressed in terms of V and $V^\perp$ via the vector space isomorphism

$$T_V G_k(W) \xleftarrow{\cong} \operatorname{Hom}(V, V^\perp) \tag{3.50}$$

given by mapping $f \in \operatorname{Hom}(V, V^\perp)$ to the tangent vector at V given by the equivalence class of the path $\mathbb{R} \to G_k(W)$, $t \mapsto \operatorname{graph}(tf)$.

The *tautological vector bundle* over $G_k(W)$ is the k-dimensional subbundle γ of the trivial bundle $\underline{W} := G_k(W) \times W$ given by

$$\gamma := \{(V, w) \mid V \in G_k(W),\ w \in V\} \ \subset\ G_k(W) \times W.$$

Similarly, we define a $n-k$-dimensional subbundle $\gamma^\perp$ by

$$\gamma^\perp := \{(V, w) \mid V \in G_k(W),\ w \in V^\perp\} \ \subset\ G_k(W) \times W.$$

By construction, γ and $\gamma^\perp$ are complementary subbundles of the trivial bundle $\underline{W}$ and hence the inclusion maps give a vector bundle isomorphism

$$\gamma \oplus \gamma^\perp \cong \underline{W}.$$

Moreover, the vector space isomorphism (3.50) depends smoothly (actually holomorphically) on V, and hence leads to an isomorphism of complex vector bundles

$$TG_k(W) \xleftarrow{\cong} \mathrm{Hom}(\gamma, \gamma^\perp). \tag{3.51}$$

Putting these vector bundle isomorphisms together, we obtain the following isomorphism

$$TG_k(W) \oplus \mathrm{Hom}(\gamma,\gamma) \cong \mathrm{Hom}(\gamma,\gamma^\perp) \oplus \mathrm{Hom}(\gamma,\gamma) \cong \mathrm{Hom}(\gamma,\gamma\oplus\gamma^\perp) \cong \mathrm{Hom}(\gamma,\underline{W}).$$

For $k=1$, the vector bundle $\mathrm{Hom}(\gamma,\gamma)$ has dimension 1, and the identity section provides a trivialization, allowing us to identify it with the trivial line bundle $\underline{\mathbb{C}}^1$. Hence for the complex projective space $\mathbb{CP}^{n-1} = \mathbb{CP}(\mathbb{C}^n) = G_1(\mathbb{C}^n)$ we have the bundle isomorphism

$$T\mathbb{CP}^{n-1} \oplus \underline{\mathbb{C}}^1 \cong \mathrm{Hom}(\gamma, \underline{\mathbb{C}}^n) = \underbrace{\gamma^* \oplus \cdots \oplus \gamma^*}_{n}, \tag{3.52}$$

where $\gamma^* = \mathrm{Hom}(\gamma, \underline{\mathbb{C}})$ is the complex line bundle dual to γ.

Example 3.53. The total Chern class of the tangent bundle of $\mathbb{CP}^{n-1}$. We recall that the cohomology ring $H^(\mathbb{CP}^{n-1};\mathbb{Z})$ is the truncated polynomial ring $\mathbb{Z}[x]/(x^n)$, where $x \in H^2(\mathbb{CP}^{n-1};\mathbb{Z})$ is the element characterized by $\langle x, [\mathbb{CP}^1]\rangle = 1$, and (x^n) is the ideal generated by x^n. Let $\gamma \to \mathbb{CP}^{n-1}$ be the tautological bundle and $\gamma^* \to \mathbb{CP}^{n-1}$ its dual. Then

$$\gamma^* \otimes \gamma \cong \mathrm{Hom}(\gamma,\gamma)$$

is trivializable (via the nowhere vanishing identity section), and hence by Exercise **??**,

$$0 = c_1(\gamma^* \otimes \gamma) = c_1(\gamma^*) + c_1(\gamma)$$

and hence the normalization condition (and the fact that the bundles γ, γ^* over $\mathbb{CP}^{n-1}$ restrict to the bundles over $\mathbb{CP}^1$ with the same names) implies

$$\langle c_1(\gamma^*), [\mathbb{CP}^1]\rangle = -\langle c_1(\gamma), [\mathbb{CP}^1]\rangle = 1.$$

Hence $c_1(\gamma^*) = x$, and $c(\gamma^*) = 1 + x \in H^*(\mathbb{CP}^{n-1})$.

Using the bundle equation (3.52) we can calculate the total Chern class of the tangent bundle $T\mathbb{CP}^{n-1}$:

$$c(T\mathbb{CP}^{n-1}) = c(T\mathbb{CP}^{n-1} \oplus \underline{\mathbb{C}}^1) = c(n\gamma^*) = c(\gamma^*)^n = (1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

The Chern class $c_k(T\mathbb{CP}^{n-1}) \in H^*(\mathbb{CP}^{n-1};\mathbb{Z})$ is the degree $2k$ component of the total Chern class, and hence

$$c_k(T\mathbb{CP}^{n-1}) = \binom{n}{k} x^k.$$

In particular, the top Chern class of this $n-1$-dimensional tangent bundle evaluated on the fundamental class gives

$$\langle c_{n-1}(T\mathbb{CP}^{n-1}), [\mathbb{CP}^{n-1}]\rangle = \langle nx^{n-1}, \mathbb{CP}^{n-1}\rangle = n$$

which agrees with the Euler characteristic of $\mathbb{CP}^{n-1}$.

Example 3.54. Let $E \to X$ be a complex vector bundle which is a sum of complex line bundles $L_1, \dots, L_n$. Let $x_i := c_1(L_i) \in H^2(X)$. Then by the exponential property of the total Chern class, we have

$$\begin{aligned} c(E) =& c(L_1)\cdots c(L_n) = (1+x_1)\cdots(1+x_n) \\ =& 1 + \sigma_1(x_1,\dots,x_n) + \sigma_2(x_1,\dots,x_n) + \cdots + \sigma_n(x_1,\dots,x_n) \end{aligned}$$

Here $\sigma_i(x_1,\dots,x_n)$ is a homogeneous polynomial of degree i in the variables $x_1,\dots,x_n$, called the i-th *elementary symmetric polynomial*. Explicitly,

$$\begin{aligned} \sigma_1(x_1,\dots,x_n) =& \sum_{1\le j\le n} x_j \\ \sigma_2(x_1,\dots,x_n) =& \sum_{1\le j<k\le n} x_j x_k \\ \sigma_3(x_1,\dots,x_n) =& \sum_{1\le j<k<l\le n} x_j x_k x_l \\ &\vdots \\ \sigma_n(x_1,\dots,x_n) =& x_1 x_2 \dots x_n \end{aligned}$$

This is a very useful statement and for future reference we state it as a lemma.

Lemma 3.55. *Let $L_1,\dots,L_n$ be complex line bundles over a space X and let $x_i := c_1(L_i) \in H^2(X;\mathbb{Z})$ be the first Chern class of L_i. Then*

$$c_k(L_1 \oplus \cdots \oplus L_n) = \sigma_k(x_1,\dots,x_n) \in H^{2k}(X;\mathbb{Z}),$$

where $\sigma_k(x_1,\dots,x_n)$ is the k-th elementary symmetric function of $x_1,\dots,x_n$.

3.5 The Leray-Hirsch Theorem and the splitting principle

Let $\pi\colon (E,E_0) \to X$ be a fiber bundle with fiber (F,F_0). Then the cohomology of the total space $H^*(E,E_0)$ is a module over the cohomology of the base space $H^*(X)$, where the module structure

$$H^*(E,E_0)\otimes H^*(X) \longrightarrow H^*(E,E_0) \qquad \text{is given by} \qquad a\otimes b \mapsto a\cup \pi^*(b).$$

If E is an oriented vector bundle of dimension k and E_0 is its complement of the zero-section, the Thom isomorphism implies that $H^*(E, E_0)$ is a *free* module over $H^*(X)$ of rank 1 with basis element given by the Thom class $U \in H^k(E, E_0)$. The following result is a generalization of the Thom isomorphism.

Theorem 3.56. (Leray-Hirsch Theorem). *Let $\pi\colon (E, E_0) \to X$ be a fiber bundle with fiber (F, F_0). Let R be a commutative ring, and assume that the cohomology with R-coefficients $H^*(F, F_0; R)$ is a free R-module. Assume further that there are (homogeneous) elements $U_1, \dots, U_n \in H^*(E, E_0; R)$ whose restriction to each fiber $(E, E_0)_x$ is an R-basis for the R-cohomology of this pair. Then the cohomology $H^*(E, E_0; R)$ is a free module over $H^*(X; R)$ with basis $\{U_1, \dots, U_n\}$. In particular, any element $a \in H^*(E, E_0; R)$ can uniquely be written in the form*

$$a = U_1 a_1 + U_2 a_2 + \cdots + U_n a_n \qquad \text{for } a_i \in H^*(X; R).$$

For notational simplicity we suppress here the cup product and the pullback π^*.

We will apply the Leray-Hirsch Theorem to the complex projective space bundle

$$\pi\colon \mathbb{CP}(W) \to X$$

associated to a complex n-dimensional vector bundle $W \to X$. The total space of this bundle is

$$\mathbb{CP}(W) := \{(x, V) \mid x \in X,\, V \subset W_x \text{ 1-dimensional subspace of the fiber } W_x\},$$

and the projection map π sends the pair (x, V) to $x \in X$. So the fiber $\mathbb{CP}(W)_x = \pi^{-1}(x)$ is the complex projective space $\mathbb{CP}(W_x)$. In other words, we can think of this bundle as a family of complex projective spaces parametrized by X.

The construction of the vector bundles γ and $\gamma^\perp$ over projective space extend to this parametrized situation. The bundles $\gamma \to \mathbb{CP}(W)$, $\gamma^\perp \to \mathbb{CP}(W)$ are now complementary subbundles of the pullback bundle $\pi^*W \to \mathbb{CP}(W)$ defined by

$$\begin{aligned} \gamma :=& \{(x, V, w) \mid x \in X,\ V \subset W_x,\ w \in V\} \\ \gamma^\perp :=& \{(x, V, w) \mid x \in X,\ V \subset W_x,\ w \in V^\perp\} \end{aligned}$$

These vector bundles generalize what we did for a single complex projective space, and on each fiber $\mathbb{CP}(W)_x = \mathbb{CP}(W_x)$, these bundles restrict to the bundles constructed earlier. In particular, the cohomology class $x := c_1(\gamma^*) \in H^2(\mathbb{CP}(W)$ restricts to the generator of $H^2(\mathbb{CP}(W); \mathbb{Z})$. Hence the cohomology classes $\{1, x, x^2, \dots, x^{n-1}\}$ restrict to a basis of $H^*(\mathbb{CP}(W); \mathbb{Z})$. Then the Leray-Hirsch Theorem implies the following result.

Corollary 3.57. *Let $W \to X$ be complex vector bundle of dimension n, and let $\pi\colon \mathbb{CP}(W) \to X$ be the associated complex projective space bundle. Let $x := c_1(\gamma^*) \in H^2(\mathbb{CP}(W); \mathbb{Z})$ be the first Chern class of the dual of the tautological line bundle γ. Then $H^*(\mathbb{CP}(W); \mathbb{Z})$ is* *a free module over $H^*(X;\mathbb{Z})$ with basis $\{1, x, \dots, x^{n-1}\}$. In particular, any element $a \in H^*(\mathbb{CP}(W);\mathbb{Z})$ can uniquely be written in the form*

$$a = a_0 + xa_1 + x^2a_2 + \cdots + x^{n-1}a_{n-1} \qquad \textit{with } a_i \in H^*(X;\mathbb{Z}).$$

Corollary 3.58. (The splitting principle.) *Let $W \to X$ be a complex vector bundle of dimension n. Then there exists a continuous map $f\colon Y \to X$ with the following properties*

- *the pullback f^*W is isomorphic to $L_1 \oplus \cdots \oplus L_n$, a sum of complex line bundles.*
- *the induced map $f^*\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is injective.*

Proof. By the previous corollary, the bundle map $\pi\colon \mathbb{CP}(W) \to X$ induces an injective map on cohomology. The pullback π^*W splits as the direct sum $\gamma \oplus \gamma^\perp$ of the complementary subbundles $\gamma, \gamma^\perp$. If $\dim W = 2$, then $\gamma^\perp$ has dimension one, and map π does the job. If $\dim W > 2$, we consider the projective bundle $\mathbb{CP}(\gamma^\perp) \to \mathbb{CP}(W)$ associated to the vector bundle $\gamma^\perp \to \mathbb{CP}(W)$ of dimension $n-1$. So applying this procedure repeatedly, we end up with a sequence of maps, each of which is injective in cohomology, such that W pulled back via their composition splits as a sum of line bundles. □

3.5.1 Construction of the Chern classes

Another application of the Leray-Hirsch Theorem, or more precisely its corollary **??**, is a construction of the Chern classes. A little care is required to avoid logical loops, since we *used* Chern classes to argue that the powers of $x = c_1(\gamma^*)$ form a basis of $H^*(\mathbb{CP}(W);\mathbb{Z})$ as a module over $H^*(X;\mathbb{Z})$.

The construction of the vector bundles γ and $\gamma^\perp$ over projective space extend to this parametrized situation. The bundles $\gamma \to \mathbb{CP}(W)$, $\gamma^\perp \to \mathbb{CP}(W)$ are now complementary subbundles of the pullback bundle $\pi^*W \to \mathbb{CP}(W)$ defined by

$$\begin{aligned}\gamma :=& \{(x, V, w) \mid x \in X,\ V \subset W_x,\ w \in V\}\\ \gamma^\perp :=& \{(x, V, w) \mid x \in X,\ V \subset W_x,\ w \in V^\perp\}\end{aligned}$$

These vector bundles generalize what we did for a single complex projective space. In particular, on each fiber $\mathbb{CP}(W)_x = \mathbb{CP}(W_x)$ these bundles restrict to the bundles constructed earlier. Let $\mathcal{V}\mathbb{CP}(W) \to \mathbb{CP}(W)$ be the *vertical tangent bundle*, which restricts to the tangent bundle on each fiber $\mathbb{CP}(W_x)$. The vector bundle isomorphism (3.51) generalizes to the parametrized situation to give a vector bundle isomorphism

$$\mathcal{V}\mathbb{CP}(W) \cong \mathrm{Hom}(\gamma, \gamma^\perp).$$

Next we want to generalize the vector bundle isomorphisms (3.52) to the parametrized version. The only difference is that γ, γ^* are no longer complementary subbundles of a trivial bundle, but rather of the pullback bundle π^*W, which leads to the isomorphism $\gamma \oplus \gamma^* \cong \pi^*W$. Hence

$$\mathcal{V}T\mathbb{CP}(W) \oplus \underline{\mathbb{C}}^1 \cong \mathrm{Hom}(\gamma, \gamma \oplus \gamma^\perp) \cong \mathrm{Hom}(\gamma, \pi^*W)$$

Let us calculate the total Chern class of the vertical tangent bundle. Let us assume that the n-dimensional complex vector bundle W splits as a sum of complex line bundles

$$W \cong L_1 \oplus \cdots \oplus L_n$$

and let $x_i := c_1(L_i) \in H^2(X;\mathbb{Z})$. Then the total Chern class of the vertical tangent bundle $\mathcal{V}\mathbb{CP}(W)$ is

$$\begin{aligned} c(\mathcal{V}\mathbb{CP}(W) =& c(\mathcal{V}\mathbb{CP}(W) \oplus \underline{\mathbb{C}}^1) \\ =& c(\gamma^* \otimes \pi^*W) \\ =& c(\gamma^* \otimes (\pi^*WL_1 \oplus \cdots \oplus \pi^*WL_n)) \\ =& c(\gamma^* \otimes \pi^*L_1) \cdots c(\gamma^* \otimes \pi^*L_n) \\ =& (1 + x + x_1) \cdots (1 + x + x_n) \end{aligned}$$

In particular, the n-th Chern class of $\mathcal{V}\mathbb{CP}(W)$ is given by

$$c_n(\mathcal{V}\mathbb{CP}(W)) = (x + x_1) \cdots (x + x_n) = x^n + x^{n-1}\sigma_1 + x^{n-2}\sigma_2 + \cdots + x\sigma_{n-1} + \sigma_n \tag{3.59}$$

where $\sigma_i = \sigma_i(x_1, \ldots, x_n)$ is the i-th elementary symmetric function of the x_i, e.g.,

$$\sigma_1 = x_1 + \cdots + x_n \qquad \sigma_2 = \sum_{1 \le i < j \le n} x_i x_j \qquad \sigma_n = x_1 \cdots x_n$$

We note that the (complex) dimension of the vertical tangent bundle is equal to the dimension of the fibers of $\mathbb{CP}(W) \to X$. These are complex projective spaces $\mathbb{CP}(W_x)$ of (complex) dimension $n-1$, and hence $c_n(\mathcal{V}\mathbb{CP}(W)) = 0$. Hence

$$-x^n = \pi^*c_n(W) + x\pi^*c_{n-1}(\pi^*(W) + \cdots + x^{n-1}\pi^*c_1(W) \tag{3.60}$$

This equation shows one way to *construct* the Chern classes of a complex vector bundle $W \to X$ of dimension n:

- form the projective bundle $\pi\colon \mathbb{CP}(W) \to X$. By the Leray-Hirsch Theorem, its cohomology $H^*(\mathbb{CP}(W);\mathbb{Z})$ is a free module over $H^*(X;\mathbb{Z})$ with basis $\{1, x, x^2, \ldots, x^{n-1}\}$ where $x = c_1(\gamma^*) \in H^2(\mathbb{CP}(W);\mathbb{Z})$ is the first Chern class of the dual of the tautological complex line bundle γ.

- Expressing the cohomology class $-x^n \in H^{2n}(\mathbb{CP}(W);\mathbb{Z})$ in terms of this basis, the coefficient of the basis element x^{n-i} is the Chern class $c_i(W) \in H^{2i}(X;\mathbb{Z})$.

3.5.2 The Chern character

Let E, F be complex vector bundles over a space X. Then the total Chern class of the direct sum $E \oplus F$ is given by the simple formula $c(E \oplus F) = c(E)c(F)$. By contrast, it is cumbersome to calculate the total Chern class of the tensor product $E \otimes F$:

- if E, F are line bundles, then $c_1(E \otimes F) = c_1(E) + c_1(F)$ by Exercise 3.49(3);
- If E and F are sum of line bundles, so is $E \otimes F$, and for each summand we can use (1) to calculate its total Chern class; multiplying them gives the total Chern class of $E \otimes F$;
- By the splitting principle, it is ok to assume that E, F split as a sum of line bundles.

The *Chern character* $\mathrm{ch}(E) \in H^{\mathrm{ev}}(X;\mathbb{Q})$ of a complex vector bundle is much more pleasant for calculations; it is designed to have the properties

1. $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$ and
2. $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$.

Let's reverse engineer $\mathrm{ch}(E)$, first defining $\mathrm{ch}(L)$ for a complex line bundle L. Since L is determined by its first Chern class $x := c_1(L) \in H^2(X;\mathbb{Z})$, the cohomology class $\mathrm{ch}(L)$ should be some function $f(x)$ of x. If L_1, L_2 are two line bundles over X with $x_i = c_1(L_i) \in H^2(X)$, then the desired property (2) and $c_1(L_1 \otimes L_2) = x_1 + x_2$ forces

$$f(x_1 + x_2) = \mathrm{ch}(L_1 \otimes L_2) = \mathrm{ch}(L_1)\,\mathrm{ch}(L_2) = f(x_1)f(x_2).$$

This shows that defining

$$\mathrm{ch}(L) := e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k \in H^{\mathrm{ev}}(X;\mathbb{Q}) \qquad \text{for a complex line bundle } L \to X \text{ with } x = c_1(L)$$

satisfies property (2). If $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles L_i with $x_i = c_1(L_i)$, then property (1) requires to define

$$\mathrm{ch}(E) = \mathrm{ch}(L_1 \oplus \cdots \oplus L_n) = \mathrm{ch}(L_1) + \cdots + \mathrm{ch}(L_n) = e^{x_1} + \cdots + e^{x_n} \in H^{\mathrm{ev}}(X;\mathbb{Q}).$$

Can we express $\mathrm{ch}(E)$ in terms of the Chern classes $c_k(E) \in H^{2k}(X;\mathbb{Z})$ of E? According to Lemma 3.55

$$c_k(E) = \sigma_k(x_1, \dots, x_n),$$

the k-th elementary symmetric function of the x_i. Let $\mathrm{ch}(E)_{2k} \in H^{2k}(X;\mathbb{Q})$ be the degree $2k$ part of $\mathrm{ch}(E)$, which is a homogeneous polynomial of $x_1, \dots, x_n$ of degree k. It is a *symmetric* polynomial, i.e., invariant under permuting the variables x_i. Then according to the *fundamental theorem of symmetric polynomials* $\mathrm{ch}(E)_{2k}$ can be expressed as a polynomial of $\sigma_1, \dots, \sigma_n$, i.e.,

$$\mathrm{ch}(L_1 \oplus \cdots \oplus L_n)_{2k} = P_k^n(\sigma_1, \dots, \sigma_n).$$

Here the polynomial P_k^n is independent of the line bundles L_i at hand. For example,

$$\begin{aligned} e^{x_1} + e^{x_2} =& 1 + x_1 + \frac{1}{2}x_1^2 + \cdots + 1 + x_2 + \frac{1}{2}x_2^2 + \dots \\ =& 2 + (x_1 + x_2) + \frac{1}{2}(x_1^2 + x_2^2) + \dots \\ =& 2 + \sigma_1 + (\frac{1}{2}\sigma_1^2 - \sigma_2) + \dots \end{aligned}$$

Here the last equality follows from $\sigma_1^2 = (x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 = (x_1^2 + x_2^2) + 2\sigma_2$. This shows that

$$P_0^2 = 2 \qquad P_1^2 = \sigma_1 \qquad P_2^2 = \frac{1}{2}\sigma_1^2 - \sigma_2.$$

Definition 3.61. Let $E \to X$ be a complex vector bundle over X with Chern classes $c_k(E) \in H^{2k}(X)$. Then the *Chern character* $\mathrm{ch}(E) \in H^{\mathrm{ev}}(X; \mathbb{Q})$ is the cohomology class whose degree $2k$ part $\mathrm{ch}(E)_{2k} \in H^{2k}(X; \mathbb{Q})$ is given by

$$\mathrm{ch}(E)_{2k} := P_k^n(c_1(E), \dots, c_n(E)).$$

By construction of the Chern character, it is clear that it has the desired properties (1) and (2). For future reference, we state this as a lemma.

Lemma 3.62. *The Chern character of a complex vector bundle $E \to X$ is an element $\mathrm{ch}(E) \in H^{\mathrm{ev}}(X; \mathbb{Q})$, which has the properties*

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F) \qquad \mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$$

for complex vector bundles E, F over X.

Remark 3.63. If $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles L_i, then by construction

$$\mathrm{ch}(E) = e^{x_1} + \cdots + e^{x_n} \qquad \text{for } x_i = c_1(L_i)$$

Motivated by the equation

$$(1 + x_1) \cdot \dots \cdot (1 + x_n) = c(L_1 \oplus \cdots \oplus L_n) = c(E),$$

the classes $x_i \in H^2(X)$ are often referred to as *Chern roots* of E. This terminology is used even in cases where E does not split as a sum of line bundles: in that case, the splitting principle guarantees that a pullback of E via some map $f: Y \to X$ splits as a sum of line bundles, and hence the Chern roots of E don't live in the cohomology of X, but rather the cohomology of Y. The injectivity of the induced homomorphism $f^*: H^*(X) \to H^*(Y)$ guarantees that this is good enough for calculations in cohomology. In particular, since the properties (1) and (2) of the Chern character hold by construction for sum of line bundles, by the splitting principle they hold in general.

3.6 Comparing Orientations in K-theory and ordinary cohomology

We recall that for a compact topological space X the elements of the K-theory group $K(X)$ are of the form $[E]-[F]$ where E, F are complex vector bundles over X (see Definition 3.31). Let

$$\mathrm{ch}\colon K(X) \longrightarrow H^{\mathrm{ev}}(X;\mathbb{Q}) \qquad \text{be defined by} \qquad [E^+]-[E^-] \mapsto \mathrm{ch}(E^+)-\mathrm{ch}(E^-).$$

The additive property $\mathrm{ch}(E\oplus F)=\mathrm{ch}(E)+\mathrm{ch}(F)$ of the Chern character guarantees that the above map is a well-defined homomorphism. The multiplicative property $\mathrm{ch}(E\otimes F) = \mathrm{ch}(E)\cup\mathrm{ch}(F)$ implies that the Chern Character is a ring homomorphism, with the product in K-theory given by the tensor product of vector bundles, and the product in cohomology given by the cup product.

We recall that we constructed K-theory orientations (also known as K-theory Thom classes) for suitable vector bundles $V\to X$. In fact, we mentioned two different K-theory orientations:

(i) If $V\to X$ is a complex vector bundle of (complex) dimension n, we constructed a K-theory orientation $U^K_{\mathbb{C}}(V)\in K_c(V)$, see Definition 3.41;

(ii) if $V\to X$ is a real vector bundle of real dimension $2n$ with spin structure, there is a K-theory orientation $U^K(V)\in K_c(V)$. This orientation we have so far only constructed for $n=1$, i.e., if V is a complex line bundle with spin structure, see Definition 5.16. The general construction will be done later.

The goal of this section is compare the images of our K-theory orientations $U^K_{\mathbb{C}}(V)$, $U^K(V)$ under the Chern character map with the usual orientation class $U^H(V)\in H^*_c(V;\mathbb{Z})$ in ordinary cohomology. We will first do this in the case where the vector bundle V is a complex line bundle; later we will deal with the general case.

Proposition 3.64. *Let $\pi\colon V\to X$ be a complex line bundle over a compact space X, and let $x=c_1(V)\in H^2(X;\mathbb{Z})$ be the first Chern class of V. Then*

1.
$$\mathrm{ch}(U^K_{\mathbb{C}}(V)) = U^H(V)\ \cup\ \pi^*\left(\frac{1-e^x}{x}\right)\ \in H^{\mathrm{ev}}_c(V;\mathbb{Q}) \tag{3.65}$$

2. If V is equipped with a spin structure, then

$$\mathrm{ch}(U^K(V)) = U^H(V)\ \cup\ \pi^*\left(-\frac{\sinh(x/2)}{x/2}\right)\ \in H^{\mathrm{ev}}_c(V;\mathbb{Q}) \tag{3.66}$$

Before delving into the proof, we start with some general remarks. Both sides of the equation above are compatible with pullback in the sense that for a map $f: Y \to X$ both sides of the above equation for the bundle f^*V are pullbacks of the corresponding side for the bundle V. In particular, if equation (3.65) holds for V, then it also holds for f^*V.

The complex line bundle $V \to X$ is the pullback of the tautological line bundle $\gamma \to \mathbb{CP}^\infty$ via some map $X \to \mathbb{CP}^\infty$. Alas, we don't want to work directly with $\mathbb{CP}^\infty$ due to its non-compacness. Fortunately, since X is compact, this map in factors in the form

$$X \xrightarrow{f} \mathbb{CP}^k \hookrightarrow \mathbb{CP}^\infty$$

for some sufficiently large k. Hence it suffices to prove the proposition in the case of the tautological line bundle $\gamma \to \mathbb{CP}^k$.

The proof of the proposition is further simplified by the following result.

Lemma 3.67. *Let $\pi: \gamma \to \mathbb{CP}^k$ be the tautological complex line bundle and let $i: \mathbb{CP}^k \to \gamma$ be the zero section. Then the induced map in cohomology $i^*: H_c^*(\gamma; \mathbb{Z}) \to H^*(\mathbb{CP}^k; \mathbb{Z})$ is injective in degree $* < 2k + 2$.*

Exercise 3.68. Prove this lemma. Hints:

- as discussed earlier this semester, the compactly supported cohomology $H_c^*(V)$ of a vector bundle $V \to X$ over a compact base X can be identified with the reduced cohomology $\widetilde{H}^*(X^V)$ of the Thom space X^V. Moreover, if $i: X \hookrightarrow V$ and $j: X \hookrightarrow X^V$ are given by the inclusion of the zero section, then the diagram

$$\begin{tikzcd} H_c^*(V) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^*(X) \\ H^*(X^V) \arrow[ur, "j^*"'] & \end{tikzcd}$$

  is commutative.

- Show that the Thom space $(\mathbb{CP}^k)^\gamma$ of the tautological bundle $\gamma \to \mathbb{CP}^k$ is homeomorphic to $\mathbb{CP}^{k+1}$ such that the diagram

$$\begin{tikzcd} H^*((\mathbb{CP}^k)^\gamma) \arrow[r, "j^*"] \arrow[d, "\cong"'] & H^*(\mathbb{CP}^k) \\ H^*(\mathbb{CP}^{k+1}) \arrow[ur, "\iota^*"'] & \end{tikzcd}$$

  is commutative, where $\iota: \mathbb{CP}^k \to \mathbb{CP}^{k+1}$ is the inclusion map.

Hence to prove Proposition 5.17 it suffices to calculate the image of both side under the map i^*.

Remark 3.69. Let $V \to X$ be a vector bundle and $U^E_V \in E^*_c(V)$ an E-theory orientation of V. Then the cohomology class $\chi^E_V := i^*U^E_V$ is referred to as *E-theory Euler class of* V. It is an obstruction to the existence of a nowhere vanishing section s of the vector bundle V. In other words, if such a section exists, then i^*U^E is trivial. This is evident, since the zero section i is homotopic to the section s, and hence $i^*U^E_V = s^*U^E_V$. By scaling, the section s can be modified such that its image is disjoint from the compact support of U^E_V and hence $i^*U^E_V = 0$.

Exercise 3.70.

1. Show that for the tautological complex line bundle $\gamma \to \mathbb{CP}^k$ its Euler class $\chi^H(\gamma) = i^*U^H(\gamma) \in H^2(\mathbb{CP}^k;\mathbb{Z})$ in ordinary cohomology is given by $i^*U^H(\gamma) = c_1(\gamma)$. Use the second hint for the previous exercise.

2. More generally, show that for any complex vector bundle $V \to X$ of dimension n the Euler class of V is equal to the n-th Chern class $c_n(V) \in H^{2n}(X;\mathbb{Z})$. Hint: The orientation U^H is exponential in the sense that for oriented vector bundles V, W over X
$$U^H(V \oplus W) = U^H(V) \cup U^H(W) \in H^*_c(V \oplus W). \tag{3.71}$$
Here we suppress the projection maps from $V \oplus W$ to V resp. W in the notation. Show the statement assuming at first that V is a sum of complex line bundles, then use the splitting principle to deal with the general case.

Proof of Proposition 5.17. We recall from Definition 3.41 that for the complex vector bundle $\pi\colon V \to X$ over a compact space X the K-theory orientation $U^K_{\mathbb{C}}(V) \in K_c(X)$ is given by the Hermitian triple of the form $\alpha\colon \pi^*\Lambda^*(V) \to \pi^*\Lambda^*(V)$, where $\Lambda^*(V)$ is the exterior algebra bundle generated by the vector bundle V, and $\pi^*\Lambda^*(V)$ is its pullback via the projection map $\pi\colon V \to X$. This implies that $i^*(\pi^*\Lambda^*(V))$, the restriction of $\pi^*\Lambda^*(V)$ to the zero section $X \hookrightarrow V$ is the $\mathbb{Z}/2$-graded vector bundle Λ^*V. Over the compact base space X the vector bundle map α becomes irrelevant, and so the K-theory element $i^*U^K_{\mathbb{C}}(V) \in K(X)$ is just given by the formal difference of the even and the odd part of the exterior algebra bundle:
$$i^*U^K_{\mathbb{C}}(V) = \Lambda^{\mathrm{ev}}V - \Lambda^{\mathrm{odd}}V \in K(X).$$
In particular, if V is a line bundle, then $\Lambda^{\mathrm{ev}}V = \Lambda^0 V = \underline{\mathbb{C}}$ is the trivial line bundle, and $\Lambda^{\mathrm{odd}}V = \Lambda^1 V = V$. Now we specialize to $V = \gamma \to \mathbb{CP}^k$ and set $x := c_1(\gamma)$. Then
$$i^*(\mathrm{ch}\, U^K_{\mathbb{C}}(\gamma)) = \mathrm{ch}(i^*U^K_{\mathbb{C}}(\gamma)) = \mathrm{ch}(\underline{\mathbb{C}} - \gamma) = \mathrm{ch}(\underline{\mathbb{C}}) - \mathrm{ch}(\gamma) = 1 - e^{c_1(\gamma)} = 1 - e^x$$
Since by Exercise 3.70(1) $i^*U^H_\gamma = c_1(\gamma) = x$, it follows that
$$i^*(\mathrm{ch}\, U^K_{\mathbb{C}}(\gamma)) = x \ \cup\ \frac{1-e^x}{x} = i^*U^H_\gamma \ \cup \frac{1-e^x}{x} = i^*\left(U^H(\gamma) \ \cup\ \pi^*\left(\frac{1-e^x}{x}\right)\right).$$

Since i^* is injective, this implies $\mathrm{ch}(U^K_{\mathbb{C}}(\gamma)) = U^H(\gamma) \cup \pi^*\left(\frac{1-e^x}{x}\right)$, thus proving the first part of the Proposition.

The second part of the proposition is proved similarly, but some care is needed due to the fact that the tautological line bundle $\gamma \to \mathbb{CP}^k$ does *not* have a spin structure. We recall from Definition 5.16 that a spin structure on a line bundle V amounts to a "square root" of V, i.e., a complex line bundle L and an isomorphism $\beta\colon L^{\otimes 2} \xrightarrow{\cong} V$. In particular, this implies that $c_1(V) = c_1(L^{\otimes 2}) = 2c_1(L)$ must be divisible by 2; this is not the case for the tautological line bundle γ for which $c_1(\gamma) \in H^2(\mathbb{CP}^k;\mathbb{Z})$ is a generator.

Let $f\colon X \to \mathbb{CP}^k$ be a map that classifies the line bundle $L \to X$ in the sense that $f^*\gamma$ is isomorphic to L. Then $f^*\gamma^{\otimes 2}$ is isomorphic to V, and hence it suffices to prove part (2) for the spin bundle $V = \gamma^{\otimes 2}$ with square root $V^{1/2} = \gamma$. Setting $x := c_1(V)$, we have

$$c_1(V) = x \qquad c_1(V^{1/2}) = x/2 \qquad c_1(V^{-1/2}) = -x/2.$$

It follows that

$$\begin{aligned} i^*\,\mathrm{ch}(U^K(V)) =& \mathrm{ch}(i^*U^K(V)) = \mathrm{ch}(V^{-1/2} - V^{1/2}) = \mathrm{ch}(V^{-1/2}) - \mathrm{ch}(V^{1/2}) \\ =& e^{c_1(V^{-1/2})} - e^{c_1(V^{1/2})} = e^{-x/2} - e^{x/2} = -2\sinh(x/2) \\ =& x \cup -\frac{\sinh(x/2)}{x/2} = i^*U^H(V) \cup -\frac{\sinh(x/2)}{x/2} \\ =& i^*\left(U^H(V) \cup \pi^*\left(-\frac{\sinh(x/2)}{x/2}\right)\right) \end{aligned}$$

Since i^* is injective, this implies

$$\mathrm{ch}(U^K(V)) = U^H(V) \cup \pi^*\left(-\frac{\sinh(x/2)}{x/2}\right) \in H^{\mathrm{ev}}_c(V;\mathbb{Q})$$

as claimed. □

3.6.1 Exponential characteristic classes

The main result so far is Proposition 5.17, which compares the orientations $\mathrm{ch}(U^K_{\mathbb{C}}(V))$, $\mathrm{ch}(U^K(V))$ and $U^H(V)$ for complex line bundle. The goal of this subsection is to do the same for more general vector bundles V. In the literature this often goes under the name *multiplicative sequences* (see for example that section in the excellent book *Characteristic Classes* by Milnor and Stasheff), but I don't like that terminology since I find it not very descriptive.

As motivation, let us first calculate $\mathrm{ch}(U^K_{\mathbb{C}}(V))$ in the special case where V is a sum of complex line bundles $V = L_1 \oplus \cdots \oplus L_n$. The K-theory orientation $U^K_{\mathbb{C}}$ is *exponential* in the sense that for complex vector bundles V, W over a compact space X we have

$$U^K_{\mathbb{C}}(V \oplus W) = U^K_{\mathbb{C}}(V) \otimes U^K_{\mathbb{C}}(W) \in K_c(V \oplus W). \tag{3.72}$$

It follows that for $x_i := c_1(L_i)$

$$\begin{aligned}
\mathrm{ch}(U^K_{\mathbb{C}}(V)) &= \mathrm{ch}\left(U^K_{\mathbb{C}}(L_1)\otimes\cdots\otimes U^K_{\mathbb{C}}(L_n)\right)\\
&= \mathrm{ch}(U^K_{\mathbb{C}}(L_1))\cdot\dots\cdot\mathrm{ch}(U^K_{\mathbb{C}}(L_n))\\
&= \left(U^H(L_1)\ \cup\ \pi^*\left(\frac{1-e^{x_1}}{x_1}\right)\right)\cdot\dots\cdot\left(U^H(L_n)\ \cup\ \pi^*\left(\frac{1-e^{x_n}}{x_n}\right)\right)\\
&= U^H(V)\ \cup\ \pi^*\left(\frac{1-e^{x_1}}{x_1}\cdot\dots\cdot\frac{1-e^{x_n}}{x_n}\right)
\end{aligned} \tag{3.73}$$

Here the last line follows from the exponential property of the orientation U^H in ordinary cohomology. Now we can argue as we did in the case of the Chern Character in section 3.5.2 that the second factor of the right hand side can be expressed in terms of the Chern classes of the complex vector bundle V.

Definition/Construction 3.74. *Let $q(x) = 1 + a_1x + a_2x^2 + \dots$ be a power series with coefficients in a ring R. Then q determines an associated* exponential characteristic class *$T_q(V) \in H^{\mathrm{ev}}(X;\mathbb{R})$ for complex vector bundles $V \to X$ with the following properties:*

normalization: *If V is a complex line bundle, then $T_q(V) = q(c_1(V)) \in H^{\mathrm{ev}}(X;R)$.*

naturality: *For any map $f\colon Y \to X$, $T_q(f^*V) = f^*T_q(V)$.*

exponential property: *$T_q(V \oplus W) = T_q(V)T_q(W)$ for complex vector bundles V, W.*

Again, the argument used to construct the exponential characteristic class $T_q(V)$ is entirely similar to the construction of the Chern Character of a complex vector bundle (despite the fact that the Chern Character is *not* exponential, but rather additive in the sense that $\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$!). First assume that V is a sum of complex line bundles $V = L_1 \oplus \cdots \oplus L_n$, and let $x_i := c_1(L_i)$. Then

$$T_q(V) = T_q(L_1)\cdots T_q(L_n) = q(x_1)\cdots q(x_n).$$

Since the degree $2k$-component $T_q(V)_{2k} \in H^{2k}(X;\mathbb{R})$ is a polynomial of the x_i which is symmetric under permutation of the x_i, the class $T_q(V)_{2k}$ is a polynomial of the elementary symmetric functions $\sigma_i(x_1,\dots,x_n)$. Since $\sigma_i(x_1,\dots,x_n) = c_i(V)$, this can then be used as the definition of $T_q(V)$ of a general complex vector bundle V.

Example 3.75. (Examples of exponential characteristic classes). Let $V \to X$ be a complex vector bundle. Then the following characteristic classes are examples of exponential exponential characteristic classes:

1. The Todd class $\mathrm{Td}(V) := T_q(V) \in H^{\mathrm{ev}}(X;\mathbb{Q})$ associated to $q(x) = \frac{x}{1-e^{-x}}$;

2. The *A-roof class* $\widehat{A}(V) := T_q(V) \in H^{\mathrm{ev}}(X;Q)$ associated to $q(x) = \frac{x/2}{\sinh(x/2)}$.

3. The *L-class* $L(V) := T_q(V) \in H^{\mathrm{ev}}(X;Q)$ associated to $q(x) = \frac{x}{\tanh(x)}$.

Proposition 3.76. *Let V be a complex vector bundle of dimension k over a compact space X. Then*

1. $\mathrm{ch}(U^K_{\mathbb{C}}(V)) = U^H(V) \cup \pi^*((-1)^k \mathrm{Td}(\bar{V})^{-1})$.

2. *If V has a spin structure, then* $\mathrm{ch}(U^K(V)) = U^H(V) \cup \pi^*((-1)^k \widehat{A}(V)^{-1})$.

Proof. If V is a sum complex line bundles $V = L_1 \oplus \cdots \oplus L_k$ and let $x_i = c_1(L_i) \in H^2(X)$. Then the calculation (3.73) shows that

$$\begin{aligned}
\mathrm{ch}(U^K_{\mathbb{C}}(V)) =& U^H(V) \cup \pi^* \left(\frac{1-e^{x_1}}{x_1} \cdot \cdots \cdot \frac{1-e^{x_k}}{x_k} \right) \\
=& U^H(V) \cup \pi^* \left(\left(\frac{x_1}{1-e^{x_1}} \right)^{-1} \cdot \cdots \cdot \left(\frac{x_k}{1-e^{x_k}} \right)^{-1} \right) \\
=& U^H(V) \cup \pi^* \left((-\mathrm{Td}(\bar{L}_1)^{-1}) \cdot \cdots \cdot (-\mathrm{Td}(\bar{L}_k)^{-1}) \right) \\
=& U^H(V) \cup \pi^*((-1)^k \mathrm{Td}(\bar{V})^{-1})
\end{aligned}$$

The general case follows by applying the splitting principle.

The statement concerning the Chern character of the orientation class $U^K(V)$ for complex vector bundles with spin structure follows similarly. If V is a line bundle, part (2) of Proposition 5.17 shows that

$$\mathrm{ch}(U^K(V)) = U^H(V) \cup \pi^*(-\widehat{A}(V)^{-1}).$$

The exponential property of the orientions U^K, U^H as well as the exponential property for the exponential characteristic class $\widehat{A}(V)$ then implies the statement in the case where V is a sum of line bundles. By the splitting principle the statement holds for any complex vector bundle V. □

3.7 Characteristic classes for real vector bundles

In the last section we showed that for a k-dimensional complex vector bundle $\pi\colon V \to X$ with spin structure the Chern character of the K-theory orientation $U^K(V) \in K_c(V)$ is given by

$$\mathrm{ch}(U^K(V)) = U^H(V) \cup (-1)^k \pi^* \widehat{A}(V)^{-1},$$

where $\widehat{A}(V) \in H^{\mathrm{ev}}(X;\mathbb{Q})$ is the $\widehat{A}$-class of V. Back when we first mentioned the orientation class $U^K(V)$ we emphasized that it can be constructed for even dimensional *real vector* *bundles* with spin structure. This leads to the question of how to extend the definition of the characteristic class $\widehat{A}(V)$ from complex bundles to real vector bundles such that the above equation holds.

More generally, we can ask whether the exponential characteristic class $T_q(V)$ associated to a power series $q(x) \in R[[x]]$ (see Definition/Construction 3.74) can be extended from complex vector bundles V to real vector bundles. More formally, we can consider T_q as the map from $\mathrm{Vect}^{\mathbb{C}}(X)$, the isomorphism classes of complex vector bundles over X to $H^{\mathrm{ev}}(X;R)$, and we ask whether this map factors through $\mathrm{Vect}^{\mathbb{R}}(X)$, the set of isomorphism classes of real vector bundles over X. In other words, we ask whether there is a map $T_q^{\mathbb{R}}$ (which is *natural in* X, i.e., compatible with pullbacks) making the following diagram commutative.

$$\begin{tikzcd}
\mathrm{Vect}^{\mathbb{C}}(X) \arrow[rr, "(\ )_{\mathbb{R}}"] \arrow[dr, "T_q"'] & & \mathrm{Vect}^{\mathbb{R}}(X) \arrow[dl, "T_q^{\mathbb{R}}"] \\
& H^{\mathrm{ev}}(X;R) &
\end{tikzcd} \tag{3.77}$$

Here $(\ )_{\mathbb{R}}$ is the map that sends a complex vector bundle V to the real vector bundle $V_{\mathbb{R}}$ obtained by forgetting the complex structure on the fibers V_x, just regarding them as real vector spaces.

Proposition 3.78. *Let $q(x) = 1 + a_1x + a_2x^2 + \cdots \in R[[x]]$ be a power series with coefficients in a ring R. Assume that $2 \in R$ is invertible in R ($R = \mathbb{Q}$ is the case we will be most interested in). Then there is a unique map $T_q^{\mathbb{R}}$ making the diagram* (3.77) *commutative if and only if $q(x)$ is an even power series, i.e., the odd coefficients a_{2i+1} vanish. Moreover, for $q(x)$ even the map $T_q^{\mathbb{R}}$ (and hence also T_q) has values in $H^{4*}(X;R) := \prod_{i=0}^{\infty} H^{4i}(X;R)$.*

In Example 3.75 we mentioned three exponential characteristic classes $T_q(V)$ defined for complex vector bundles V, namely the Todd class $\mathrm{Td}(V)$ (for $q(x) = \frac{x}{1-e^{-x}}$), the $\widehat{A}$-class $\widehat{A}(V)$ (for $q(x) = \frac{x/2}{\sinh(x/2)}$), and the L-class $L(V)$ (for $q(x) = \frac{x}{\tanh x)}$. The last two power series are even, which implies that the corresponding exponential characteristic classes can be defined for real vector bundles.

For reference purposes, we state this explicitly.

Definition 3.79. Let $V \to X$ be a real vector bundle. Then the

$\widehat{\mathbf{A}}$-class $\widehat{A}(V) := T_q^{\mathbb{R}}(V) \in H^{4*}(X;\mathbb{Q})$ is the exponential characteristic class determined by the even power series $q(x) = \frac{x/2}{\sinh(x/2)} \in \mathbb{Q}[[x]]$, and

L-class $L(V) := T_q^{\mathbb{R}}(V) \in H^{4*}(X;\mathbb{Q})$ is the exponential characteristic class determined by the even power series $q(x) = \frac{x}{\tanh x} \in \mathbb{Q}[[x]]$.

In particular, now we can conclude:

Proposition 3.80. *Let $\pi\colon V \to X$ be real vector bundle of dimension $n = 2k$ equipped with a spin structure, and let $U^K(V) \in K_c(V)$ be the K-theory orientation. Then*

$$\mathrm{ch}(U^K(V)) = U^H(V) \ \cup \ (-1)^k \pi^*(\widehat{A}(V)^{-1}).$$

The Thom isomorphism in K-theory (resp. cohomology) is given by multiplication by the K-theory orientation U^K (resp. cohomology orientation U^H). Hence the interplay between these orientations via the Chern character expressed by the previous result allows us to express how the Chern character interacts with the Thom isomorphisms.

Corollary 3.81. *For $\pi\colon V \to X$ as above, the following diagram is commutative.*

$$\begin{tikzcd}
K(X) \arrow[rr, "U^K(V) \,\cup\, \pi^*(\ )"] \arrow[d, "\mathrm{ch}"'] & & K_c(V) \arrow[d, "\mathrm{ch}"] \\
H^*(X;Q) \arrow[r, "(-1)^k\widehat{A}(V)^{-1} \,\cup\, -"] & H^*(X;Q) \arrow[r, "U^H(V) \,\cup\, \pi^*(\ )"] & H_c^*(V;\mathbb{Q})
\end{tikzcd}$$

Proof. Let $E \to X$ be a complex vector bundle and let $[E] \in K(X)$ the element it represents in $K(X)$. Then

$$\begin{aligned}
\mathrm{ch}(U^K(V) \cup \pi^*([E])) &= \mathrm{ch}(U^K(V)) \cup \pi^*(\mathrm{ch}(E)) \\
&= U^H(V) \cup \pi^*((-1)^k \widehat{A}(V)^{-1}) \cup \pi^*(\mathrm{ch}(E)) \\
&= U^H(V) \cup \pi^*((-1)^k \widehat{A}(V)^{-1} \cup \mathrm{ch}(E)),
\end{aligned}$$

where the second equation is a consequence of the proposition. It is evident that chasing $[E] \in K(X)$ along the other path in the diagram (first down to $H^*(X;\mathbb{Q})$ and then horizontally to $H^*(V;\mathbb{Q})$) we obtain the same element. $\square$

The rest of this section is devoted to the proof of Proposition 3.78. It is easy to see that the requirement that $q(x)$ is an even power series is a necessary condition for T_q to factor in the form (3.77). We prove this now. Then we introduce characteristic classes for real vector bundles known a *Pontryagin classes* and use them to construct $T_q^{\mathbb{R}}$.

Let $\gamma \to \mathbb{CP}^\infty$ be the tautological complex line bundle and $\bar{\gamma} \to \mathbb{CP}^\infty$ its complex conjugate bundle. These two bundles are equal as *real* vector bundles, and hence assuming that T_q factors through $T_q^{\mathbb{R}}$, then $T_q(L) = T_q^{\mathbb{R}}(L_{\mathbb{R}}) = T_q^{\mathbb{R}}(\bar{L}_R) = T_q(\bar{L})$. But by construction of T_q (see Definition/Construction 3.74), the exponential characteristic class T_q for the line bundles γ, $\bar{\gamma}$ is given by evaluating q on their first Chern classes. Let $x = c_1(\gamma)$; then $c_1(\bar{\gamma}) = -x$ and hence

$$T_q(\gamma) = q(x) \in H^*(\mathbb{CP}^\infty;\mathbb{Q}) = \mathbb{Q}[[x]] \qquad \text{and} \qquad T_q(\bar{\gamma}) = q(-x).$$

It follows that $q(-x) = q(x)$, i.e., the power series $q(x)$ is even.

The basic idea for obtaining characteristic classes for a real vector bundle $V \to X$ is to take the Chern classes of the *complexfication* $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V. This is the complex vector bundle whose fiber $(V_{\mathbb{C}})_x$ over a point $x \in X$ is the complexification $(V_x)_{\mathbb{C}} := (V_x) \otimes_{\mathbb{R}} \mathbb{C}$ of the fiber V_x (which is a real vector space). We observe that the complex conjugate bundle $\overline{V_{\mathbb{C}}}$ is isomorphic to $V_{\mathbb{C}}$; the isomorphism

$$V_{\mathbb{C}} \longrightarrow \overline{V_{\mathbb{C}}} \qquad \text{is given by} \qquad v \otimes_{\mathbb{R}} z \mapsto v \otimes_{\mathbb{R}} \bar{z}.$$

Lemma 3.82. *Let $W \to X$ be a complex vector bundle. Then the k-th Chern class of the conjugate bundle $\bar{W}$ is given by $c_k(\bar{W}) = (-1)^k c_k(W) \in H^{2k}(X; \mathbb{Z})$. In particular, if $W = V_{\mathbb{C}}$, then $2c_k(W) = 0 \in H^{2k}(X; \mathbb{Z})$ for k odd.*

Proof. If W is a complex line bundle, then $\bar{W}$ is isomorphic to the dual bundle W^* (via the choice of a hermitian metric on W), and hence $c_1(\bar{W}) = c_1(W^*) = -c_1(W)$. If W is a direct sum of complex line bundles $W = L_1 \oplus \cdots \oplus L_n$ with $x_i := c_1(L_i)$, then the total Chern class of $\bar{W}$ is given by

$$c(\bar{W}) = c(\bar{L}_1 \oplus \cdots \oplus \bar{L}_n) = c(\bar{L}_1) \cdots c(\bar{L}_n) = (1 - x_1) \cdots (1 - x_n)$$

It follows that

$$c_k(\bar{W}) = \sigma_k(-x_1, \ldots, -x_n) = (-1)^k \sigma_k(x_1, \ldots, x_n) = (-1)^k c_k(W),$$

where the second equality follows from the fact that the elementary symmetric polynomial $\sigma_k(x_1, \ldots, x_n)$ is a homogeneous polynomial of degree k in the variables x_i. □

Definition 3.83. Let $V \to X$ be a real vector bundle of dimension n. Then the characteristic class $p_i(V) := (-1)^i c_{2i}(V_{\mathbb{C}}) \in H^{4i}(X; \mathbb{Z})$ is the *i-th Pontryagin class* of V. This implies $p_0(V) = 1 \in H^0(X; \mathbb{Z})$ and $p_i(V) = 0$ for $2i > n$ from the corresponding properties of the Chern classes. The total Pontryagin class $p(V)$ is defined by

$$p(V) = 1 + p_1(V) + p_2(V) + \cdots \in H^{4*}(X; \mathbb{Z}).$$

The sign $(-1)^i$ in the definition of $p_i(V)$ is motivated by a more pleasant formula for the first Pontryagin class $p_1(L_{\mathbb{R}}) \in H^4(X; \mathbb{Z})$ of a complex line bundle $L \to X$ in terms of its first Chern class $c_1(L) \in H^2(X; Z)$. Here $L_{\mathbb{R}}$ is the complex line bundle L considered as a real vector bundle of dimension 2. More precisely and generally, if $W \to X$ is a complex vector bundle of dimension n, we can forget the complex structure on the fibers W_x, and can regard W as real vector bundle of dimension $2n$, which we denote by $W_{\mathbb{R}}$.

Exercise 3.84. Show that the complexification $(W_{\mathbb{R}})_{\mathbb{C}}$ is isomorphic to $W \oplus \bar{W}$.

Let L be a complex line bundle and let $x := c_1(L)$ be its first Chern class. Then

$$c((L_{\mathbb{R}})_{\mathbb{C}}) = c(L \oplus \bar{L}) = c(L)c(\bar{L}) = (1+x)(1-x) = 1-x^2.$$

This implies $p_1(L_{\mathbb{R}}) = -c_1((L_{\mathbb{R}})_{\mathbb{C}}) = x^2$ and $p(L_{\mathbb{R}}) = 1 + x^2$.

Exercise 3.85. Let V_1, V_2 be real vector bundles over X. Show that

$$p(V_1 \oplus V_2) = p(V_1)p(V_2) \quad \text{modulo 2-torsion}$$

Let V be a real vector bundle of the form $V = (L_1)_{\mathbb{R}} \oplus \cdots \oplus (L_n)_{\mathbb{R}}$ where L_i are complex line bundles. Let $x_i := c_1(L_i)$. Then

$$\begin{aligned} p(V) =& p((L_1)_{\mathbb{R}} \oplus \cdots \oplus (L_n)_{\mathbb{R}}) \equiv p((L_1)_{\mathbb{R}}) \cdots p((L_n)_{\mathbb{R}}) \\ =& (1+x_1^2)\cdots(1+x_n^2) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n \in H^{4*}(X;\mathbb{Z}) \end{aligned}$$

where $\equiv$ stands for equality modulo 2-torsion, and $\sigma_i = \sigma_i(x_1^2, \ldots, x_n^2)$ is the i-th elementary symmetric function. It follows that for $V = (L_1)_{\mathbb{R}} \oplus \cdots \oplus (L_n)_{\mathbb{R}}$

$$p_i(V) \equiv \sigma_i(x_1^2, \ldots, x_n^2) \in H^{4i}(X;\mathbb{Z}). \tag{3.86}$$

Proof of Proposition 3.78. Let $q(x) = 1 + a_1 x + a_2 x^2 + \ldots$ be an even power series with coefficients in a ring R with $2 \in R^\times$. Our goal is to construct the characteristic class $T_q^{\mathbb{R}}(V) \in H^*(X;R)$ for real vector bundles V making the diagram 3.77 commutative. In other words, it should have the property that for any complex vector bundle W we have the equality

$$T_q(W) = T_q^{\mathbb{R}}(W_{\mathbb{R}}).$$

Let us use reverse engineering to find a formula for $T_q^{\mathbb{R}}(W_{\mathbb{R}})$ in terms of the Pontryagin classes of $W_{\mathbb{R}}$. First assume that $W = L$ is a line bundle with $c_1(L) = x$. Then

$$T_q(L) = q(c_1(L)) = q(x) = 1 + a_2 x^2 + a_4 x^4 + \cdots = r(x^2),$$

where $r(x) \in R[[x]]$ is the power series

$$r(x) = 1 + a_2 x + a_4 x^2 + \ldots.$$

This shows that for real vector bundles of the form $L_{\mathbb{R}}$ for a complex line bundle L, we can express $T_q(L)$ in terms of $x^2 = p_1(L_{\mathbb{R}})$, the first Pontryagin class of the real vector bundle $L_{\mathbb{R}}$. Hence for real vector bundles of the form $L_{\mathbb{R}}$, we define

$$T_q^{\mathbb{R}}(L_{\mathbb{R}}) := r(p_1(L_{\mathbb{R}})).$$

More generally, if W is the sum $W = L_1 \oplus \cdots \oplus L_n$ of complex line bundles L_i with $c_1(L_i) = x_i \in H^2(X;\mathbb{Z})$, then

$$T_q^{\mathbb{R}}(W_{\mathbb{R}}) = T_q^{\mathbb{R}}((L_1)_{\mathbb{R}}) \cdots T_q^{\mathbb{R}}((L_n)_{\mathbb{R}}) = r(x_1^2) \cdots r(x_n^2).$$

To define $T_q^{\mathbb{R}}(V)$ for a general real vector bundle V, we proceed as in the construction of the Chern character in Definition 3.61. The component $(r(x_1^2) \cdots r(x_n^2))_{4k} \in H^{4k}(X;R)$ of $r(x_1^2) \cdots r(x_n^2)$ of degree $4k$ is a polynomial of x_i^2 which is invariant under permuting the x_i^2. Hence by the fundamental theorem of symmetric functions,

$$(r(x_1^2) \cdots r(x_n^2))_{4k} = Q_k^n(\sigma_1, \ldots, \sigma_n),$$

where Q_k^n is a polynomial of the elementary symmetric functions $\sigma_i = \sigma_i(x_1^2, \ldots, x_n^2)$ of the variables x_i^2. By equation (3.86) and our assumption that 2 is invertible in the ring R, we have

$$\sigma_i(x_1^2, \ldots, x_n^2) = p_i(V) \in H^{4i}(X;R).$$

This suggests to define for any real vector bundle V

$$(T_q^{\mathbb{R}}(V))_{4k} := Q_k^n(p_1(V), \ldots, p_n(V)) \in H^{4k}(X;R),$$

which generalizes what we have done above for real vector bundles of the form

$$V = (L_1 \oplus \cdots \oplus L_n)_{\mathbb{R}}.$$

□

3.8 Relating the umkehr maps in K-theory and cohomology

Theorem 3.87. *Let X be a closed n-manifold, n even, with a spin structure. Let*

$$p_!\colon K(X) \to K(\mathrm{pt}) = \mathbb{Z}$$

be the umkehr map in K-theory, and let $E \to X$ be a complex vector bundle and $[E] \in K(X)$ the class represented by E. Then

$$p_!([E]) = \langle \widehat{A}(TX) \,\cup\, \mathrm{ch}(E), [X] \rangle,$$

where $\widehat{A}(TX) \in H^{4*}(X;\mathbb{Q})$ is the $\widehat{A}$-class of the tangent bundle TX (see 3.79), $\mathrm{ch}(E) \in H^{\mathrm{ev}}(X;\mathbb{Q})$ is the Chern character of E (see 3.61), and the right hand side above is the evaluation of the cup product of the classes on the fundamental class $[X] \in H_n(X;\mathbb{Z})$.

We recall that at the beginning of section 3 we stated two versions of the Index Theorem for twisted Dirac operators D_E on a closed spin manifold X of dimension $n = 2k$

K-theory version: $\operatorname{index}(D_E^+) = (-1)^k p_!([E]) \in K(\mathrm{pt}) = \mathbb{Z}$, where $p_!\colon K(X) \to K(\mathrm{pt})$ is the umkehr map in K-theory for spin manifolds;

Cohomology version: $\operatorname{index}(D_E^+) = \langle \widehat{A}(TX) \cup \operatorname{ch}(E), [X]\rangle$

The K-theory formulation is the slick, abstract formulation good for generalizations and the proof of the index theorem, the cohomological formulation is well-adapted to explicit calculations. So the tremendous amount of time we spend to talk about characteristic classes and the Chern character is well-spent, since it provides the necessary toolbox for many explicit calculations. In particular, the theorem above shows that the K-theory version of the Index Theorem is equivalent to its cohomological formulation.

Proof of Theorem 3.87. Let $X \hookrightarrow \mathbb{R}^{n+k}$ be the imbedding of X into Euclidean space with k even. Let $\pi V \to X$ be the k-dimensional normal bundle of this embedding, let $i\colon V \hookrightarrow \mathbb{R}^{n+k}$ be the embedding of the total space of V as a tubular neighborhood of $X \subset \mathbb{R}^{n+k}$. Consider the following commutative diagram.

$$
\begin{tikzcd}
K(X) \arrow[rr, "U^K(V)\,\cup\,\pi^*(\ )", "\cong"'] \arrow[d, "\mathrm{ch}"'] \arrow[rrrr, bend left, "p_!^K"] & & K_c(V) \arrow[r, "i_!"] \arrow[d, "\mathrm{ch}"] & K_c(\mathbb{R}^{n+k}) \arrow[d, "\mathrm{ch}"] & K(\mathrm{pt}) \arrow[l, "\cong"'] \arrow[d, "\mathrm{ch}"] \\
H^*(X;Q) \arrow[r, "(-1)^k\widehat{A}(V)^{-1}\,\cup"] & H^*(X;Q) \arrow[r, "U^H(V)\,\cup\,\pi^*(\ )", "\cong"'] \arrow[rrr, bend right, "p_!^H=\langle\ ,[X]\rangle"'] & H_c^{*+k}(V;\mathbb{Q}) \arrow[r, "i_!"] & H_c^{*+k}(\mathbb{R}^{n+k}) & H^{*-n}(\mathrm{pt};\mathbb{Q}) \arrow[l, "\cong"']
\end{tikzcd}
$$

The top row is the definition of the umkehr map $p_!^K$ in K-theory: the composition of the Thom isomorphism for the normal bundle $V \to X$, followed by the map $i_!$ induced on K-theory with compact support by the proper embedding $i\colon V \hookrightarrow \mathbb{R}^{n+k}$, followed by the (inverse of the) suspension isomorphism. Similarly, the composition of the last three maps in the bottom row is the umkehr map $p_!^H$ in ordinary cohomology, which alternatively can be described as $\langle\ ,[X]\rangle$, i.e, as evaluation on the fundamental class $[X] \in H_n(X;\mathbb{Z})$. The vertical maps are given by the Chern character and the commutativity of the right two squares is evident. The commutativity of the left part of this diagram is the statement of Corollary 3.81. $\square$

4 Calculations with characteristic classes

4.1 Hirzebruch's Signature Theorem

Let X be a closed oriented manifold of dimension $n = 2\ell$ with ℓ even. We recall from section 1.4.1 that

$$I_X \colon H^\ell_{\mathrm{dR}}(X) \times H^\ell_{\mathrm{dR}}(X) \longrightarrow \mathbb{R} \qquad \text{given by} \qquad ([\alpha],[\beta]) \mapsto \int_X \alpha \wedge \beta$$

is a non-degenerate symmetric bilinear form. The *signature of* X $\operatorname{sign}(X)$ is defined to be the signature of this symmetric bilinear form, i.e.,

$$\operatorname{sign}(X) = \dim H^\ell_+ - \dim H^\ell_-,$$

where $H^\ell_+, H^\ell_- \subset H^\ell_{\mathrm{dR}}$ are complementary subspace such that the form I_X is positive definite on H^ℓ_+ and negative definite on H^ℓ_-.

Theorem 4.1. (Hirzebruch's Signature Theorem). *Let X be a closed oriented manifold of dimension $n \equiv 0 \mod 4$. Then the signature of X is equal to $L(X)$ the L-genus of X, defined by*

$$L(X) = \langle L(TX), [X] \rangle.$$

Here $L(TX) = T^{\mathbb{R}}_q(TX) \in H^{4*}(X;\mathbb{Q})$ is the exponential characteristic class associated to the power series $q(x) = \frac{x}{\tanh x}$ (which makes sense for the real vector bundles like TX since $q(x)$ is an even power series; see Proposition 3.78 and Definition 3.79).

We would like to emphasize the different flavors of both sides of the equation

$$\operatorname{sign}(X) = L(X).$$

- The signature of X is an integer by construction, while $L(X) = \langle L(TX), [X] \rangle$ is a priori only a rational number, since the construction of the L-class $L(TX)$ involves denominators (coming from the fact that the coefficients of the power series $\frac{x}{\tanh x}$ are rational), and hence its evaluation on the fundamental class $[X] \in H_n(X;\mathbb{Z})$ can at first only expected to be in $\mathbb{Q}$.
- The signature of X depends *only* on the homotopy type of the n-manifold X (plus a choice of the generator of $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$). By contrast, the construction of $L(X)$ involves the tangent bundle of TX, i.e., it a priori depends on the smooth structure of X.

Proposition 4.2. *The signature* $\operatorname{sign}(X)$ *of oriented closed manifolds of dimension divisible by 4 has the following properties:*

1. $\operatorname{sign}(X \amalg Y) = \operatorname{sign}(X) + \operatorname{sign}(Y)$, *where* $X \amalg Y$ *is the disjoint union of the manifolds* X, Y *of the same dimension.*

2. $\operatorname{sign}(X \times Y) = \operatorname{sign}(X) \cdot \operatorname{sign}(Y)$.

3. *If* X *is the boundary of an oriented manifold* W, *then* $\operatorname{sign}(X) = 0$.

For the proof of these properties of the signature, it will be useful to have a good criterium for the vanishing of the signature of a non-degenerate symmetric bilinear form.

$$I \colon V \times V \longrightarrow \mathbb{R}$$

on a finite dimensional real vector space V.

Definition 4.3. Let $I \colon V \times V \longrightarrow \mathbb{R}$ be a non-degenerate symmetric bilinear form on a finite dimensional real vector space V.

1. A subspace $L \subset V$ is *isotropic* if $I(\ell, \ell') = 0$ for all $\ell, \ell' \in L$. Equivalently, if
$$L \subset L^{\perp} := \{v \in V \mid I(v, \ell) = 0 \text{ for all } \ell \in L\}.$$

2. A subspace $L \subset V$ is *Lagrangian* if $L = L^{\perp}$.

Proposition 4.4. *Let* $I \colon V \times V \to \mathbb{R}$ *and* $J \colon W \times W \to \mathbb{R}$ *be non-degenerate symmetric bilinear forms.*

1. $\operatorname{sign}(V \oplus W, I \oplus J) = \operatorname{sign}(V, I) + \operatorname{sign}(W, J)$.

2. $\operatorname{sign}(V \otimes W, I \otimes J) = \operatorname{sign}(V, I) \cdot \operatorname{sign}(W, J)$

3. *Let* $I \colon V \times V \to \mathbb{R}$ *be a non-degenerate symmetric bilinear form. If* $L \subset V$ *is Lagrangian, then* $\operatorname{sign}(I) = 0$.

The proof of this lemma is left to the reader.

Proof of Proposition 4.4. The property $\operatorname{sign}(X \amalg Y) = \operatorname{sign}(X) + \operatorname{sign}(Y)$ follows from Lemma 5.40(1).

To prove the multiplicative property $\operatorname{sign}(X \times Y) = \operatorname{sign}(X) \cdot \operatorname{sign}(Y)$, we need to analyze the cup product pairing on the middle dimensional cohomology of the product $X \times Y$. For $\dim X = 2k$ and $\dim Y = 2\ell$ with k, ℓ even, we decompose the middle dimensional cohomology $H := H^{k+\ell}(X \times Y)$ (working with coefficients in $\mathbb{R}$ throughout) as follows:

$$\begin{aligned} H^{k+\ell}(X \times Y) &\cong \bigoplus_{i+j=k+\ell} H^i(X) \otimes H^j(Y) \\ &= \underbrace{H^k(X) \otimes H^\ell(Y)}_{U} \oplus \underbrace{\bigoplus_{0 \le i < k} H^i(X) \otimes H^{k+\ell-i}(Y)}_{V} \oplus \underbrace{\bigoplus_{k < i \le 2k} H^i(X) \otimes H^{k+\ell-i}(Y)}_{W} \end{aligned}$$

Let $I_{X\times Y} = I\colon H\times H\to\mathbb{R}$ be the cup product pairing. We note that for $u\in U$ the pairing $I(u,v)$ vanishes if $v\in V$ or $v\in W$; if $u=u_1\otimes u_2$ with $u_1\in H^k(X)$, $u_2\in H^\ell(Y)$, and $v=v_1\otimes v_2$ with $v_1\in H^i(X)$, $v_2\in H^{k+\ell-i}(Y)$, then

$$I(u,v)=\langle u\cup v,[X\times Y]\rangle=\langle u_1\cup v_1,[X]\rangle\cdot\langle u_2\cup v_2,[Y]\rangle=0,$$

since the cohomology class $u_1\cup v_1\in H^*(X)$ is in degree $k+i<2k=\dim X$. For $v\in W$ the class $u_1\cup v_1$ is in degree $>2k$ and hence again $I(u,v)=0$. This shows that (H,I) splits as a direct sum

$$(H,I)=(U,I_{|U})\ \oplus\ (V\oplus W,I_{V\oplus W}),$$

and hence $\operatorname{sign}(H,I)=\operatorname{sign}(U,I_{|U})+\operatorname{sign}(V\oplus W,I_{V\oplus W})$. It should be emphasized that $(V\oplus W,I_{V\oplus W})$ does not split as a direct sum of V and W equipped with the restriction of the form I, since the pairing $I(v,w)$ for $v\in V$, $w\in W$ is typically non-zero. Rather, we note that for $v=v_1\otimes v_2\in V$ and $v'=v_1'\otimes v_2'\in V$ the pairing $I(v,v')$ vanishes (again since $v_1\cup v_1'\in H^*(X)$ are in degree <0). In fact, V is a Lagrangian subspace of $(V\oplus W,I_{V\oplus W})$, since if w belongs to $V^\perp\subset V\oplus W$, then Poincaré duality implies that the W-component of w must be trivial, i.e., $w\in V$. By part (3) of Lemma 5.40 this implies that $\operatorname{sign}(V\oplus W,I_{V\oplus W})=0$, and hence

$$\begin{aligned}\operatorname{sign}(X\times Y)=&\operatorname{sign}(H,I)=\operatorname{sign}(U,I_{|U})+\operatorname{sign}(V\oplus W,I_{V\oplus W})\\ =&\operatorname{sign}(U,I_{|U})=\operatorname{sign}(H^k(X)\otimes H^\ell(Y),I_X\otimes I_Y)\\ =&\operatorname{sign}(H^k(X),I_X)\cdot\operatorname{sign}(H^\ell(Y),I_Y)=\operatorname{sign}(X)\cdot\operatorname{sign}(Y)\end{aligned}$$

Here the second to last equation follows from part (2) of Lemma 5.40.

To prove the third part, suppose the $2k$-manifold X is the boundary of an oriented $2k+1$-manifold W. Consider the following commutative diagram, known as *Poincaré-Lefschetz duality* whose rows are the portion of the long exact (co)homology sequence of the pair (W,X), and whose top vertical isomorphisms are the Poincaré duality isomorphisms and whose bottom vertical isomorphisms come from the universal coefficient theorem.

$$\begin{tikzcd}
H^k(W;\mathbb{R}) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^k(X;\mathbb{R}) \arrow[r, "\delta"] \arrow[d, "\cong"'] & H^{k+1}(W,X;\mathbb{R}) \arrow[d, "\cong"'] \\
H_{k+1}(W,X;\mathbb{R}) \arrow[r, "\partial"] \arrow[d, "\cong"'] & H_k(X;\mathbb{R}) \arrow[r, "i_*"] \arrow[d, "\cong"'] & H_k(W;\mathbb{R}) \arrow[d, "\cong"'] \\
\operatorname{Hom}(H^{k+1}(W,X),\mathbb{R}) & \operatorname{Hom}(H^k(X),\mathbb{R}) & \operatorname{Hom}(H^k(W),\mathbb{R})
\end{tikzcd}$$

Unwinding the definitions, we can check that the compositions of the vertical isomorphisms are given by the appropriate non-degenerate cup product pairings:

$$\begin{aligned}&I_X\colon H^k(X;\mathbb{R})\times H^k(X;\mathbb{R})\longrightarrow\mathbb{R} &&(\alpha,\alpha')\mapsto\langle\alpha\cup\alpha',[X]\rangle\\ &I_W\colon H^k(W;\mathbb{R})\times H^{k+1}(W,X;\mathbb{R})\longrightarrow\mathbb{R} &&(\beta,\gamma)\mapsto\langle\beta\cup\gamma,[W,\partial W]\rangle\end{aligned}$$

Moreover, these pairings are related by

$$\langle i^*\beta \cup \alpha, [X]\rangle = \langle \beta \cup \delta(\alpha), [W, \partial W]\rangle \qquad \alpha \in H^k(X;\mathbb{R}),\ \beta \in H^k(W;\mathbb{R}).$$

We claim that $L := \operatorname{im}(i^*)$ is a Lagrangian subspace of $H^k(X;\mathbb{R})$ which by part (3) of Lemma 5.40 then implies $\operatorname{sign}(X) = 0$.

So let $i^*\beta, i^*\beta' \in L$. Then

$$I_X(i^*\beta, \imath^*\beta') = \langle \imath^*\beta \cup i^*\beta', [X]\rangle = \langle \beta \cup \delta(i^*\beta'), [W, \partial W]\rangle = 0$$

shows that L is isotropic. To show that L is Lagrangian, let $\alpha \in L^\perp$, i.e.,

$$0 = I_X(i^*\beta, \alpha) = \langle i^*\beta \cup \alpha, [W, \partial W]\rangle = \langle \beta \cup \delta(\alpha), [W, \partial W]\rangle \qquad \text{for all } \beta \in H^k(W;\mathbb{R}).$$

By the vertical isomorphisms on the right side, this implies $\delta(\alpha) = 0$ which by the exactness of the top sequence implies that α is in the image of i^*, i.e., it belongs to L. □

Proposition 4.5. *Let $q(x) = 1 + a_2x^2 + a_4x^4 + \cdots \in R[[x]]$ be an even power series with coefficients in a ring R with $2 \in R^\times$. For a closed oriented manifold $[X]$, let $T_q(X) \in R$ be the T_q-*genus *of X, defined by*

$$T_q(X) := \langle T_q(TX), [X]\rangle \in R.$$

The T_q-genus has the following properties:

1. $T_q(X \amalg Y) = T_q(X) + T_q(Y)$
2. $T_q(X \times Y) = T_q(X) \cdot T_q(Y)$
3. *If $X = \partial W$ for a compact oriented manifold W, then $T_q(X) = 0$.*

Exercise 4.6. Prove this proposition.

Definition 4.7. Let X, Y be closed oriented n-manifolds. They are *bordant* if there is an oriented $(n+1)$-manifold W whose boundary ∂W is diffeomorphism to the disjoint union $X \amalg \bar{Y}$ as oriented manifold. Here the orientation on W induces an orientation on ∂W, and $\bar{Y}$ denotes the manifold Y, but equipped with the opposite orientation. Let Ω_n be the set of bordism classes of closed oriented n-manifolds.

Disjoint union of n-manifolds gives Ω_n the structure of an abelian group, called the *oriented bordism group* of dimension n. The unit of Ω_n is represented by the empty set, and the inverse of the bordism class $[X] \in \Omega_n$ of a closed oriented n-manifold X is given by $[\bar{X}]$.

Older source often use "cobordant" rather than "bordant", and refer to Ω_n as "cobordism group". Since bordism groups lead in a natural way to a *homology* rather than a *cohomology* theory, the "bordism" terminology has become more prevalent.

The cartesian product of manifolds gives a bilinear map

$$\Omega_m \times \Omega_n \to \Omega_{m+n} \qquad \text{defined by} \qquad ([X],[Y]) \mapsto [X \times Y].$$

In other words, the cartesian product gives

$$\Omega_* := \bigoplus_{n=0}^{\infty} \Omega_n$$

the structure of a $\mathbb{Z}$-graded ring. Then Proposition 4.5 implies.

Corollary 4.8. *The map $L\colon \Omega_* \to \mathbb{Q}$ defined by $[X] \mapsto L(X)$ is a ring homomorphism.*

Similarly, we want to say that proposition 4.4 implies that assigning to a closed oriented n-manifold X its signature $\operatorname{sign}(X) \in \mathbb{Z}$ gives a ring homomorphism from Ω_* to $\mathbb{Z}$. This is not clear at this point, since so far we have defined the signature of a closed oriented manifold X only for manifolds of dimension $n \equiv 0 \mod 4$. Now we extend this definition to oriented closed n-manifolds for any n by declaring $\operatorname{sign}(X) = 0$ if $n \not\equiv 0 \mod 4$.

Corollary 4.9. *The map $\operatorname{sign}\colon \Omega_* \to \mathbb{Z}$ defined by $[X] \mapsto \operatorname{sign}(X)$ is a ring homomorphism.*

It should be pointed out that this result is *not* a corollary of proposition 4.4 alone. That proposition does not exclude the possibility that there are manifolds X, Y of dimension $\not\equiv 0 \mod 4$ such that $X \times Y$ has dimension $\equiv 0 \mod 4$ and $\operatorname{sign}(X \times Y) \neq 0$. It turns out that this is not possible as we will argue below.

The proof of Hirzebruch's signature theorem is based on the calculation of $\Omega_* \otimes \mathbb{Q}$ due to Thom.

Theorem 4.10. (Thom). *The rational oriented bordism ring $\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to the polynomial algebra $\mathbb{Q}[[\mathbb{CP}^2],[\mathbb{CP}^4],\dots,[\mathbb{CP}^{2n}],\dots]$ generated by the bordism classes $[\mathbb{CP}^{2n}] \in \Omega_{2n}$ of even dimensional complex projective spaces.*

Remark 4.11. The proof to this important result is based on the Pontryagin-Thom isomorphism

$$\Omega_n \cong \pi_n \,\mathrm{MSO}$$

which expresses the oriented bordism groups as the homotopy groups of the *Thom spectrum* MSO (which is a special case of a statement expressing very general bordism groups as homotopy groups of associated Thom spectra). The k-th space MSO_k in the spectrum MSO is the Thom space of the universal oriented vector bundle $V^k \to \mathrm{BSO}(k)$ of dimension k; the universality of V^k means that the map

$$[X, \mathrm{BSO}_k] \longrightarrow \{\text{vector bundles over } X \text{ of dimension } k\}/\text{isomorphism}$$

which sends a map $f\colon X \to \mathrm{BSO}_k$ to the pullback bundle $f^*V^k \to X$, is a bijection. In more elementary terms, the Pontryagin-Thom construction yields an isomorphism

$$\Omega_n \cong \lim_k \pi_{n+k}\, \mathrm{BSO}(k)^{V^k},$$

where $\mathrm{BSO}(k)^{V^k}$ is the Thom space of $V^k \to \mathrm{BSO}(k)$.

Then the calculation of $\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q}$ then boils down to calculating the rational homology groups of $\mathrm{BSO} = \bigcup_k \mathrm{BSO}(k)$ thanks to the isomorphisms

$$\Omega_* \otimes \mathbb{Q} \cong \pi_*\, \mathrm{MSO} \otimes \mathbb{Q} \cong H_*(\mathrm{MSO}; \mathbb{Q}) \cong H_*(\mathrm{BSO}; \mathbb{Q})$$

the last which is the Thom isomorphism.

We note that Theorem 4.10 in particular implies that Ω_n is a torsion group for $n \not\equiv 0 \mod 4$, since all the generators of $\Omega_* \otimes \mathbb{Q}$ have degree $\equiv 0 \mod 4$. This implies that $\mathrm{sign}\colon \Omega_* \to \mathbb{Z}$ is in fact a ring homomorphism as claimed in Corollary 4.9.

To show that $\mathrm{sign}(X) = L(X)$ for all oriented closed manifolds, we will now view sign and L as ring homomorphisms from the oriented bordism ring $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$, and hence it suffices to show that $\mathrm{sign}(X) = L(X)$ for $X = \mathbb{CP}^{2n}$, the generators of the ring $\Omega_* \otimes \mathbb{Q}$. Hence the Hirzebruch signature theorem follows from the following explicit calculations of the signature and the L-genus for complex projective spaces.

Lemma 4.12. $\mathrm{sign}(\mathbb{CP}^n) = 1$ *for n even.*

Proof. We recall that the cohomology ring of $\mathbb{CP}^n$ is given by $\mathbb{H}^*(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n-1})$, where $x \in H^2(\mathbb{CP}^n; \mathbb{Z})$ and (x^{n-1}) is the ideal generated by x^{n-1}. As generator x we will choose the first Chern class of γ^*, the dual of the tautological line bundle. We claim that $\langle x^n, [\mathbb{CP}^n] \rangle = 1$, where $[\mathbb{CP}^n] \in H_n(\mathbb{CP}^n; \mathbb{Z})$ is the fundamental class of $\mathbb{CP}^n$ determined by the orientation of $\mathbb{CP}^n$ as a complex manifold.

To prove this, we use the fact that for a complex manifold X of complex dimension n its Euler characteristic $\chi(X)$ is given by

$$\chi(X) = \langle c_n(TX), [X] \rangle.$$

We recall from (3.52) that

$$T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong \underbrace{\gamma^* \oplus \cdots \oplus \gamma^*}_{n+1}, \tag{4.13}$$

which implies

$$c(T\mathbb{CP}^n) = (1+x)^{n+1} \qquad \text{and hence} \qquad c_n(TX) = (n+1)x^n.$$

On the other hand, $\chi(\mathbb{CP}^n) = n+1$ and hence $\langle x^n, [X]\rangle = 1$ as claimed.

For $n = 2\ell$, the middle dimensional cohomology group $H^n(\mathbb{CP}^n)$ is generated by x^ℓ, and since $\langle x^\ell \cup x^\ell, [\mathbb{CP}^n]\rangle = 1$, it follows that the cup product pairing on $H^n(\mathbb{CP}^n)$ is positive definite, and hence $\operatorname{sign}(\mathbb{CP}^n) = 1$. $\square$

Lemma 4.14. $L(\mathbb{CP}^n) = \begin{cases} 1 & \textit{for } n \textit{ even} \\ 0 & \textit{for } n \textit{ odd} \end{cases}$

Proof. To compute $L(\mathbb{CP}^n) = \langle L(T\mathbb{CP}^n), [\mathbb{CP}^n]\rangle \in \mathbb{Q}$, we recall that $L = T_q^{\mathbb{R}}$ is the exponential characteristic class for real vector bundles associated to the power series $q(x) = \frac{x}{\tanh x}$. Due to (3.52) we have

$$L(T\mathbb{CP}^n) = T_q(\bigoplus_{n+1} \gamma^*) = T_q(\gamma^*)^{n+1} = q(x)^{n+1} = \left(\frac{x}{\tanh x}\right)^{n+1}$$

Then

$$\begin{aligned} L(\mathbb{CP}^n) =& \langle \left(\frac{x}{\tanh x}\right)^{n+1}, [\mathbb{CP}^n]\rangle \\ =& \text{coefficient of } x^n \text{ in } \left(\frac{x}{\tanh x}\right)^{n+1} \\ =& \text{residue at 0 of } \left(\frac{1}{\tanh x}\right)^{n+1}. \end{aligned}$$

By the residue theorem

$$L(\mathbb{CP}^n) = \frac{1}{2\pi i}\oint \left(\frac{1}{\tanh x}\right)^{n+1}$$

where the contour integral is taken over a small circle with center the origin of the complex x-line. Substituting $z = \tanh x$ and using $dz = (1-\tanh^2 x)dx = (1-z^2)dx$, this integral can be simplified to

$$\begin{aligned} \frac{1}{2\pi i}\oint \frac{dz}{(1-z^2)z^{n+1}} &= \frac{1}{2\pi i}\oint \frac{1+z^2+z^4+\dots}{z^{n+1}}\, dz \\ &= \text{coefficient of } z^n \text{ in } 1+z^2+z^4+\dots \\ &= \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \end{aligned}$$

$\square$

4.2 The topology of the Kummer surface

The *Kummer surface* is defined by

$$K = \{[z_0, z_2, z_2, z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}.$$

It is easy to check that this a smooth complex hypersurface of the complex projective space $\mathbb{CP}^3$. Hence it is a closed complex manifold of complex dimension 2 and real dimension 4. Our goal here is to completely determine the (co)homology groups of K and in particular to compute the intersection form of K. Many of the techniques we use can be applied to much more general situations, and so where appropriate, we make more general statements than necessary to analyze K.

Definition 4.15. Let $f(z_0, \ldots, z_{n+1})$ be a homogeneous polynomial of degree d. Then

$$X(f) := \{[z_0, \ldots, z_{n+k}] \in \mathbb{CP}^{n+1} \mid f(z_0, \ldots, z_{n+1}) = 0\}$$

is *hyperplane of degree* d. If $X(f)$ is smooth, then it is a closed complex manifold of complex dimension n.

Theorem 4.16. (Lefschetz hyperplane theorem). *For a smooth hyperplane X the embedding $i\colon X \hookrightarrow \mathbb{C}^{n+1}$ is an n-equivalence, i.e.,*

$$i_*\colon \pi_j X \longrightarrow \pi_j \mathbb{CP}^{n+1} \qquad \textit{is an isomorphism for } j < n \textit{ and surjective for } j + n.$$

There are different approaches to prove this. Appealing to topologists might be the proof by Bott using Morse theory.

Now let us explore what the Lefschetz hyperplane theorem implies for the homology groups of the hyperplane X. First off, the fact that $i\colon X \hookrightarrow \mathbb{CP}^{n+1}$ is an n-equivalence implies that

$$i_*\colon H_j(X;\mathbb{Z}) \longrightarrow H_j(\mathbb{CP}^{n+1};\mathbb{Z}) \qquad \text{is an isomorphism for } j < n \text{ and surjective for } j + n.$$

It follows that

$$H_j(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \le j < n,\ j \text{ even} \\ 0 & 0 \le j < n,\ j \text{ odd} \end{cases}.$$

Next, we determine the homology groups $H_j(X;\mathbb{Z})$ for $j > n$ by using Poincaré duality and the universal coefficient theorem to obtain the isomorphisms

$$H_j(X;\mathbb{Z}) \cong H^{2n-j}(X;\mathbb{Z}) \cong \mathrm{Hom}(H_{2n-j}(X;\mathbb{Z}),\mathbb{Z}).$$

We note that the term $\mathrm{Ext}(H_{2n-j-1}(X;\mathbb{Z}),\mathbb{Z})$ in the universal coefficient theorem vanishes since the homology groups of X below the middle dimension n are either $\mathbb{Z}$ or 0.

Finally, we look at the middle dimensional homology group $H_n(X;\mathbb{Z})$:

$$H_n(X;\mathbb{Z}) \cong H^n(X;\mathbb{Z}) \cong \mathrm{Hom}(H_n(X;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),\mathbb{Z}).$$

Here the first isomorphism is Poincaré duality and the second isomorphism comes from the universal coefficient theorem. The latter is usually formulated as a short exact sequence, which splits (not naturally), thus leading to the isomorphism above. Since $H_{n-1}(X;\mathbb{Z})$ is $\mathbb{Z}$ or 0 by our considerations above, the Ext-term vanishes. So we conclude

$$H_n(X;\mathbb{Z}) \cong \mathrm{Hom}(H_n(X;\mathbb{Z}),\mathbb{Z}).$$

In particular, $H_n(X;\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module, i.e., isomorphic to a direct sum of copies of $\mathbb{Z}$ (but the Lefschetz theorem does give no information about the rank of $H_n(X;\mathbb{Z})$).

Summarizing our discussion of the homology groups of the hyperplane $X \hookrightarrow \mathbb{CP}^{n+1}$ we have:

$$H_j(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \leq j \leq 2n,\ j \text{ even},\ j \neq n \\ \mathbb{Z}^r & j = n \\ 0 & \text{otherwise} \end{cases}. \tag{4.17}$$

Specializing to the Kummer surface $K \subset \mathbb{CP}^3$ we obtain

$$H_j(K;\mathbb{Z}) = \begin{cases} \mathbb{Z} & j = 0,4 \\ \mathbb{Z}^r & j = 2 \\ 0 & \text{otherwise} \end{cases}. \tag{4.18}$$

Besides the Lefschetz hyperplane theorem we will need the following fact to evaluate cohomology classes on the fundamental class $[X] \in H_{2n}(X;\mathbb{Z})$. Let $\gamma \to \mathbb{CP}^{n+k}$ be the tautological line bundle and γ^* its dual. Let $x := c_1(\gamma^*) \in H^2(\mathbb{CP}^{n+k};\mathbb{Z})$ the first Chern class of γ^* and $y := i^*x \in H^2(X;\mathbb{Z})$ its pullback to the hyperplane X. Then

$$\langle y^n, [X] \rangle = d, \tag{4.19}$$

where d is the degree of the hyperplane $X = X(f)$, i.e., the degree of the homogeneous polynomial f whose zero locus is X.

Question: What is the rank of the middle dimensional homology group $H_n(X;\mathbb{Z})$?

The idea is to look at the Euler characteristic of X. By the calculations of the homology groups of X above

$$\chi(X) = \sum_{j=0}^{2n} \mathrm{rk} H_j(X;\mathbb{Z}) = n + \mathrm{rk} H_n(X;\mathbb{Z}).$$

On the other hand, the Euler characteristic can be calculated in terms of the Euler class $\chi^H(TX) \in H^{2n}(X;\mathbb{Z})$, which for the complex vector bundle TX agrees with the the n-th Chern class $c_n(TX)$. Hence

$$\chi(X) = \langle \chi^H(TX), [X] \rangle = \langle c_n(TX), [X] \rangle.$$

In order to calculate $c_n(TX)$ we need to understand the tangent bundle TX. Since X is a submanifold of $\mathbb{CP}^{n+1}$, we have the vector bundle isomorphism

$$TX \oplus \nu(X \hookrightarrow \mathbb{CP}^{n+1}) \cong T\mathbb{CP}^{n+1}_{|X},$$

where $\nu(X \hookrightarrow \mathbb{CP}^{n+1})$ is the normal bundle of X in $\mathbb{CP}^{n+1}$. Previously we have determined the tangent bundle of $\mathbb{CP}^{n+1}$ (in terms of γ^*, the dual of the tautological line bundle). So our next goal is to determine the normal bundle $\nu(X \hookrightarrow \mathbb{CP}^{n+1})$.

We note that the homogenous polynomial f is *not a function* on $\mathbb{CP}^{n+1}$, but it can be interpreted as a section s_f of $(\gamma^*)^{\otimes d}$:

$$s_f \colon \mathbb{CP}^{n+1} \to (\gamma^*)^{\otimes d}.$$

If $L \in \mathbb{CP}^{n+1}$, i.e., $L \subset \mathbb{C}^{n+2}$ is a 1-dimensional complex subspace, then

$$s_f(L) \in (\gamma^*_L)^{\otimes d} = (L^*)^{\otimes d} = \operatorname{Hom}(L^{\otimes d}, \mathbb{C})$$

is given by

$$s_f(L) \colon L^{\otimes d} \longrightarrow \mathbb{C} \qquad \underbrace{z \otimes \cdots \otimes z}_{d} \mapsto f(z) \qquad \text{for } z = (z_0, \dots, z_{n+1}) \in L \subset \mathbb{C}^{n+2}$$

We note that the assumption that $f(z)$ is homogeneous of degree d implies that the map $z \otimes \cdots \otimes z \mapsto f(z)$ is a complex linear map, i.e., an element of $(\gamma^*_L)^{\otimes d}$.

This means that the hypersurface $X(f)$ can be interpreted as the vanishing locus of the section $s_f \colon \mathbb{CP}^{n+1} \to (\gamma^*)^{\otimes d}$. The assumption of f implying that $X(f)$ is smooth translates into requiring s_f to be transversal to the zero section. This implies the vector bundle isomorphism

$$\nu(X(f) \hookrightarrow \mathbb{CP}^{n+1}) \cong (\gamma^*)^{\otimes d}_{X(f)},$$

leading to

$$TX(f) \oplus i^*(\gamma^*)^{\otimes d} \cong i^* T\mathbb{CP}^{n+1}.$$

We recall from (3.52) that

$$T\mathbb{CP}^{n+1} \oplus \underline{\mathbb{C}} \cong \underbrace{\gamma^* \oplus \cdots \oplus \gamma^*}_{n+2}$$

Putting these isomorphisms together, we have

$$TX(f) \oplus i^*(\gamma^*)^{\otimes d} \oplus \underline{\mathbb{C}} \cong i^*(\underbrace{\gamma^* \oplus \cdots \oplus \gamma^*}_{n+2}). \tag{4.20}$$

Specializing to the Kummer surface with of complex dimension $n = 2$ and degree $d = 4$ we obtain

$$TK \oplus i^*(\gamma^*)^{\otimes 4} \oplus \underline{\mathbb{C}} \cong i^*(\underbrace{\gamma^* \oplus \cdots \oplus \gamma^*}_{4}). \tag{4.21}$$

This bundle equation allows us to calculate the total Chern class of the tangent bundle TK in terms of the total Chern class of the line $i^*\gamma^*$ with $c_1(i^*\gamma^*) = y \in H^2(K;\mathbb{Z})$. Using the exponential property of the total Chern class we obtain

$$\begin{aligned} c(TK) =& c(i^*\gamma^*)^4 c(i^*(\gamma^*)^{\otimes 4})^{-1} \\ =& (1+y)^4(1+4y)^{-1} \\ =& (1+4y+6y^2)(1-4y+16y^2) \\ =& 1+6y^2 \end{aligned} \tag{4.22}$$

In particular, $c_2(TK) = 6$ and hence by (4.19)

$$\chi(K) = \langle c_2(TK), [K]\rangle = \langle 6y^2, [K]\rangle = 6\langle y^2, [K]\rangle = 24.$$

From the calculation (4.18) of the homology groups of K it follows that

$$\chi(K) = \mathrm{rk} H_0(K) + \mathrm{rk} H_2(K) + \mathrm{rk} H_4(K) = 2 + \mathrm{rk} H_2(K),$$

and hence $\mathrm{rk} H_2(K) = 22$.

Question: What is the signature of K?

By the signature theorem, it suffices to calculate the L-class $L(TK)$. We recall that $L(V)$ is the exponential characteristic class $T_q^{\mathbb{R}}(V)$ for a real vector bundles associated to the even power series

$$q(x) = \frac{x}{\tanh x} = 1 + \frac{1}{3}x^2 + \dots$$

In particular from the bundle isomorphism (4.21),

$$\begin{aligned} L(TK) =& L(i^*\gamma^*)^4 L(i^*(\gamma^*)^{\otimes 4}) = (1+\frac{1}{3}y^2)^4(1+\frac{1}{3}(4y)^2)^{-1} \\ =& (1+\frac{4}{3}y^2)(1+\frac{16}{3}y^2)^{-1} = (1+\frac{4}{3}y^2)(1-\frac{16}{3}y^2) \\ =& 1-\frac{12}{3}y^3 = 1-4y^2 \end{aligned}$$

Hence
$$\operatorname{sign}(K) = \langle -4y^2, [K]\rangle = -4\langle y^2, [K]\rangle = -16$$

Question: What is the intersection form I_K on $H^2(K;\mathbb{Z}) \cong \mathbb{Z}^{22}$?

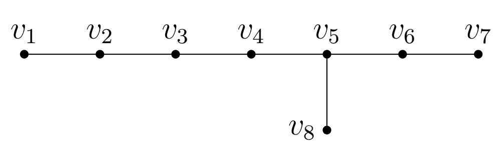
Definition 4.23. A *lattice* is a free $\mathbb{Z}$-module Λ equipped with a symmetric bilinear form
$$I\colon \Lambda \times \Lambda \to \mathbb{Z}.$$

The lattice (Λ, I) is

- *unimodular* if the map
$$\Lambda \longrightarrow \operatorname{Hom}(\Lambda, \mathbb{Z}) \qquad \text{given by} \qquad \lambda \mapsto (\lambda' \mapsto I(\lambda, \lambda'))$$
is an isomorphism. Equivalently, (Λ, I) is unimodular if and only if for a basis $\{e_i\}_{1\le i\le n}$ of Λ then the associated $n \times n$ matrix $I(e_i, e_j)$ has determinant ± 1.
- *even* if $I(\lambda, \lambda) \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$; otherwise it is *odd*.
- *positive definite* (resp. *negative definite*) if $I(\lambda, \lambda) > 0$ for all $0 \neq \lambda \in \Lambda$ (resp. $I(\lambda, \lambda) < 0$ for all $0 \neq \lambda \in \Lambda$). If (Λ, I) is not (positive or negative) definite, then it is *indefinite*.

Example 4.24. (Examples of unimodular lattices).

1. The integers $\mathbb{Z}$ with the form $I(m, n) := m \cdot n$. This is unimodular, positive definite and odd.
2. The hyperbolic form $H = (\mathbb{Z}^2, I)$ with matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This form is unimodular and even (since its diagonal entries are even). It is indefinite since $I(e_1 + e_2, e_1 + e_2) = 2$ and $I(e_1 - e_2, e_1 - e_2) = -2$.
3. The form $E_8 = (\mathbb{Z}^8, I)$ with matrix
$$I(e_i, e_j) = \begin{cases} 2 & \text{for } i = j \\ \#\{\text{edges connecting vertices } v_i,\, v_j \text{ in the graph } E_8\} & \text{for } i \neq j \end{cases}$$
This form is unimodular, even and positive definite.



The graph E_8

Theorem 4.25. *Let (Λ, I) be an indefinite unimodular lattice of rank $m+n$ and signature $m-n$ (i.e., on the real vector space $\Lambda \otimes \mathbb{R}$ the associated $\mathbb{R}$-valued symmetric bilinear form is positive (resp. negative) definite on a subspace of dimension m (resp. n).*

1. *If (Λ, I) is odd, then $(\Lambda, I) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m} \oplus \underbrace{-\mathbb{Z} \oplus \cdots \oplus -\mathbb{Z}}_{n}$*

2. *If (Λ, I) is even, then $m-n = \pm 8k$ for $k \in \mathbb{N}_0$, and*

$$(\Lambda, I) = \begin{cases} \underbrace{E_8 \oplus \cdots \oplus E_8}_{k} & \text{for } m-n = 8k,\ k \in \mathbb{N}_0 \\ \underbrace{-E_8 \oplus \cdots \oplus -E_8}_{k} & \text{for } m-n = -8k,\ k \in \mathbb{N}_0 \end{cases}$$

In particular, any indefinite unimodular lattice is determined by its rank, signature and type (i.e., even or odd).

For the lattice $(H^2(K;\mathbb{Z}), I_K)$ we have already calculated its rank to be 22 and its signature to be -16. Below we will argue that I_K is even. Using the classification of indefinite unimodular lattices we then obtain the main result of this section.

Theorem 4.26. *The intersection form I_K of the Kummer surface is isomorphic to $-E_8 \oplus -E_8 \oplus 3H$.*

Proposition 4.27. *Let X be a closed oriented manifold of dimension $2k$. Let $U, V \subset X$ be oriented submanifolds of dimension k, and let $u, v \in H^k(X)$ be the cohomology classes Poincaré dual to the homology classes $i_*^U[U], i_*^V[V] \in H_k(X)$. Here $[U] \in H_k(U)$ is the fundamental class of U, and $i_*^U \colon H_k(U) \to H_k(X)$ is the homomorphism induced by the inclusion map $i_U \colon U \hookrightarrow X$; and similarly for V.*

1. *If U and V intersect transversally, then $I_X(u, v)$ is the number of intersection points, counted with signs.*

2. *If $\nu \to U$ is the normal bundle of $U \subset X$, and $s \colon U \to \nu$ is a section transverse to the zero section, then $I_X(u, u)$ is the number of zeroes of s, counted with signs.*

3. *$I(u, u) = \langle \chi^H(\nu), [U] \rangle$.*

Any $u \in H^2(K)$ is Poincaré dual to $i_*^U[U] \in H^2(K)$ for a codimension 2 oriented submanifold $U \subset K$. This can be seen by using the fact that any 2-dimensional cohomology class u is of the form $u = f^*x$ for some map $f \colon K \to \mathbb{CP}^n$ and x the generator of $H^2(\mathbb{CP}^k) \cong \mathbb{Z}$. Then making f smooth and transversal to $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ then produces the oriented submanifold $U := f^{-1}(\mathbb{CP}^{n-1})$ with the desired properties.

Let $\nu \to U$ be the normal bundle of $U \subset K$, which inherits an orientation, and hence we can regards it as complex line bundle. The same remark applies to the tangent bundle TU. The bundle equation

$$TU \oplus \nu \cong TK_{|U}$$

then implies that $c(TU)c(\nu) = i_U^* c(TK)$ and hence

$$c_1(TU) + c_1(\nu) = i_U^* c_1(TK) = 0$$

by our calculation (4.22) of the total Chern class of TK. By Proposition 4.27 (3) it follows that

$$I(u,u) = \langle \chi^H(\nu), [U]\rangle = \langle c_1(\nu), [U]\rangle = -\langle c_1(TU), [U]\rangle = -\langle \chi^H(TU), [U]\rangle = -\chi(U)$$

Since the Euler characteristic of a Riemann surface is even, this implies that the intersection form I_K of the Kummer surface is even as claimed.

4.3 Exotic 7-spheres

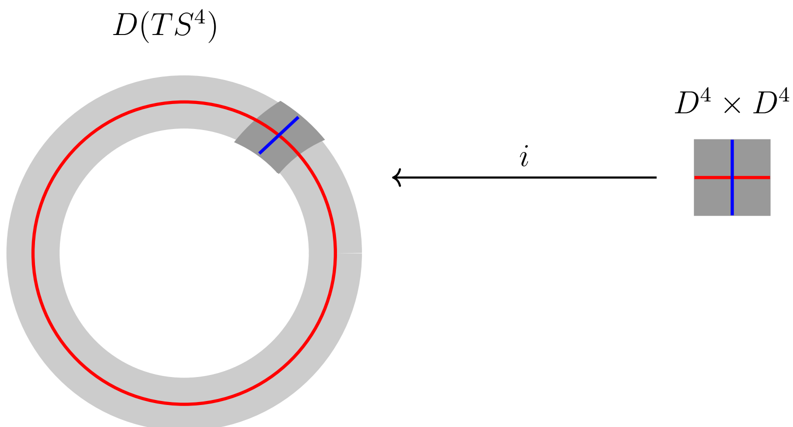
The strategy for constructing exotic 7-spheres, that is, smooth manifolds Σ that are homeomorphic, but not diffeomorphic to the 7-sphere S^7, is as follows.

- Construct a smooth 8-manifold W and show that its boundary $\Sigma := \partial W$ is homeomorphic to S^7.
- Show that Σ is not diffeomorphic to S^7 using proof by contradiction: If Σ were diffeomorphic to S^7, we could construct a closed smooth 8-manifold

  $$X = W \cup_{S^7} D^8$$

  by attaching an 8-disk to the boundary ∂W. This leads to a contradiction by contemplating the L-genus $L(X)$ and the $\widehat{A}$-genus $\widehat{A}(X)$.

Warmup Construction. Let $D(TS^4)$ be the disk bundle of the tangent bundle of S^4. This is an 8-manifold whose boundary is the unit tangent bundle (aka sphere bundle) $S(TS^4)$. Restricted to a disk $D^4 \subset S^4$ the vector bundle TS^4 can be trivialized, which allows us to identify the disk bundle $D(TS^4)$ restricted to $D^4 \subset S^4$ with the product $D^4 \times D^4$. Here the points $(x,0) \in D^4 \times D^4$ correspond to points in the zero section of TS^4, and (x,y) is a point in the fiber T_xS^4. Let $i\colon D^4 \times D^4 \hookrightarrow D(TS^4)$ be the inclusion map. It is illustrated in the picture below, where the diskbundle $D(TS^4)$ is drawn as a band with the zero section represented by the red circle (so the picture in an honest representation of the disk bundle of the 1-dimensional vector bundle TS^1). The blue line represents the fiber of the disk bundle over the origin of the disk $D^4 \subset S^4$, and the darker square is the disk bundle restricted to D^4, which is diffeomorphic to $D^4 \times D^4$.



Let W be the smooth 8-manifold with boundary constructed by gluing two copies of $D(TS^4)$ by identifying a point $i(x, y)$ in the first copy of $D(TS^4)$ with the point $i(y, x)$ in the second copy of $D(TS^4)$. In other words, W is constructed as pushout in the diagram

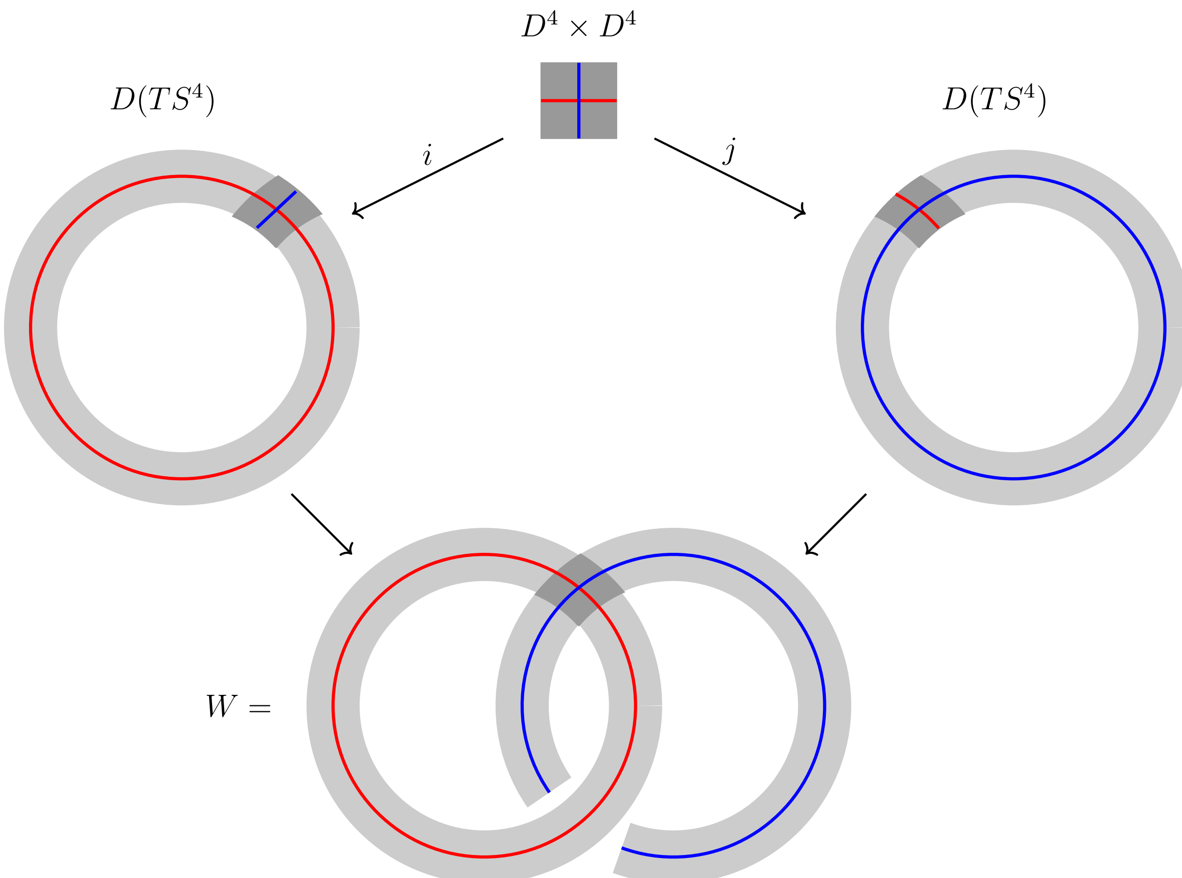
$$\begin{tikzcd} D^4 \times D^4 \arrow[r, "i"] \arrow[d, "j"'] & D(TS^4) \arrow[d] \\ D(TS^4) \arrow[r] & W \end{tikzcd} \tag{4.28}$$

where $j\colon D^4 \times D^4 \to D(TS^4)$ is given by $(x, y) \mapsto i(y, x)$. Strictly speaking, the manifold W has *corners*, consisting of the subset

$$S^3 \times S^3 \subset D^4 \times D^4 \subset D(TS^4)$$

(note that via gluing the two copies of $D(TS^4)$, this subspace of the first copy of $D(TS^4)$ is identified with this subspace in the second copy).

Here is an illustration of the situation. We note that the fiber over the origin of the disk in the first copy of $D(TS^4)$ (drawn as a blue line) corresponds in W to the zero section of the second copy of $S(TS^4)$ restricted to the disk $D^4 \subset S^4$ and vice versa.



Construction of W as a pushout

Proposition 4.29. *1. The smooth 8-manifold W is homotopy equivalent to the wedge $S^4 \vee S^4$; its intersection form is given by the matrix*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

2. The boundary ∂W is a simply connected smooth manifold with homology groups

$$H_k(\partial W) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 7 \\ \mathbb{Z}/3 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $S^4 \subset D(TS^4)$ be the zero section. The two copies of S^4 inside W (shown as red resp. blue circle in the picture above) intersect in one point. In other words, $S^4 \vee S^4$ embeds into W such that its image is the union of these two 4-spheres. We claim that $S^4 \vee S^4 \subset W$ is a deformation retract of W; that deformation is literally given by collapsing the disk bundles in radial direction to the zero section (a little care is needed to write down the correct deformation of $D^4 \times D^4$ into the subspace $D^4 \times \{0\} \cup \{0\} \times D^4$).

It follows that $H_4(W) = H_4(S^4 \vee S^4) \cong \mathbb{Z}^2$ has a basis $\{e_1, e_2\}$, where e_1 resp. e_2 is the homology class represented by the zero sections of the two copies of TS^4 used in our construction of W. From the construction of W and the geometric interpretation of the intersection pairing given by Proposition 4.27, it is easy to see that the intersection form for this basis is given by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Here the diagonal entry 2 comes from the fact that the normal bundle ν of $S^4 \hookrightarrow TS^4$ is the tangent bundle of S^4 and hence

$$\langle \chi^H(\nu), [S^4] \rangle = \langle \chi^H(TS^4), [S^4] \rangle = \chi(S^4) = 2.$$

To show that the boundary ∂W is simply connected, we note that ∂W is a deformation retract of $W \setminus (S^4 \vee S^4)$. Explicitly, for a point $w \in W \setminus (S^4 \vee S^4)$ the deformation path connecting w with a point in ∂W is given as follows. For $w = (x, y) \in D^4 \times D^4 \setminus (D^4 \times \{0\} \cup \{0\} \times D^4)$, take the straight line connecting it with $(x/||x||, y/||y||) \in S^3 \times S^3$ ($S^3 \times S^3 \subset \partial W$ are the original corner points of W; smoothing the corners does not change the underlying topological space). For a point w in one of the copies of $D(TS^4) \setminus S^4$, but away from $(D^4 \times D^4)$ the path starting at w goes radially outward until it reaches the unit sphere bundle $S(TS^4)$.

Let $\gamma \colon S^1 \to \partial W$ be a based loop. Since $W \sim S^4 \vee S^4$ is simply connected, γ extends to a map $\Gamma \colon D^2 \to W$. By a small deformation this is homotopic (as a map of pairs $(D^2, S^1) \to (W, \partial W)$) to a smooth map, which after another small deformation can be assumed to be transversal to both submanifolds $S^4 \subset W$ (given by the zero sections of the two copies of $D(TS^4)$). Since $\dim D^2 + \dim S^4 < \dim W$, there are no transversal intersection points, and hence Γ maps D^2 to $W \setminus (S^4 \vee S^4)$. Since ∂W is a deformation retract of $W \setminus (S^4 \vee S^4)$, this map is then homotopic (without changing it on $S^1 \subset D^2$), to a map $D^2 \to \partial W$. This shows that ∂W is simply connected.

To calculate the homology groups of ∂W we will first determine the homology groups of W and $(W, \partial W)$ and then use the long exact homology sequence of the pair $(W, \partial W)$. Since W is homotopy equivalent to $S^4 \vee S^4$, the homology groups of W are trivial except $H_0(W) \cong \mathbb{Z}$ and $H_4(W) \cong \mathbb{Z}^2$. By Poincaré duality and universal coefficient theorem we obtain

$$H_q(W, \partial W) \cong H^{8-q}(W) \cong \mathrm{Hom}(H_{8-q}(W), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & q = 8 \\ \mathbb{Z}^2 & q = 4 \end{cases}$$

Now we consider the long exact homology sequence of the pair $(W, \partial M)$:

$$\longrightarrow H_{q+1}(W) \longrightarrow H_{q+1}(W, \partial W) \xrightarrow{\partial} H_q(\partial W) \longrightarrow H_q(W) \longrightarrow H_q(W, \partial W) \longrightarrow$$

This implies that $H_q(\partial W) = \mathbb{Z}$ for $q = 0, 7$ and $H_q(\partial W) = 0$ for $q \neq 0, 3, 4, 7$. The interesting portion of this sequence is:

$$\begin{tikzcd}
0 \arrow[r] & H_4(\partial W) \arrow[r] & H_4(W) \arrow[r] \arrow[dr, "I_W"'] & H_4(W, \partial W) \arrow[r] \arrow[d, "\cong"] & H_3(\partial W) \arrow[r] & 0 \\
& & & \operatorname{Hom}(H_4(W), \mathbb{Z}) & &
\end{tikzcd} \tag{4.30}$$

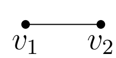
The nice feature here is that the map $H_4(W) \to H_4(W, \partial W)$ in that sequence turns out to be the intersection form I_W on the manifold W after we use Poincaré duality and the universal coefficient theorem to identify $H_4(W, \partial W)$ and $\operatorname{Hom}(H_4(W), \mathbb{Z})$. Exercise: prove this. It follows that

$$H_4(\partial W) \cong \ker I_W \qquad H_3(\partial W) \cong \operatorname{coker} I_W \tag{4.31}$$

The map $I_W \colon H_4(W) \to \operatorname{Hom}(H_4(W), \mathbb{Z})$ with respect to the basis $\{e_1, e_2\}$ for $H_4(W)$ constructed above and the dual basis for $\operatorname{Hom}(H_4(W), \mathbb{Z})$ is given by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ as proved in part (1) of the proposition. Since the determinant of this matrix is 3, the map I_W is injective with cokernel $\mathbb{Z}/3$, which implies $H_4(\partial W) = 0$ and $H_3(\partial W) \cong \mathbb{Z}/3$ as claimed. $\square$

Proposition 4.29 shows that ∂W has rationally the homology groups of S^7, but not integrally. In particular, ∂W is not homeomorphic to S^7. We observe that by (4.31) the origin for the failure of ∂W to be a homology sphere is the failure of the intersection form I_W to be unimodular. So the idea is to tweak the construction above in order to produce an 8-manifold W with boundary whose intersection pairing I_W is *unimodular*.

The construction above is known as *plumbing construction*. We note that the graph



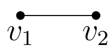
can be thought of as *blueprint* for the construction above by taking for each vertex a copy of the disk bundle $D(TS^4)$ and interpret the edge as the prescription to glue those two copies in the way we did above. We further note that the same graph determines the matrix of the intersection form of the manifold W thus produced by the recipe of Example 4.24 (i.e., $I(e_i, e_i) = 2$ and $I(e_i, e_j)$ is the number of edges connecting the vertices v_i and v_j).

This suggests to start with a general graph Γ and use it as a "blue print" to construct an associated smooth 8-manifold with boundary $W(\Gamma)$ in the following way:

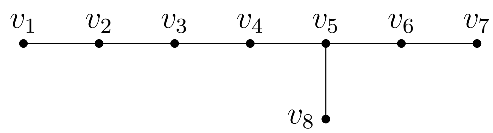
- Take the disjoint union $\coprod_i D(TS^4)_i$ of copies of the disk bundle $D(TS^4)$, one for each vertex v_i of the graph Γ;

- let $W(\Gamma)$ be the quotient of this disjoint union modulo the equivalence relation $\sim$ generated by the edges of Γ as follows. For each edge connecting v_i and v_j choose embeddings $D^4 \times D^4 \subset D(TS^4)_k$ for $k = i$ and $k = j$ given by choosing a disk D^4 in the base space S^4 and trivializing the disk bundle over these (the images of all these embeddings are chosen to be mutually disjoint). Then declare the point

$$(x, y) \in D^4 \times D^4 \subset D(TS^4)_i \qquad \text{to be equivalent to} \qquad (y, x) \in D^4 \times D^4 \subset D(TS^4)_j.$$

In our "warm-up" example, for the graph 

, this gives the manifold W (4.28) constructed above. For the E_8-graph

$E_8 =$ 

the manifold $W = W(E_8)$ is shown in the following picture.

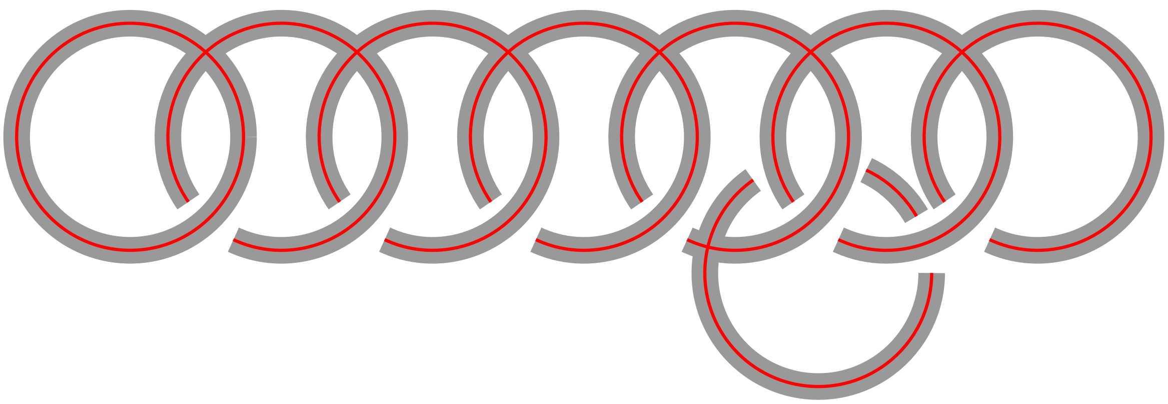


Figure 1: The manifold $W(E_8)$ constructed via plumbing from the graph E_8

In the picture each of the grey annuli represent one of the 8 copies of the disk bundle $D(TS^4)$, and the red circles in the center of each annulus depict the zero section $S^4 \subset D(ST^4)$. As in the previous example, the union of the 8 copies of S^4 is a deformation retract of the manifold W, and hence W is homotopy equivalent to a wedge $S^4 \vee \cdots \vee S^4$ of eight copies of S^4. In particular,

$$H_q(W) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}^8 & q = 4 \\ 0 & q \neq 0, 4 \end{cases},$$

and a basis $\{e_i\}_{i=1,\dots,8}$ of $H_4(W)$ is given by the images of the fundamental class $[S^4_i] \in H_4(W)$ under the homomorphism $H_4(S^4_i) \to H_4(W)$ induced by the inclusion maps $S^4_i \hookrightarrow W$, $i = 1, \dots, 8$ of the eight copies of $S^4 \subset D(TS^4)$.

It is straightforward to calculate the intersection form $I_W(e_i, e_j)$. As discussed in the warmup example, the self-intersection number $I_W(e_i, e_i) = 0$ for $i = 1, \dots, 8$, and the intersection number $I_W(e_i, e_j)$ for $i \neq j$ is the number of intersection points between $S^4_i \subset W$ and $S^4_j \subset W$. From the construction of W is it clear that for each edge in the graph E_8 connecting vertices v_i and v_j, the corresponding gluing of $D^4 \times D^4 \subset D(TS^4)_i$ and $D^4 \times D^4 \subset D(TS^4)_j$ results in an additional intersection point between the zero sections S^4_i and S^4_j (look at the intersections of the red central circles in the pictures!). In other words, $I_W(e_i, e_j)$ is the number of edges connecting v_i and v_j. Arguing as in the warmup example we then obtain the following results.

Proposition 4.32. 1. *The manifold $W(E_8)$ is a smooth simply connected 8-manifold with boundary, which is homotopy equivalent to a wedge of 8 copies of S^4 and its intersection form $I_{W(E_8)}$ is equal to the lattice $I(E_8)$ determined by the graph E_8 (see 4.24(3)).*

2. *The boundary $\partial W(E_8)$ is a simply connected smooth 7-manifold which is a* homology 7-sphere*, i.e., its homology groups are isomorphic to those of S^7.*

Exercise 4.33. For a general graph Γ, what is the homotopy type of $W(\Gamma)$? What is the intersection form of $W(\Gamma)$? What conditions on Γ guarantee that its boundary is a simply connected homology sphere? Hint: which properties of Γ guarantee that $W(\Gamma)$ is simply connected?

Definition 4.34. A *homotopy sphere* of dimension n is a closed smooth n-manifold which is homotopy equivalent to S^n.

Lemma 4.35. *Any simply connected homology n-sphere is a homotopy sphere.*

Proof. Let Σ be a homology n-sphere. Then the lowest degree non-trivial homology group of Σ is $H_n(\Sigma) \cong \mathbb{Z}$. Since Σ is simply connected, the Hurewicz theorem implies that the lowest degree non-trivial homotopy group of Σ is $\pi_n(\Sigma) \cong H_n(\Sigma) \cong \mathbb{Z}$. Let $f\colon S^n \to \Sigma$ be a basepoint preserving map that represents a generator of $\pi_n(\Sigma)$. Then the induced map f_* on homology groups is an isomorphism. Since f is a map between simply connected spaces, the Hurewicz theorem implies that then also the induced map on homotopy groups is an isomorphism. Since manifolds can be given a CW structure, e.g., by Morse theory, the Whitehead theorem then implies that $f\colon S^n \to \Sigma$ is in fact a homotopy equivalence. □

It turns out that in dimension $n \geq 5$ any homotopy sphere is homeomorphic to S^n. This is a corollary of the h-cobordism theorem due to Smale. Let W be a bordism between smooth n-manifolds M and N, i.e., the boundary of W is the disjoint union of M and N. A simple example of a bordism is the *product bordism* $W = M \times I$ which is a bordism between M and itself. An obvious property of the product bordism is that the inclusion maps $M \hookrightarrow W$ of either copy of M into W is a homotopy equivalence. More generally, any bordism W between M and N is called an *h-cobordism* if both inclusion maps $i^M : M \hookrightarrow W$ and $i^N : N \hookrightarrow W$ are homotopy equivalences. As a slogan we could say that a cobordism is an h-cobordism if it looks homotopically (that's what the h stands for) like a product cobordism.

Theorem 4.36. (h-cobordism theorem; Smale). *Let W be an h-cobordism between manifolds M and N which are simply connected and of dimension $n \geq 5$. Then W is diffeomorphic to the product bordism $M \times [0,1]$. More precisely, there is a diffeomorphism*

$$F : M \times [0,1] \longrightarrow W$$

whose restriction to $M \times \{0\}$ is the identity on M, and whose restriction to $M \times \{1\}$ is some diffeomorphism $f : M \to N$.

The dimension restriction $n \geq 5$ is crucial, while the assumption that M, N are simply connected can be relaxed: there is a more result known as the *s-cobordism theorem* without the simply connectivity assumption on M, N, but assuming that the maps $i^M : M \hookrightarrow W$ and $i^N : N \hookrightarrow W$ are *simple homotopy equivalences* which means that they are homotopy equivalences, but in addition their *torsion*, an element in the Whitehead group $\mathrm{Wh}(\pi)$ of the fundamental of these manifolds, vanishes.

Corollary 4.37. *If Σ is a homotopy sphere of dimension $n \geq 6$, then Σ is homeomorphic to S^n.*

Proof. Let $D_i^n \subset \Sigma$, $i = 1, 2$ two disjoint disks in Σ, and let W be the manifold with boundary obtained by removing the interiors of these disks. Then the boundary of W consists of two copies of S^{n-1}; in other words, W is a bordism from S^{n-1} to S^{n-1}. This is in fact an h-cobordism (proof: exercise!). Then the h-cobordism theorem gives a diffeomorphism

$$W \xrightarrow{F} S^{n-1} \times I$$

which on the boundary $\partial W = S^{n-1} \amalg S^{n-1}$ restricts to the identity of the first copy of S^{n-1} and to some diffeomorphism $f : S^{n-1} \to S^{n-1}$ on the second copy. This shows

$$\Sigma = D_1^n \cup_{\mathrm{id}} W \cup_{\mathrm{id}} D_2^n \cong \underbrace{D_1^n \cup_{\mathrm{id}} (S^{n-1} \times [0,1])}_{D^n} \cup_f D_2^n \cong D^n \cup_f D^n$$

This shows that every homotopy sphere of dimension $n \geq 6$ is diffeomorphic to the manifold $D^n \cup_f D^n$ obtained by gluing two disks via a diffeomorphism $f : S^{n-1} \to S^{n-1}$ of their boundaries. So it suffices to produce a homomorphism

$$G : S^n = D^n \cup_{\mathrm{id}} D^n \longrightarrow D^n \cup_f D^n.$$

We take G to be the identity on the first disk; restricted to the second disk we let

$$G\colon D^n \longrightarrow D^n \qquad \text{be defined by} \qquad G(tv) := tf(v) \quad \text{for } t \in [0,1],\, v \in S^{n-1}$$

Note that this is a continuous bijection and hence a homeomorphism (since its domain is compact and its codomain is Hausdorff), but it is *not smooth* (the differential at the origin of the second disk is the problem). □

Definition 4.38. Let Θ_n be the set of oriented homotopy n-spheres up to h-cobordism. In other words, the elements of Θ_n are equivalence classes of oriented homotopy n-spheres, where we declare oriented homotopy n-spheres Σ, Σ' to be equivalent if there is an oriented h-cobordism between them.

The connected sum of oriented homotopy n-spheres gives Θ_n the structure of an abelian group. The unit element given by the standard n-sphere S^n, and the inverse of an oriented homotopy n-sphere Σ is given by $\bar{\Sigma}$, which is the homotopy n-sphere Σ equipped with the opposite orientation.

The h-cobordism theorem implies that for $n \geq 5$ two oriented homotopy n-spheres Σ, Σ' represent the same element in Θ_n if and only if there is an orientation preserving diffeomorphism between them. So for $n \geq 5$, the group Θ_n can alternatively be interpreted as the group of oriented homotopy n-spheres up to oriented diffeomorphisms.

Next we want to understand the oriented homotopy sphere $\partial W^8(E_8)$. In particular, what can we say about the order of $[\partial W^8(E_8)] \in \Theta_7$?

Theorem 4.39. *The element $[\partial W^8(E_8)] \in \Theta_7$ generates a subgroup of Θ_7 whose order is a multiple of* 28.

The main tool in the proof of this result will be calculations of the L-genus and the $\widehat{A}$-genus of certain closed 8-manifolds. For that calculation it will be necessary to have explicit formulas for the associated characteristic classes $L(V)$, $\widehat{A}(V)$ of a real vector bundle V in term of the Pontryagin classes of V.

Proposition 4.40. *Let $V \to X$ be a real vector bundle. Then the L-class $L(V)$ and the $\widehat{A}$-class $\widehat{A}(V)$ are elements in $H^{4*}(X;\mathbb{Q})$ which are polynomials in the Pontryagin classes $p_i = p_i(V) \in H^{4i}(X;\mathbb{Z})$. In degree ≤ 8, these are given explicitly by*

$$\begin{aligned} L(V) &= 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \dots \\ \widehat{A}(V) &= 1 - \frac{1}{24}p_1 + \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2) + \dots \end{aligned} \tag{4.41}$$

Corollary 4.42. *If X is a closed oriented manifold of dimension 4, then*

$$\begin{aligned} L(X) &= \frac{1}{3}\langle p_1(TX), [X]\rangle \\ \widehat{A}(X) &= -\frac{1}{24}\langle p_1(TX), [X]\rangle. \end{aligned}$$

If X is a closed oriented manifold of dimension 8, then

$$\begin{aligned} L(X) &= \frac{1}{45}\langle 7p_2(TX) - p_1^2(TX), [X]\rangle \\ \widehat{A}(X) &= \frac{1}{2^7 \cdot 3^2 \cdot 5}\langle -4p_2(TX) + 7p_1^2(TX), [X]\rangle. \end{aligned}$$

Proof. Let $T_q^R(V) \in H^{4*}(X;\mathbb{Q})$ be the exponential characteristic class for real vector bundles V associated to an even power series

$$q(x) = 1 + a_2x^2 + a_4x^4 + \cdots \in \mathbb{Q}[[x]].$$

Then if L_1, L_2 are complex line bundles over X with first Chern class $x_i \in H^2(X;\mathbb{Z})$ and first Pontyagin class $y_i := x_i^2 \in H^4(X;\mathbb{Z})$, then

$$T_q^R(L_i) = 1 + a_2x_i^2 + a_4x_i^4 + \cdots = 1 + a_2y_i + a_4y_i^2 + \ldots$$

Hence by the exponential property

$$\begin{aligned} T_q^{\mathbb{R}}(L_1 \oplus L_2) =&(1 + a_2y_1 + a_4y_1^2 + \ldots)(1 + a_2y_2 + a_4y_2^2 + \ldots) \\ =&1 + a_2\underbrace{(y_1 + y_2)}_{\sigma_1} + a_2^2\underbrace{y_1y_2}_{\sigma_2} + a_4\underbrace{(y_1^2 + y_2^2)}_{\sigma_1^2 - 2\sigma_2} + \ldots \end{aligned}$$

Here $\sigma_i = \sigma(y_1, y_2)$ are the elementary symmetric polynomials of $y_i = p_1(L_1)$, which are equal to the Pontyagin class $p_i(L_1 \oplus L_2) \in H^{4i}(X;\mathbb{Z})$. Hence

$$T_q^{\mathbb{R}}(L_1 \oplus L_2) = 1 + a_2p_1 + \big((a_2^2 - 2a_4)p_2 + a_4p_1^2\big) + \ldots.$$

In particular, for the L-class,

$$q(x) = \frac{x}{\tanh x} = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \ldots.$$

and hence

$$\begin{aligned} L(V) &= 1 + \frac{1}{3}p_1 + (\frac{1}{9} - \frac{2}{45})p_2 - \frac{1}{45}p_1^2 + \ldots \\ &= 1 + \frac{1}{3}p_1 + \frac{7}{45}p_2 - \frac{1}{45}p_1^2 + \ldots \end{aligned}$$

Similarly, the $\widehat{A}$-class is associated to the even power series

$$q = \frac{x/2}{\sinh(x/2)} = 1 - \frac{1}{24}x^2 + \frac{7}{2^7 \cdot 3^2 \cdot 5}x^4 + \dots$$

and hence

$$\begin{aligned}
\widehat{A}(V) &= 1 - \frac{1}{24}p_1 + \left(\frac{1}{2^6 \cdot 3^2} - \frac{2 \cdot 7}{2^7 \cdot 3^2 \cdot 5}\right)p_2 + \frac{7}{2^7 \cdot 3^2 \cdot 5}p_1^2 + \dots \\
&= 1 - \frac{1}{24}p_1 + \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2) + \dots
\end{aligned}$$

□

Before proving theorem 4.39, let us prove the following simpler statement.

Proposition 4.43. *The homotopy sphere $\partial W(E_8)$ is not diffeomorphic to S^7.*

Proof. Let $W := W(E_8)$ and assume that ∂W is diffeomorphic to S^7. Let $X := W \cup_{S^7} D^8$ be the closed smooth 8-manifold obtained by gluing W and D^8 along their boundaries. In order to calculate the L-genus $L(X)$ in terms of the Pontryagin classes of TX, we first show $p_1(TX) = 0$.

Since the inclusion $i\colon W \hookrightarrow X$ induces an isomorphism on H^4, and i^*TX is isomorphic to TW (via the differential of the embedding i), it suffices to show $p_1(TW) = 0$. Since W is a homotopy equivalent to a wedge of 8 copies of S^4, is suffices to show that the restriction of TW to each of these eight spheres $S^4 \subset W$ is stably trivial. This follows from the vector bundle isomorphisms

$$TW_{|S^4} \cong TS^4 \oplus \nu(S^4 \hookrightarrow W) \cong TS^4 \oplus TS^4,$$

where $\nu(S^4 \hookrightarrow W)$ is the normal bundle of S^4 in W, which by construction is isomorphic to TS^4.

Due to the vanishing of $p_1(TX)$, the formula for the L-genus $L(X)$ of Corollary 4.42 simplifies to $L(X) = 7/45\langle p_2(TX), [X]\rangle$. Using Hirzebruch's signature theorem we then obtain

$$7/45\langle p_2(TX), [X]\rangle = L(X) = \operatorname{sign}(X) = \operatorname{sign}(W(E_8)) = 8.$$

This is the desired contradiction since evaluating $p_2(TX) \in H^4(X;\mathbb{Z})$ on the fundamental class $[X] \in H_4(X;\mathbb{Z})$ results in an *integer* $\langle p_2(TX), [X]\rangle$, but the above equation claims that integer to be $\frac{8 \cdot 45}{7}$. □

The stronger statement about the order of the element of Θ_7 represented by the homotopy sphere $\Sigma = \partial W(E_8)$ will be proved in a similar way. It relies on the fact that the connected sum of k-copies of Σ is the boundary of a manifold W that can be manufactured from k

copies of $W(E_8)$. More generally, if $M = \partial V$ and $N = \partial W$ are manifolds of the same dimension n, then the connected sum $M\#N$ is the boundary of a manifold $V\natural W$ (called the *boundary connected sum* of V and W) manufactured from V and W.

We recall that some care is need when defining the connected sum $M\#N$ of two n-manifolds. The reason is that the connected sum construction involves the choice of embeddings of discs $D^n \hookrightarrow M$ and $D^n \hookrightarrow N$. If M is connected and non-orientable, there is only one such embedding up to isotopy, but if M is oriented, there are two such embeddings up to isotopy, one of which is orientation preserving, and the other orientation reversing.

Similarly, the details of the construction of $V\natural W$ depend on orientability assumptions. Since we will be only interested in oriented manifolds, we define $V\natural W$ only in that case.

Definition 4.44. Let V and W be oriented manifolds of dimension $n+1$ with non-empty, connected boundaries. Let

$$i^V\colon (D^n \times (-\epsilon, 0], D^n \times \{0\}) \ \hookrightarrow\ (V, \partial V)$$

and

$$i^W\colon (D^n \times [1, 1+\epsilon), D^n \times \{1\}) \ \hookrightarrow\ (V, \partial V)$$

be orientation preserving embeddings (these are maps of pair; e.g., i^V is an embedding of $D^n \times (-\epsilon, 0]$ into V, whose restriction to $D^n \times \{0\}$ is an embedding of $D^n \times \{0\}$ into ∂V). Let $V\natural W$ be the quotient space

$$(V \amalg D^n \times (-\epsilon, 1+\epsilon) \amalg W) \,/\, \sim,$$

where the equivalence relation $\sim$ identifies a point $(x,t) \in D^n \times (-\epsilon, 1+\epsilon)$ with $i^V(x,t) \in V$ for $t \in (-\epsilon, 0]$ and with $i^W(x,t) \in W$ for $t \in [1, 1+\epsilon)$. After smoothing the corners, $V\natural W$ is a smooth oriented manifold of dimension $n+1$ called the *boundary connected sum of* V *and* W. Its boundary is

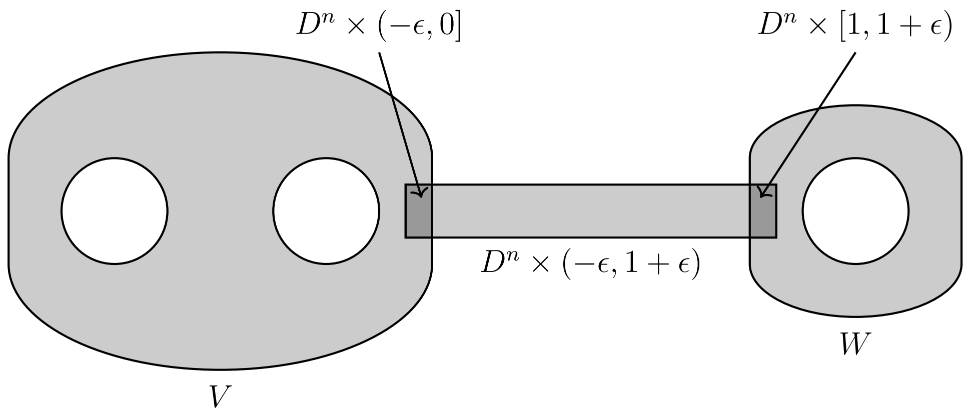
$$\partial(V\natural W) = \partial V \# \partial W$$

the connected sum of ∂V and ∂W.

Example 4.45. (Examples of boundary connected sums).

1. Here is a picture of the boundary connected sum $V\natural W$ of $(n+1)$-manifolds V and W for $n = 1$. The darker shaded areas are those pieces of the strip $D^n \times (-\epsilon, 1+\epsilon)$ where it is glued with V resp. W. We are cheating here slightly in that the boundaries of V and W in this example are not connected. Ideally, instead of drawing W as annulus, I would love to draw $T \setminus \mathring{D}^2$, a torus T with an open disk removed, but that is too hard to draw. Similarly, use your imagination to remove the two interior boundary circles

of V by gluing a copy of $T \setminus \mathring{D}^2$ into each of these holes. By this move, V becomes a surface of genus 2 with one boundary circle.



2. Let Σ_g be a surface of genus g. The standard picture of Σ_g in $\mathbb{R}^3$ shows that Σ_g is a boundary of a compact 3-manifold W_g whose interior is the bounded component of the complement of Σ_g in $\mathbb{R}^3$. Then the boundary connected sum $W_g \natural W_h$ is diffeomorphic to W_{g+h} with boundary $\Sigma_g \# \Sigma_h \cong \Sigma_{g+h}$. Here is a picture of the boundary connected sum of the solid torus W_1 and the solid double torus W_2.

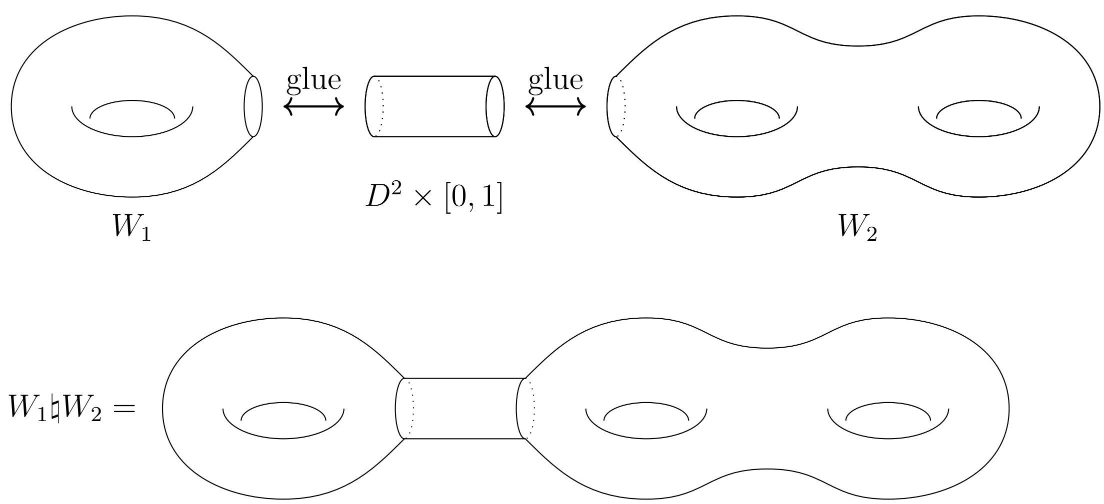


Figure 2: The boundary connected sum $W_1 \natural W_2$

Proof of Theorem 4.39. Assume that $\Sigma := \partial W^8(E_8)$ is an element of order k in Θ_7, i.e., the connected sum $\#_k \Sigma$ of k copies of Σ is diffeomorphic to S^7. Since $\#_k \Sigma$ is the boundary ∂W of the boundary connected sum

$$W := \underbrace{W^8(E_8) \natural \ldots \natural W^8(E_8)}_{k}$$

it follows that as before we obtain a closed smooth manifold $X := W \cup_{S^7} D^8$. There are isomorphisms

$$H_4(X) \cong H^4(W) \cong \underbrace{H_4(W(E_8)) \oplus \cdots \oplus H_4(W(E_8))}_{k},$$

induced the inclusion maps $W \hookrightarrow X$ resp. $W(E_8)_i \hookrightarrow W$, where $W(E_8)_i$, $i = 1, \ldots, k$ is the i-th copy of $W(E_8)$. This implies that the intersection form on X is the direct sum of k copies of the intersection form on $W(E_8)$ which is the lattice $I(E_8)$ associated to the graph E_8. In particular,

$$\mathrm{sign}(X) = \mathrm{sign}(W) = k\mathrm{sign}(W(E_8)) = k\mathrm{sign}(I(E_8)) = 8k.$$

The argument as in the proof of Proposition 4.43 shows that the Pontryagin class $p_1(TX)$ vanishes, and hence the formula for the L-genus of X (see Cor. 4.42) simplifies to $L(X) = 7/45\langle p_2(TX), [X]\rangle$. Using Hirzebruch's signature theorem we then obtain

$$7/45\langle p_2(TX), [X]\rangle = L(X) = \mathrm{sign}(X) = 8k.$$

This implies $\langle p_2(TX), [X]\rangle = \frac{45 \cdot 8k}{7}$, which shows that k must be a multiple of 7, since $\langle p_2(TX), [X]\rangle$ is an *integer*! (the evaluation of the second Pontryagin class $p_2(TX) \in H^8(X; \mathbb{Z})$ on the fundamental class $[X] \in H_8(X; \mathbb{Z})$).

To obtain the stronger statement that k must be a multiple of 28, we bring the $\widehat{A}$-genus of X into the mix. Thanks to the vanishing of $p_1(TX)$, the $\widehat{A}$-genus of X can be expressed solely in terms of $\langle p_2(TX), [X]\rangle$, which in turn can be expressed in terms of $L(X) = \mathrm{sign}(X)$. This results in the following equalities:

$$\begin{aligned}\widehat{A}(X) &= -\frac{1}{2^5 \cdot 3^2 \cdot 5}\langle p_2(TX), [X]\rangle = -\frac{1}{2^5 \cdot 3^2 \cdot 5} \cdot \frac{45}{7} L(X) \\ &= -\frac{1}{2^5 \cdot 7}\mathrm{sign}(X) = -\frac{1}{2^5 \cdot 7} 8k = -\frac{k}{2^2 \cdot 7}\end{aligned}$$

By the index theorem, $\widehat{A}(X)$ is equal to the index of the Dirac operator on X. In particular, $\widehat{A}(X)$ is an integer, and hence we arrive at a contradiction unless k is a multiple of 28. □

4.3.1 Survey on the group Θ_n of homotopy n-spheres

The goal of this section is to mention the main results on the groups Θ_n, where by results of Kervaire-Milnor are intimately related to stable homotopy groups of spheres.

Proposition 4.46. (Kervaire-Milnor 1963). *Let Σ be a homotopy n-sphere. Then its tangent bundle is stably trivial, i.e., there is a vector bundle isomorphism*

$$\alpha \colon T\Sigma \oplus \underline{\mathbb{R}}^k \xrightarrow{\cong} \underline{\mathbb{R}}^{n+k}$$

for some k.

Definition 4.47. A *framing* for an n-manifold M is a vector bundle isomorphism

$$\alpha\colon TM \oplus \underline{\mathbb{R}}^k \xrightarrow{\cong} \underline{\mathbb{R}}^{n+k}$$

for some k. More precisely, a framing is an equivalence class of such vector bundle isomorphisms. The equivalence relation is generated by

- a homotopy between vector bundle isomorphisms α, α', and
- α is equivalent to the isomorphism $\alpha \oplus \mathrm{id}_{\underline{\mathbb{R}}}\colon TM \oplus \underline{\mathbb{R}}^{k+1} \xrightarrow{\cong} \underline{\mathbb{R}}^{n+k+1}$.

A *framed manifold* is a pair (M, α) consisting of a manifold M and a framing α.

Example 4.48. (Framings of S^n).

1. The n-disk D^{n+1} has an obvious framing. The *standard framing* α_0 for S^n is the framing obtained as boundary of the framed manifold D^{n+1}. More explicitly, α_0 is the composition

$$TS^n \oplus \underline{\mathbb{R}} \cong TS^n \oplus \nu(S^n \hookrightarrow D^{n+1}) \xrightarrow{\cong} TD^{n+1}_{|S^n} \cong \underline{\mathbb{R}}^{n+1}$$

   Here $\nu(S^n \hookrightarrow D^{n+1})$ is the normal bundle of S^n in D^{n+1}, which is isomorphic to the trivial line bundle $\underline{\mathbb{R}}$ by the choice of an outward pointing normal vector field on S^n (this is unique up to homotopy).

2. Let $f\colon S^n \to O(n+k)$ be a smooth map to the orthogonal group. Let α_f be the new framing of S^n obtained by composing the standard framing α_0 with the bundle automorphism of the trivial bundle $\underline{\mathbb{R}}^{n+k}$ determined by f. More explicitly, α_f is the composition

$$TS^n \oplus \underline{\mathbb{R}}^k = TS^n \oplus \underline{\mathbb{R}}^1 \oplus \underline{\mathbb{R}}^{k-1} \xrightarrow{\alpha_0 \oplus \mathrm{id}_{\underline{\mathbb{R}}^{k-1}}} \underline{\mathbb{R}}^{n+1} \oplus \underline{\mathbb{R}}^{k-1} = \underline{\mathbb{R}}^{n+k} \xrightarrow{\widehat{f}} \underline{\mathbb{R}}^{n+k}$$

   Here $\widehat{f}\colon S^n \times \mathbb{R}^{n+1} \longrightarrow S^n \times \mathbb{R}^{n+1}$ is given by $(x, v) \mapsto (x, f_x(v))$, where $f_x \in O(n+k)$ is the image of $x \in S^n$ under $f\colon S^n \to O(n+k)$.

3. The framing α_f depends only on the homotopy class $[f] \in \pi_n(O(n+k))$ of f. In fact, it depends only on the image in $\pi_n(O)$, where $O = \bigcup_k O(k)$ is the infinite orthogonal group. Moreover, the map

$$\pi_n(O) \longrightarrow \{\text{framings of } S^n\} \qquad \text{given by} \qquad [f] \mapsto \alpha_f$$

   is a bijection. We note Bott showed that $\pi_n(O) = \begin{cases} \mathbb{Z} & \text{for } n = 3 \mod 4 \\ \mathbb{Z}/2 & \text{for } n = 0, 1 \mod 4 \\ 0 & \text{otherwise} \end{cases}$.

Let

$$J_n\colon \pi_n(O) \longrightarrow \Omega_n^{\mathrm{fr}} \qquad \text{be defined by} \qquad f \mapsto [S^n, \alpha_f].$$

This is a homomorphism known as *J-homomorphism* in stable homotopy theory.

Let

$$\Theta_n \longrightarrow \Omega_n^{\mathrm{fr}}/\operatorname{im} J_n \qquad \text{be the map given by} \qquad \Sigma \mapsto [\Sigma, \alpha], \tag{4.49}$$

where α is some framing on Σ. This map is well-defined and a homomorphism. Its kernel is denoted by bP_{n+1}, which stands for *boundaries of parallelizable* $(n+1)$*-manifolds*. Kervaire and Milnor calculated the kernel and cokernel of the above map up to the *Kervaire invariant* which is a homomorphism

$$K_n\colon \Omega_n^{\mathrm{fr}}/\operatorname{im} J_n \to \mathbb{Z}/2$$

for $n \equiv 2 \mod 4$. Like signature of a manifold, which was originally defined only for manifolds of dimension $n \equiv 0 \mod 4$, and later extended to n-manifolds for any n, we declare $K_n = 0$ for $n \not\equiv 2 \mod 4$.

Theorem 4.50. (Kervaire-Milnor).

1. *The sequence*

$$0 \longrightarrow \mathrm{bP}_{n+1} \longrightarrow \Omega_n^{\mathrm{fr}}/\operatorname{im} J_n \xrightarrow{K_n} \begin{cases} \mathbb{Z}/2 & n \equiv 2 \mod 4 \\ 0 & n \not\equiv 2 \mod 4 \end{cases}$$

is an exact sequence (note that there is no claim of surjectivity of K_n).

2. *For $k \geq 2$ the group bP_{4k} is cyclic of order $a_k \cdot 2^{2k-2} \cdot (2^{2k-1} - 1) \cdot \mathrm{Num}(4B_{2k}/k)$, where*
   - *$a_k = 1$ for k even, $a_k = 2$ for k odd,*
   - *B_{2k} is the Bernoulli number (using number theorist conventions) defined via the Taylor expansion*
   $$\frac{x}{1 - e^{-x}} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m,$$
   - *$\mathrm{Num}(4B_{2k}/k)$ is the numerator of the rational number $4B_{2k}/k$.*

3. *bP_{n+1} is trivial for $n+1$ even. For $n+1 \equiv 2 \mod 4$ this group is $\mathbb{Z}/2$ if the Kervaire invariant homomorphism K_{n+1} is trivial; otherwise bP_{n+1} is trivial.*

Remark 4.51. K_n is known to be non-trivial for $n = 2, 6, 14, 30, 62$ (note that these numbers are of the form $n = 2^k - 2$ for $k = 2, 3, 4, 5, 6$). Browder has shown that $K_n = 0$ for $n \neq 2^k - 2$, and recently, Hill-Hopkins-Ravenel showed $K_n = 0$ for $n = 2^k - 2$, $k \geq 8$.

5 Dirac operator, the index theorem and applications

Motivated by the dual goal of constructing the Dirac operator on spin manifolds and the K-theory orientation of spin vector bundles, we introduce Clifford algebras in the first section 5.1 and use them to give an algebraic construction of the group $\mathrm{Spin}(n)$ and the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. In the second section 5.2 we use Clifford algebras to construct the Dirac operator and the K-theory orientation for spin vector bundles. The Dirac operator is then used in 5.3 to show that the index of the Dirac operator on a closed spin manifold X is an obstruction to the existence of a Riemannian metric with positive scalar curvature. In section 5.4 we state the Index Theorem for a general elliptic operator P on a closed manifold X, which expresses the index of P in terms of principal symbol of P which represents an in the K-theory group $K_c(T^*X)$ with compact support. In the last section 5.5 we outline the proof of the Index Theorem for Dirac operators.

5.1 Constructions with Clifford algebras

The goal of this section and the next section is twofold:

1. The construction of the spinor bundle $S \to X$ and the Dirac operator $D\colon \Gamma(S) \to \Gamma(S)$ for an even dimensional spin manifold X.

2. The construction of the K-theory orientation (aka Thom class) $U^K(V) \in K_c(V)$ for an even dimensional spin vector bundle $V \to X$.

While these goals might sound very different from each other, they are very closely related as the following result shows.

Proposition 5.1. *Let X be spin manifold of dimension $n = 2k$ and let $D\colon \Gamma(S) \to \Gamma(S)$ be the Dirac operator on X. Let $\pi\colon T^*X \to X$ be the cotangent bundle, and let*

$$\sigma^D\colon \pi^*S \longrightarrow \pi^*S$$

be the principal symbol of the Dirac operator (a vector bundle morphism which is an isomorphism for all $0 \neq \xi \in T^*X$, since D is elliptic). Then the K-theory class

$$[\pi^*S, \tau, \sigma^D] \in K_c(T^*X)$$

represented by the graded Hermitian triple (π^*S, τ, σ^D) (see Definition 3.37) is equal to the orientation class $U^K(T^*X) \in K_c(T^*X)$.

Slogan: The Dirac operator D is the square root of the Laplace operator Δ.

This is a very imprecise statement since D acts on the sections $\Gamma(S)$ of the spinor bundle, and we have not defined what we mean by the Laplace operator acting to $\Gamma(S)$. Moreover, this is statement is not true for the operators themselves, but rather for their principle symbols. In other words, the principal symbol of D^2 is equal to the principal symbol of the Laplace operator Δ^S acting on spinors.

Example 5.2. (Dirac and Laplace operators and their principal symbols).

1. Let X be a Riemannian spin manifold of dimension $2k$, let $S \to X$ be the $\mathbb{Z}/2$-graded spinor bundle and let ∇^S be the connection on S induced by the Levi-Civita connection on the tangent bundle TX. We recall that the Dirac operator is the first order elliptic operator given by the composition

   $$D\colon \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*X \otimes S) \xrightarrow{c} \Gamma(S),$$

   where the *Clifford multiplication map* c is induced by a vector bundle map

   $$T^*X \otimes S \to S$$

   which abusing notation we again denote by c. The principal symbol $\sigma^D_\xi \in \mathrm{End}(S_x)$ of D for $\xi \in T^*_xX$ is given by

   $$\sigma^D_\xi = ic_\xi,$$

   where $c_\xi\colon S_x \to S_x$ is given by $v \mapsto c(\xi \otimes v)$. The square D^2 of the Dirac operator is a second order elliptic differential operator whose principal symbol is then

   $$\sigma^{D^2}_\xi = -c^2_\xi.$$

2. Let X be a Riemannian manifold and let $E \to X$ be a complex vector bundle equipped with a Hermitian metric and a metric connection ∇. Then

   $$\nabla\colon \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E) \qquad \text{and its adjoint} \qquad \nabla^*\colon \Gamma(T^*X \otimes E) \longrightarrow \Gamma(E)$$

   are both first order differential operators. The composition

   $$\Delta^E\colon \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*X \otimes E) \xrightarrow{\nabla^*} \Gamma(E)$$

   is then a second order differential operator called the *(rough or Bochner) Laplacian* on E. If E is the trivial line bundle, Δ^E is the usual Laplace operator on functions on X (but it is *not* equal to the Laplace-Beltrami operator on $\Omega^k(X)$ for $k > 0$!). Its principal symbol is given by

   $$\sigma^{\Delta^E}_\xi = ||\xi||^2;$$

   more precisely, for $\xi \in T^*_xX$, it is the endomorphism of E_x given by multiplication by $||\xi||^2$. In particular, $\sigma^{\Delta^E}_\xi$ is an isomorphism for $\xi \neq 0$, and hence the Laplace operator Δ^E is elliptic.

This shows that for $\xi \in T_x^*X$ the endomorphism $c_\xi \in \mathrm{End}(S_x)$ should have the property $c_\xi^2 = -||\xi||^2$; this would guarantee that the principal symbol of the Dirac operator D is the square root of the principal symbol of Δ^S. So we will do some reverse engineering to produce the Clifford multiplication map $c\colon T_x^*X \otimes S_x \to S_x$.

Observation. Let V be the real inner product space $V = T_x^*X$, and let M be the $\mathbb{Z}/2$-graded vector space $M := S_x$. Then M is a module over the tensor algebra $T(V) := \bigoplus_{k=0}^\infty V^{\otimes k}$ by defining the action

$$T(V) \otimes M \longrightarrow M \qquad \text{by} \qquad \xi_1 \otimes \cdots \otimes \xi_k \otimes \psi \mapsto c_{\xi_1} \dots c_{\xi_k}\psi.$$

Moreover, the relation $c_\xi^2 = -||\xi||^2$ for $\xi \in V$ implies that M is a module over the quotient algebra of $T(V)$ modulo the ideal generated by $\xi \otimes \xi + ||\xi||^2$.

Definition 5.3. Let V be a real vector space equipped with a symmetric bilinear form $q\colon V \times V \to \mathbb{R}$. Then the *Clifford algebra* $C\ell(V,q)$ is the quotient of the tensor algebra $T(V)$ modulo the ideal generated by $v \otimes v + q(v,v)$. This is a $\mathbb{Z}/2$-graded algebra, with the grading induced from the $\mathbb{Z}$-grading on the tensor algebra $T(V)$.

We note that for $q \equiv 0$, the Clifford algebra $C\ell(V,q)$ is equal to the exterior algebra $\Lambda^*(V)$. If $\{e_i\}_{i=1,\dots,n}$ is a basis of V, then $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $1 \le i_1 < \cdots < i_k \le n$ is a basis for $\Lambda^*(V)$. In particular, the dimension of $\Lambda^*(V)$ is $2^{\dim(V)}$. This is still true for the Clifford algebra as the following result shows.

Lemma 5.4. *The map*

$$\Lambda^*(V) \longrightarrow C\ell(V,q) \qquad \textit{given by} \qquad v_1 \wedge \cdots \wedge v_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)}$$

is a vector space isomorphism. In particular, $\dim C\ell(V,q) = 2^{\dim V}$.

Exercise 5.5. Prove this lemma. Hint: there is natural filtration $F_0 \subset F_1 \subset F_k \subset \dots$ on $\Lambda^*(V)$ resp. $C\ell(V,q)$ by defining F_0 to be the scalar multiples of the identity element, and defining F_k to be the subspace spanned by the product of up to k elements of V. Show that the map above is compatible with these filtrations and induces an isomorphism on filtration quotients.

Let $v, w \in V$. Then for $v, w \in V$ in $C\ell(V,q)$, we have the relation

$$(v+w)^2 = -q(v+w, v+w).$$

Expanding both sides we obtain

$$\begin{aligned}(v+w)^2 &= v^2 + vw + wv + w^2 \\ -q(v+w, v+w) &= -q(v,v) - 2q(v,w) - q(w,w)\end{aligned}$$

Using the relations $vv = -q(v,v)$ and $ww = -q(w,w)$, the relation

$$vw + wv = -2q(v,w) \tag{5.6}$$

follows. In particular, if v, w are perpendicular, i.e., $q(v,w) = 0$, then v and w anti-commute as elements of the Clifford algebra $C\ell(V,q)$.

From now on we assume that q is a positive definite inner product on V, which we will typically denote as $\langle\ ,\ \rangle$. We will suppress the form q in the notation, writing $C\ell(V)$ instead of $C\ell(V,q)$ for the associated Clifford algebra. The considerations above show that if $\{e_i\}_{i=1,\dots,n}$ is an orthonormal basis for V, then $C\ell(V)$ is the algebra generated by elements

$$e_1, \dots, e_n \qquad \text{subject to the relations} \qquad e_i^2 = -1 \quad e_ie_j + e_je_1 = 0 \text{ for } i \neq j.$$

Example 5.7. (Examples of Clifford algebras). Let $C\ell_n = C\ell(\mathbb{R}^n)$, where $\mathbb{R}^n$ is equipped with its standard inner product.

1. $C\ell_1$ is an algebra of dimension 2 generated by e_1 with the relation $e_1^2 = -1$. It follows that $C\ell_1$ is isomorphic to $\mathbb{C}$ by sending $e_1 \in C\ell_1$ to $i \in \mathbb{C}$.

2. $C\ell_2$ is an algebra of dimension $2^2 = 4$ generated by e_1, e_2 with relations $e_1^2 = e_2^2 = -1$ and $e_1e_2 = -e_2e_1$. It follows that $C\ell_2$ is isomorphic to the quaternions $\mathbb{H}$ by sending $e_1 \in \mathbb{R}^2 \subset C\ell_2$ to $i \in \mathbb{H}$ and $e_2 \in \mathbb{R}^2 \subset C\ell_2$ to $j \in \mathbb{H}$.

Let $v \in V$ be a non-zero vector in V. Then in $C\ell(V)$ the relation $vv = -||v||^2$ shows that v belongs to the group $C\ell(V)^\times$ of invertible elements of the Clifford algebra. Moreover, its inverse $v^{-1} \in C\ell(V)$ is given by $v^{-1} = -v/||v||^2$; in particular, if v is a unit vector, then $v^{-1} = -v$.

Definition 5.8. Let $\mathrm{Pin}(V)$ be the subgroup of $C\ell(V)^\times$ generated by unit vectors in V, and $\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap C\ell(V)^{\mathrm{ev}}$. The *twisted adjoint action*

$$\mathrm{Pin}(V) \times C\ell(V) \longrightarrow C\ell(V) \qquad \text{is defined by} \qquad (g,x) \mapsto \tau(g)xg^{-1},$$

where τ is the grading involution on $C\ell(V)$.

Lemma 5.9. *The twisted adjoint action restricts to an action ρ of* $\mathrm{Pin}(V)$ *on $V \subset C\ell(V)$. For a unit vector v the map $\rho(v)\colon V \to V$ is reflection at the hyperplane perpendicular to v.*

Proof. To calculate $\rho(v)w$ for a vector $w \in V$, decompose w in the form $w = w_1 + w_2$, where w_1 belongs to the span of v and w_2 is perpendicular to v. Then

$$\begin{aligned}\rho(v)(w) &= \tau(v)wv^{-1} = -vw(-v) = vwv = v(w_1 + w_2)v = vw_1v + vw_2v \\ &= vvw_1 - vvw_2 = -||v||^2w_1 - (-||v||^2)w_2 = -w_1 + w_2\end{aligned}$$

This is the reflection of $w = w_1 + w_2$ at the hyperplane $v^\perp$ which proves the lemma. $\square$

Proposition 5.10. *The homomorphisms*

$$\rho\colon \mathrm{Pin}(V) \to O(V) \qquad \textit{and} \qquad \rho\colon \mathrm{Spin}(V) \to SO(V)$$

are surjective with kernel $\{\pm 1\} \subset \mathrm{Spin}(V) \subset C\ell(V)$*. The space* $\mathrm{Spin}(V)$ *is connected for* $\dim V \geq 2$*, and hence* $\rho\colon \mathrm{Spin}(V) \to SO(V)$ *is the universal covering for* $\dim V \geq 3$.

Proof. It is well-known that every element of $O(V)$ can be written as a composition of reflections at some hyperplanes of V, and hence $\rho\colon \mathrm{Pin}(V) \to O(V)$ is surjective. Moreover, the product $v_1 \cdots v_k \in \mathrm{Pin}(V)$ of k unit vectors $v_i \in V$ maps to an element in $SO(V)$ if and only if k is even, in other words, if $v_1 \cdots v_k$ belongs to $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C\ell(V)^{\mathrm{ev}}$. This shows that also $\rho\colon \mathrm{Spin}(V) \to \mathrm{SO}(V)$ is surjective.

Let $x \in \mathrm{Pin}(V)$ be an element in the kernel of ρ. We will show that $x = \pm 1 \in C\ell(V)$ in three steps (see below for definitions of unitary and graded center):

(1) x is a unitary element of $C\ell(V)$;

(2) x belongs to the graded center of $C\ell(V)$;

(3) The graded center of $C\ell(V)$ consists of the scalar multiples of the unit element $1 \in C\ell(V)$, and hence the unitary elements of the graded center are just ± 1.

Concerning (1) we recall that a $*$-*algebra* is an algebra A together with an anti-involution $*\colon A \to A$; in other words, $*$ is a linear map which is an involution, i.e., $(a^*)^* = a$ for all $a \in A$, and an anti-automorphism, i.e., $(ab)^* = b^*a^*$ for $a, b \in A$. The prototypical example is the algebra $A = \mathrm{End}(V)$ of endomorphisms of a Hermitian vector space V, equipped with the anti-involution that takes an Endomorphism $a\colon V \to V$ to its adjoint a^*. An element a of a $*$-algebra is *unitary* if $a^*a = 1$ and $aa^* = 1$.

Let $*\colon C\ell(V) \to C\ell(V)$ be the anti-involution such that $v^* = -v$ for $v \in V \subset C\ell(V)$. For a unit vector $v \in V$ its inverse $v^{-1} = -v$ and hence $v^*v = (-v)v = v^{-1}v = 1$, which shows that every unit vector in V is a unitary element. Since the product of unitary elements is again unitary, it follows that every element of $\mathrm{Pin}(V) \subset C\ell(V)$ is unitary.

Concerning (2), let A be a $\mathbb{Z}/2$-graded algebra. The *(graded) commutator* of homogeneous elements $a, b \in A$ of degrees $|a|, |b| \in \mathbb{Z}/2$ is defined by

$$[a, b] := ab - (-1)^{|a||b|}ba.$$

For non-homogeneous elements the graded commutator is determined by requiring that $[\ ,\ ]$ is linear in each slot. The *(graded) center* of A is defined by

$$Z(A) := \{z \in A \mid [z, a] = 0\}.$$

In other words, a element $z \in A$ belongs to the center of A if $za = (-1)^{|z||a|}az$ for all $a \in A$, i.e., z commutes in the graded sense with any $a \in A$.

If $x \in \mathrm{Pin}(V)$ is in the kernel of $\rho\colon \mathrm{Pin}(V) \to O(V)$, then $v = \rho(x)v = \tau(x)vx^{-1}$ for any $v \in V$. This implies

$$vx = \tau(x)v = (-1)^{|x|}xv = (-1)^{|v||x|}xv,$$

where the first equation holds by definition of the grading involution τ, and the second holds since $|v| = 1$ for any $v \in V \subset C\ell(V)$. The above equation says that x commutes in the graded sense with any $v \in V$ (and shows that what is generally called the "twisted adjoint action" is the version of the adjoint action appropriate for $\mathbb{Z}/2$-graded algebras). Since the Clifford algebra $C\ell(V)$ is generated by V, this implies that x commutes with any element $a \in C\ell(V)$ in the graded sense and hence $x \in Z(C\ell(V))$.

To prove statement (3) let $x \in Z(C\ell(V))$ be an element of the graded center of $C\ell(V)$, which without loss of generality we may assume to be homogeneous. Let $e_1, \dots, e_n$ be an orthonormal basis of V. Using the fact that the products $e_{i_1}e_{i_2}\cdots e_{i_k}$ for $1 \le i_1 < i_2 < \cdots < i_k$, $k \ge 0$ form a basis for $C\ell(V)$, we can write x in the form

$$x = a + e_1 b,$$

where $a, b \in C\ell(V)$ are linear combinations of the basis elements $e_{i_1}e_{i_2}\cdots e_{i_k}$ that do not involve a factor of e_1. Using that $|x| = |a| = |b| + 1$ we obtain

$$xe_1 = ae_1 + e_1be_1 = (-1)^{|a|}e_1a + (-1)^{|b|}e_1e_1b = (-1)^{|a|}e_1a - (-1)^{|b|}b = (-1)^{|x|}(e_1a + b)$$

and $e_1x = e_1a + e_1e_1b = e_1a - b$. It follows that $[x, e_1] = (-1)^{|x|}2b$, and hence $b = 0$ since x belongs to the center. Thus x does not involve e_1. Similarly, it does not involve any other basis element, and hence it is a scalar. If x is unitary, then $1 = x^*x = x^2$ and hence $x = \pm 1$. □

5.2 Construction of spinor bundles

Definition 5.11. Let G be a Lie group, $\pi\colon P \to X$ a smooth principal G-bundle over a manifold X, and V a representation of G. Then the vector bundle

$$E(V) := P \times_G V \to X$$

is the *associated vector bundle* over X. Here $P \times_G V$ is the quotient of $P \times V$ given by the equivalence relation $(pg, v) \sim (p, gv)$ for $p \in P$, $g \in G$ and $v \in V$. The projection map $P \times_G V \to X$ is given by $[p, v] \mapsto \pi(p)$.

Let V, W be representations of G and let $f\colon V \to W$ be a G-equivariant linear map. Then f induces a vector bundle map

$$E(f)\colon E(V) = P \times_G V \longrightarrow P \times_G W = E(W) \qquad \text{given by} \qquad [p, v] \mapsto [p, f(w)].$$

Notice that the equivariance of f implies that the map $E(f)$ is well-defined.

Example 5.12. (Examples of associated vector bundles).

(i) Let $F \to X$ be an oriented real vector bundle of dimension n equipped with a metric, and let

$$\mathrm{SO}(F) := \{(x, f) \mid x \in X,\ f\colon \mathbb{R}^n \to F_x \text{ orientation preserving isometry}\} \longrightarrow X$$

be the oriented frame bundle of F. This is a principal $\mathrm{SO}(n)$-bundle. The right action

$$\mathrm{SO}(F) \times \mathrm{SO}(n) \longrightarrow \mathrm{SO}(F) \qquad \text{is given by} \qquad (x, f), g \mapsto (x, f \circ g).$$

Then the associated vector bundle $\mathrm{SO}(F) \times_{\mathrm{SO}(n)} \mathbb{R}^n$ is isomorphic to F. The isomorphism

$$\mathrm{SO}(F) \times_{\mathrm{SO}(n)} \mathbb{R}^n \xrightarrow{\cong} F \qquad \text{is given by} \qquad [(x, f), v] \mapsto f(v).$$

(ii) Assume that F has a spin structure given by the double covering $\mathrm{Spin}(F) \xrightarrow{q} \mathrm{SO}(F)$. Then $\mathrm{Spin}(F)$ is a principal $\mathrm{Spin}(n)$-bundle, and we can form the associated vector bundle $\mathrm{Spin}(F) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \to X$ where $\mathrm{Spin}(n)$ acts on $\mathbb{R}^n$ via the double covering map $\rho\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$. An isomorphism

$$\mathrm{Spin}(F) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \xrightarrow{\cong} \mathrm{SO}(F) \times_{\mathrm{SO}(n)} \mathbb{R}^n \qquad \text{is given by} \qquad [p, v] \mapsto [q(p), v].$$

(iii) Let Δ be a $\mathbb{Z}/2$-graded module over the Clifford algebra $C\ell_n$, and let $\mathrm{Spin}(F) \to X$ be the principal $\mathrm{Spin}(n)$-bundle of a spin vector bundle F. Then regarding Δ as a $\mathbb{Z}/2$-graded representation of the group $\mathrm{Spin}(n) \subset C\ell_n^\times$, we obtain the associated vector bundle

$$S_\Delta(F) := \mathrm{Spin}(F) \times_{\mathrm{Spin}(n)} \Delta \longrightarrow X$$

called the *spinor bundle* associated to F, Δ.

(iv) Specializing the previous example, if X is a Riemannian spin n-manifold, then

$$S_\Delta := \mathrm{Spin}(T^*X) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \longrightarrow X$$

is the *spinor bundle of* X associated to the $C\ell_n$-module Δ.

Lemma 5.13. *Let Δ be a $\mathbb{Z}/2$-graded module over $C\ell_n$. Then the map*

$$c\colon \mathbb{R}^n \otimes \Delta \longrightarrow \Delta \qquad v \otimes \lambda \mapsto v\lambda,$$

given by the left $C\ell_n$-module structure of Δ is $\mathrm{Spin}(n)$-equivariant, where $\mathrm{Spin}(n)$ acts on $\mathbb{R}^n$ via the double covering map $\rho\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ and on Δ via the embedding $\mathrm{Spin}(n) \hookrightarrow C\ell_n$.

Corollary 5.14. *Let $F \to X$ be a vector bundle with metric and spin structure of dimension n and let* $\mathrm{Spin}(F) \to X$ *be the corresponding spin frame bundle. Then the* $\mathrm{Spin}(n)$*-equivariant map c induces a map of associated vector bundles*

$$E(c)\colon E(\mathbb{R}^n \otimes \Delta) = E(\mathbb{R}^n) \otimes E(\Delta) \longrightarrow E(\Delta).$$

Via the vector bundle isomorphism discussed above, this yields a vector bundle map

$$c\colon F \otimes S_\Delta(F) \longrightarrow S_\Delta(F),$$

that we will refer to as Clifford multiplication.

Proof of lemma 5.13. The group $\mathrm{Spin}(n)$ is a subgroup of $\mathrm{Pin}(n)$, and we observe that the action of $\mathrm{Spin}(n)$ on $\mathbb{R}^n$ and Δ extends to $\mathrm{Pin}(n)$ via the double covering map $\rho\colon \mathrm{Pin}(n) \to O(n)$ resp. the inclusion $\mathrm{Pin}(n) \subset C\ell_n$. Let us check whether $c\colon \mathbb{R}^n \otimes \Delta \to \Delta$ is $\mathrm{Pin}(n)$-equivariant. We recall that $\mathrm{Pin}(n)$ is generated by unit vectors $v \in \mathbb{R}^n$. So let $v \in S^{n-1}$ and $w \otimes \lambda \in \mathbb{R}^n \otimes \Delta$. Then v acts on $w \otimes \lambda$ by

$$v(w \otimes \lambda) = (\tau(v)wv^{-1}) \otimes (v\lambda) = (-vwv^{-1}) \otimes (v\lambda).$$

Applying c, we obtain

$$c(v(w \otimes \lambda)) = c((-vwv^{-1}) \otimes (v\lambda)) = -vwv^{-1}v\lambda = -vw\lambda \in \Delta,$$

while $v \cdot c(w \otimes \lambda) = v(w\lambda) = vw\lambda \in \Delta$. This shows that the map c is *not* $\mathrm{Pin}(n)$-equivariant, due to the minus sign. However, it is equivariant for the subgroup $\mathrm{Spin}(n) \subset \mathrm{Pin}(n)$ whose elements are products in $C\ell_n$ of an *even* number of unit vectors. □

Definition 5.15. Let Δ be a $\mathbb{Z}/2$-graded module over the Clifford algebra $C\ell_n$, and let X be a Riemannian spin manifold of even dimension n. Then the *Dirac operator on X associated to* Δ is the operator

$$D_\Delta\colon \Gamma(S_\Delta) \xrightarrow{\nabla} \Gamma(T^*X \otimes S_\Delta) \xrightarrow{c} \Gamma(S_\Delta)$$

where $S_\Delta = \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \Delta$ is the spinor bundle associated to Δ, ∇ is the connection on S_Δ induced by the Levi-Civita connection on TX, and c is Clifford multiplication.

Definition 5.16. Let X be a compact space and let $\pi\colon V \to X$ be a vector bundle of even dimension n equipped with a metric and a spin structure. Let $U_\Delta(V) \in K_c(V)$ be the element given by the odd vector bundle endomorphism

$$\pi^* S_\Delta(F) \xrightarrow{c} \pi^* S_\Delta(F) \qquad \text{given by} \qquad (x, v \in F_x, \psi \in S_x) \mapsto (x, v, c(v, \psi)).$$

The two constructions described in the definition above yields our goal of constructing the Dirac operator on even dimensional spin manifolds, and the K-theory orientation class for even dimensional spin vector bundles, provided we make the correct choice for the graded Clifford module Δ. It is clear that we should use an *irreducible module* Δ, since the direct sum of say two copies of Δ in the construction of $U_\Delta(V)$ would lead to $U_{\Delta\oplus\Delta}(V) = 2U_\Delta(V) \in K_c(V)$ which would not be an orientation class.

Proposition 5.17. *If Δ is an irreducible $\mathbb{Z}/2$-graded module over $C\ell_{2k}$, then the class $U_\Delta(V)$ constructed above is a K-theory orientation for real spin vector bundles V of dimension $2k$.*

Exercise 5.18. Prove the above statement using the following steps.

1. Show that the graded tensor product $C\ell_n \otimes C\ell_{n'}$ is isomorphic as $\mathbb{Z}/2$-graded algebra to $C\ell_{n+n'}$.
2. Show that if Δ, Δ' are irreducible $\mathbb{Z}/2$-graded complex modules over $C\ell_{2k}$ resp. $C\ell_{2k'}$, then the graded tensor product $\Delta \otimes \Delta'$ is an irreducible $\mathbb{Z}/2$-graded complex module over $C\ell_{2k} \otimes C\ell_{2k'} \cong C\ell_{2k+2k'}$.
3. Prove the proposition above for $k = 1$. Hint: It suffices to show that the restriction of U_Δ^V to each fiber $V_x \cong \mathbb{R}^2$ gives a generator of $K_c(\mathbb{R}^2)$.
4. Prove the statement for general k by using part (2) to reduce to the case $k = 1$.

This leaves us with the question about the number of irreducible modules, addressed by the next proposition.

Proposition 5.19. *There are two irreducible $\mathbb{Z}/2$-graded complex modules Δ over $C\ell_{2k}$ up to isomorphism. The dimension of both of them is 2^k.*

Example 5.20. We have observed before that the Clifford algebra $C\ell_n$ is isomorphic to the quaternions $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, but we want to pay attention to the $\mathbb{Z}/2$-grading now, and would like to set up the isomorphism such that $C\ell_2^0$ corresponds to $\mathbb{C} \subset \mathbb{H}$ and $C\ell_2^1$ corresponds to $j\mathbb{C} \subset \mathbb{H}$. We observe that the standard basis elements $e_1, e_2 \in \mathbb{R}^2 \subset C\ell_2$ have degree 1, while e_1e_2 and the unit $1 \in C\ell_2$ have degree 0. Hence defining

$$C\ell_2 \longrightarrow \mathbb{H} \qquad \text{by} \qquad e_1 \mapsto j,\ e_2 \mapsto ji$$

determines a graded algebra isomorphism. In particular, $\mathbb{H}$ is a graded left module over $\mathbb{H}$. In fact, this is a *complex module* if we are careful with the definition of the complex structure on $\mathbb{H}$: it commutes with the left action of the algebra on $\mathbb{H}$ if we us *right* multiplication by $z \in \mathbb{C}$ to define the complex structure on $\mathbb{H}$.

Lemma 5.21. *Let $e_1, \dots, e_{2k}$ be the standard basis of $\mathbb{R}^{2k}$. Then the element $\omega := i^k e_1 \cdots e_k \in C\ell_{2k} \otimes \mathbb{C}$ has the properties*

$$\omega^2 = 1 \qquad \text{and} \qquad \omega v = -v\omega \text{ for all } v \in \mathbb{R}^{2k} \subset C\ell_{2k}.$$

In particular, the grading involution on $C\ell_{2k} \otimes \mathbb{C}$ is an inner involution, given by conjugation by the element ω.

Lawson and Michelsohn refer to ω as *complex volume element* [LM, Ch. I, equation (5.12)].

Corollary 5.22. *Every (ungraded) module Δ over $C\ell_{2k}$ has a preferred grading involution, given by multiplication by $\omega \in C\ell_{2k}$.*

Combing this result with Proposition 5.19 we conclude the following statement.

Corollary 5.23. *There is a unique irreducible $\mathbb{Z}/2$-graded complex module Δ over $C\ell_{2k}$ whose $\mathbb{Z}/2$-grading is given by multiplication by $\omega \in C\ell_{2k} \otimes \mathbb{C}$.*

Lemma 5.24. *There is a commutative diagram of group homomorphisms*

$$\begin{tikzcd} S^1 \arrow[r, "\phi", "\cong"'] \arrow[d, two heads, "\pi"'] & \mathrm{Spin}(2) \arrow[d, two heads, "\rho"'] \\ S^1 \arrow[r, "\psi", "\cong"'] & \mathrm{SO}(2) \end{tikzcd}$$

Here the vertical maps are double coverings with π defined by $\pi(z) := z^2$, and ρ is the double covering map of Lemma 5.9. The homomorphism ϕ sends $z = a + ib \in S^1$ to $\phi(a + ib) = a + be_1e_2 \in \mathrm{Spin}(2) \subset C\ell_2$, and ψ sends $e^{i\theta} \in S^1$ to the isometry $\psi(\theta) \in \mathrm{SO}(2)$ given by rotation by θ.

Proof. We note that for $a + ib = e^{i\theta} \in S^1$

$$\phi(a + ib) = a + be_1e_2 = -ae_1^2 + be_1e_2 = e_1(-ae_1 + be_2) \in C\ell_2$$

is the product in the Clifford algebra $C\ell_2$ of the two unit vectors e_1 and $-ae_1 + be_2$. Hence it belongs to $\mathrm{Pin}(2)$, the group generated by products of unit vectors in $C\ell_2^\times$; in fact it belongs to $\mathrm{Spin}(2) = \mathrm{Pin}(2) \cap C\ell_2^{\mathrm{ev}}$. It follows that

$$\rho(\phi(a + ib)) = \rho(e_1)\rho(v) \in \mathrm{SO}(2), \qquad \text{for } v := -ae_1 + be_2 \in S^1$$

the composition of the reflection at $v^\perp$ followed by reflection at $e_1^\perp$. Arguing either geometrically (by drawing the situation), or by using the formula for the isometry given by the reflection at a line in $\mathbb{R}^2$, it is easy to check that $\rho(e_1)\rho(v)$ is rotation by 2θ. This proves the commutativity of the diagram. Since the bottom horizontal map is an isomorphism, and the vertical maps are both double coverings, it follows that the top horizontal map is an isomorphism. □

Corollary 5.25. *Let $\Delta = \Delta^+ \oplus \Delta^-$ be the irreducible complex $C\ell_2$-module (which has complex dimension 2) whose grading involution is given by multiplication by $\omega = ie_1e_2 \in C\ell_2 \otimes \mathbb{C}$. Then as complex 1-dimensional representations of $S^1 \cong \mathrm{Spin}(2)$, $\Delta^\pm \cong \mathbb{C}_{\mp 1}$(where $\mathbb{C}_k$ is the representation of S^1 given by letting $z \in S^1$ act on $\mathbb{C}$ by multiplication by z^k).*

Proof. The grading involution on Δ is given by $\lambda \mapsto \omega\lambda = ie_1e_2\lambda$, and hence $\lambda \in \Delta^\pm$ if and only if $ie_1e_2\lambda = \pm\lambda$. It follows that for $z = a + ib \in S^1$ and $\lambda \in \Delta^\pm$,

$$\begin{aligned}\phi(z)\lambda = \phi(a+ib)\lambda = (a + be_1e_2)\lambda &= a\lambda - ibie_1e_2\lambda \\ &= a\lambda - ib(\pm\lambda) = (a \mp ib)\lambda = z^\mp\lambda\end{aligned}$$

□

Corollary 5.26. *Let $V \to X$ be complex line bundle with spin structure and let $\mathrm{Spin}(V) \to X$ be the associated principal bundle with structure group $\mathrm{Spin}(2) \cong S^1$. Let $V^{1/2}$ be the square root of V determined by the spin structure on V; in other words, $V^{1/2}$ is the associated vector bundle $V^{1/2} = \mathrm{Spin}(V) \times_{\mathrm{Spin}(2)} \mathbb{C}_1$. Then there are isomorphisms of complex line bundles*

$$\mathrm{Spin}(V) \times_{\mathrm{Spin}(2)} \Delta^+ \cong V^{-1/2} \qquad \mathrm{Spin}(V) \times_{\mathrm{Spin}(2)} \Delta^- \cong V^{1/2}.$$

Remark 5.27. An important reference for the use of modules over Clifford algebras for the construction of K-theory orientations is the 1963 paper *Clifford modules* by Atiyah-Bott-Shapiro [ABS]. In section 5 of that paper they analyze in detail modules over real and complex Clifford algebras. In particular, their table 2 on page 12 shows the Grothendieck groups $M(C\ell_n)$ and $M^c(C\ell_n)$ of real (resp. complex) graded modules over the Clifford algebra $C\ell_n$ (which they denote by C_n). For example, our statement that there are *two* irreducible complex graded modules over $C\ell_{2k}$ even means that the Grothendieck group $M^c(C\ell_{2k}) \cong \mathbb{Z} \oplus \mathbb{Z}$ as shown in the table.

They distinguish the two irreducible complex $\mathbb{Z}/2$ graded $C\ell_{2k}$-modules $M = M^{\mathrm{ev}} \oplus M^{\mathrm{odd}}$ by the action of the element $e_1 \cdots e_{2k} \in C\ell_{2k}$ on M^{ev}, calling M an *ϵ-module* if $e_1 \cdots e_{2k}$ acts on M^{ev} by multiplication by $\epsilon \in \mathbb{C}$ (see p. 16, paragraph after Theorem 6.10). Since $(e_1 \cdots e_{2k})^2 = (-1)^k$, the possible values of ϵ is $\pm i^k$, distinguishing the two irreducible modules. They work with the module $\mu_k^c \in M^c(C\ell_{2k})$ that is the irreducible i^k-module. Earlier, in Proposition 5.11 they show that the exterior algebra $\Lambda(\mathbb{C}^k)$ is an irreducible $\mathbb{Z}/2$-module over $C\ell_{2k}$, which is an $(-i)^k$-module.

In terms of the complex volume element $\omega_{\mathbb{C}} := i^k e_1 \cdots e_{2k}$ that we use to distinguish the two irreducible modules, it means that $\omega_{\mathbb{C}}$ acts on the even part M^{ev} of such a module M

- by $+1$ for $M = \Lambda(\mathbb{C}^k)$ (with its standard $\mathbb{Z}/2$-grading $\Lambda = \Lambda^{\mathrm{ev}} \oplus \Lambda^{\mathrm{odd}}$), and
- by $(-1)^k$ for the module $M = \mu_{2k}^c$.

5.3 Scalar curvature and index theory

Let X be a Riemannian n-manifold. Then the *scalar curvature* is a smooth function $s\colon X \to \mathbb{R}$ which controls the volume of small balls. It is usually defined as a contraction of the Riemannian curvature tensor, but here we describe $s(x)$, the scalar curvature at the point $x \in X$, in terms of the volume of balls

$$B_r(x, X) := \{y \in X \mid \mathrm{dist}(x, y) < r\}$$

of radius r centered at x. The volume $\mathrm{vol}\, B_r(x, M)$ of this ball can be compared to the volume $\mathrm{vol}\, B_r(0, \mathbb{R}^n)$ of the ball of the radius r in Euclidean space $\mathbb{R}^n$. The scalar curvature $s(x)$ then appears as a coefficient in the Taylor expansion of the quotient of these two volumes as a function of r:

$$\frac{\mathrm{vol}\, B_r(x, M)}{\mathrm{vol}\, B_r(0, \mathbb{R}^n)} = 1 - \frac{s(x)}{6(n+2)} r^2 + \dots$$

In particular, if the volume of a small balls of radius r around x is *smaller* than the volume of the balls of the same radius in $\mathbb{R}^n$, then the volume ratio is smaller than 1, hence the coefficient of r^2 must be negative, and hence $s(x) > 0$. This is the case for example for spheres with their standard metric. If the volume of small balls in X is larger, for example if X is a hyperbolic space (mountain pass), then $s(x) < 0$.

Theorem 5.28. (Lichnerowicz, 1963) *Let X be a closed Riemannian spin manifold of dimension $4k$ with positive scalar curvature (i.e, $s(x) > 0$ of all $x \in X$). Then $\widehat{A}(X) = 0$.*

This result shows that the $\widehat{A}$-genus is an obstruction for the existence of a positive scalar curvature metric on spin manifolds. For example, the Kummer surface K is a spin manifold with $\widehat{A} = 2$, and hence there is no positive scalar curvature metric on K.

Proof. We recall that the Dirac operator on X is a first order differential operator D acting on $\Gamma(S)$, the sections of the spinor bundle. The principal symbol of the second order operator D^2 by construction agrees with the principal symbol of the rough Laplacian $\nabla^*\nabla = \Delta\colon \Gamma(S) \to \Gamma(S)$ (see Example 5.2). This means that these two second order differential operators can only differ by a differential operator of order 1. Lichnerowicz showed that the difference between these two operators is actually of order 0, and simply given by multiplication by the function $s/4$ (where $s \in C^\infty(X)$ is the scalar curvature function). The equation

$$D^2 = \nabla^*\nabla + \frac{s}{4}$$

is known as the Lichnerowicz formula (also as Weizenböck-Bochner-Lichnerowicz formula; Weizenböck and Bochner proved analogous formulas comparing natural second order operators).

We claim that $s(x) > \epsilon$ for all $x \in X$ implies that the kernel of D is trivial. To prove this, let

$$\langle \psi, \phi \rangle := \int_X \langle \psi(x), \phi(x) \rangle_S \operatorname{vol}_X$$

be the L^2 inner product of spinors $\psi, \phi \in \Gamma(S)$, and let $||\psi||^2 = \langle \psi, \psi \rangle$ be the corresponding L^2-norm. Then for $\psi \in \ker D$

$$\begin{aligned} 0 = (D\psi, D\psi) = (D^2\psi, \psi) &= (\nabla^*\nabla\psi, \psi) + (\frac{s}{4}\psi, \psi) \\ &\geq (\nabla\psi, \nabla\psi) + \frac{\epsilon}{4}(\psi, \psi) \geq ||\nabla\psi||^2 + \frac{\epsilon}{4}||\psi||^2 \end{aligned}$$

implies $||\psi|| = 0$ and hence $\psi = 0$.

Since the kernel of D is trivial, its super dimension

$$\operatorname{sdim} \ker D = \dim \ker D^+ - \dim \ker D^- = \dim \ker D^+ - \dim \operatorname{coker} D^+ = \operatorname{index}(D^+)$$

is zero. Since $\operatorname{index}(D^+) = \widehat{A}(X)$ by the index theorem for the Dirac operator, it follows that $\widehat{A}(X) = 0$. □

Theorem 5.29. (Hitchin, 1974) *Let X be a closed spin manifold of dimension n with positive scalar curvature metric. Then the Atiyah invariant*

$$\alpha(X) \in \mathrm{KO}^{-n}(\mathrm{pt}) = \begin{cases} \mathbb{Z} & n \equiv 0 \mod 4 \\ \mathbb{Z}/2 & n \equiv 1, 2 \mod 8 \\ 0 & \textit{otherwise} \end{cases}$$

vanishes.

Here $\mathrm{KO}^*(X)$ is the real K-theory which is defined completely analogous to complex K-theory, but using real vector bundles instead of complex vector bundles. Real K-theory is 8-periodic. Using modules over Clifford algebras as in the construction of the K-orientation $U^K(V)$ (see Definition 5.16 and Proposition 5.17), Atiyah has constructed KO-theory orientations for real vector bundles V of dimension $\equiv 0 \mod 8$ with spin structure. This can be used to define an umkehr map

$$p_! \colon \mathrm{KO}^0(X) \longrightarrow \mathrm{KO}^{-n}(\mathrm{pt})$$

for closed spin n-manifolds X. The Atiyah invariant $\alpha(X) \in \mathrm{KO}^{-n}(\mathrm{pt})$ is the image of the unit $1 \in \mathrm{KO}^0(X)$ (represented by the trivial real line bundle) under the umkehr map $p_!$. The Atiyah invariant α-invariant $\alpha(X)$ has a neat description as the *Clifford index* of the *Clifford linear Dirac operator* on X [LM, Ch. III, section 10]. With that interpretation of $\alpha(X)$, the argument above for the vanishing of $\widehat{A}(X)$, the index of the Dirac operator on X, immediately generalizes to prove the vanishing of the Clifford index, thus proving Hitchin's generalization (Hitchin used a different argument).

Remark 5.30. For $n \equiv 0 \mod 4$ the Atiyah invariant $\alpha(X)$ for a closed spin n-manifold X agrees with $\widehat{A}(X)$ (up to a factor of 2 for $n \equiv 4 \mod 8$. For $n \equiv 1, 2 \mod 8$, $n \geq 9$, there are manifolds Σ^n of dimension n homeomorphic to S^n with $\alpha(\Sigma^n) \neq 0$, and hence Σ^n does not admit a positive scalar curvature metric. The homotopy spheres Σ^n in Ω_n^{fr} correspond via the map (4.49) to elements in the n-th stable homotopy group of spheres constructed by Adams.

Theorem 5.31. (Stolz, 1990) *Let X be a closed simply connected spin manifold of dimension $n \geq 5$. Then X admits a positive scalar curvature metric if and only if $\alpha(X) = 0$.*

5.4 A general index theorem

Let $D\colon \Gamma(E) \to \Gamma(F)$ be a differential operator on a closed manifold X with principal symbol

$$\sigma^D\colon \pi^*E \longrightarrow \pi^*F.$$

Here $\pi\colon T^*X \to X$ is the cotangent bundle. If D is elliptic, then for each non-zero cotangent vector $\xi \in T_x^*X$ the linear map $\sigma_\xi^D\colon E_x \to F_x$ is an isomorphism, and hence we obtain a K-theory element

$$\boldsymbol{\sigma}(D) := [\pi^*E, \pi^*F; \sigma^D] \in K_c(T^*M) = K_c(TM).$$

Lemma 5.32. *The index of D depends only on $\boldsymbol{\sigma}(D) \in K_c(TM)$.*

Definition 5.33. (Short hand notation for Thom isomorphisms). Let $\pi\colon V \to X$ be a smooth vector bundle and let $i\colon X \to V$ be the zero section. Assume either that

(i) V is a complex vector bundle of complex dimension k (which guarantees the existence of the K-theory orientation $U^K_{\mathbb{C}}(V)$), or that

(ii) V is a real vector bundle of dimension $2k$ with a spin structure (which guarantees the existence of the K-theory orientation $U^K(V)$).

Let

$$i_!\colon K_c(X) \xrightarrow{\cong} K_c(V)$$

be the resulting Thom isomorphism given by multiplication by the orientation class, and let

$$\pi_!\colon K_c(V) \xrightarrow{\cong} K_c(X)$$

be its inverse. If there is need to emphasize which orientation class on V is used, we write $i_!^{\mathbb{C}}$, $\pi_!^{\mathbb{C}}$ in case (i) and $i_!^{\mathrm{spin}}$, $\pi_!^{\mathrm{spin}}$ in case (ii).

Let $f\colon X \hookrightarrow Y$ be a proper embedding of manifolds with normal bundle $N \to X$ satisfying assumptions (i) or (ii) above. Let

$$f_!\colon K_c(X) \longrightarrow K_c(Y)$$

be the composition of $i_!\colon K_c(X) \to K_c(N)$ followed by the map $K_c(N) \to K_c(Y)$ obtained by identifying a tubular neighborhood of X in Y with N.

Let X be a closed n-manifold and let $f\colon X \hookrightarrow \mathbb{R}^{n+k}$ be an embedding with normal bundle N. Then the differential

$$Tf\colon TX \longrightarrow T\mathbb{R}^{n+k}$$

is a proper embedding. Its normal bundle is the pullback of $N \oplus N \cong N \otimes_{\mathbb{R}} \mathbb{C}$ via the projection map $TX \to X$. The complex structure on the normal bundle then gives a homomorphism

$$Tf_! = Tf_!^{\mathbb{C}}\colon K_c(TX) \longrightarrow K_c(T\mathbb{R}^{n+k})$$

Thinking of $q\colon T\mathbb{R}^{n+k} = \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k} = \mathbb{C}^{n+k} \to \mathrm{pt}$ as a complex vector bundle, let

$$q_!\colon K_c(T\mathbb{R}^{n+k}) \to K(\mathrm{pt})$$

be the inverse of the Thom isomorphism.

Definition 5.34. The *topological index* of an elliptic differential operator D on a closed manifold X is the image of $\boldsymbol{\sigma}(D) \in K_c(TX)$ under the composition

$$\text{top-ind}\colon K_c(TX) \xrightarrow{Tf_!} K_c(T\mathbb{R}^N) \xrightarrow{q_!} K(pt) = \mathbb{Z}$$

Theorem 5.35. (The index theorem for general elliptic operators). *Let P be an elliptic operator on a compact n-manifold X with principal symbol $\boldsymbol{\sigma}(P) \in K_c(TX)$. Then*

$$\mathrm{index}(P) = \textit{top-ind}(\boldsymbol{\sigma}(P)).$$

Our next goal is to show that the above result implies the index theorem for the Dirac operator that we stated in section 3. More precisely, we will prove the K-theory version of the Index Theorem 3.1 according to which

$$\mathrm{index}(D_E^+) = (-1)^m p_!([E]) \in K(\mathrm{pt}) = \mathbb{Z}. \tag{5.36}$$

Here $D_E\colon \Gamma(S \otimes E) \to \Gamma(S \otimes E)$ is the Dirac operator on a compact n-manifold, $n = 2m$, twisted by a complex vector bundle $E \to X$, and $p_!\colon K(X) \to K(\mathrm{pt})$ is the umkehr map associated to the projection map $p\colon X \to \mathrm{pt}$ (see Definition 3.27). We recall that the construction of $p_!$ is based on choosing an embedding $f\colon X \hookrightarrow \mathbb{R}^{n+k}$, k even, with normal bundle N. Using the notation from Definition 5.33, the map $p_!$ is the composition

$$p_!\colon K(X) \xrightarrow{f_!^{\mathrm{spin}}} K_c(\mathbb{R}^{n+k}) \xrightarrow[\cong]{(\iota_!)^{-1}} K(\mathrm{pt}) = \mathbb{Z}$$

Here $\mathbb{R}^{n+k}$ is viewed as a vector bundle over pt, the map $\iota\colon \mathrm{pt} \to \mathbb{R}^{n+k}$ is the inclusion of the zero-section, and $\iota_!\colon K(\mathrm{pt}) \xrightarrow{\cong} K_c(\mathbb{R}^{n+k})$ is the Thom isomorphism aka suspension isomorphism aka Bott isomorphism.

Proof of Theorem 3.1/equation (5.36). We recall from 5.2(1) that the principal symbol of the Dirac operator D^+ is given by

$$\pi^* S^+ \xrightarrow{ic} \pi^* S^-$$

where $\pi\colon T^*X \to X$ is the cotangent bundle and c is the Clifford multiplication map. This represents an element in $K_c(T^*X)$, which after a homotopy in order to remove the factor i is precisely the orientation class

$$U^K(T^*X) = [\pi^*S^+, \pi^*S^-; c] \in K_c(T^*X).$$

In other words, the principal symbol class $\boldsymbol{\sigma}(D)$ of the Dirac operator is the K-theory orientation class $U^K(T^*X)$ and More generally, In other words, the princ

Let $f\colon X \hookrightarrow \mathbb{R}^{n+k}$ be an embedding and consider the following commutative diagram of embeddings

$$\begin{tikzcd}
X \arrow[r, hook, "f"] \arrow[d, hook, "i"'] & \mathbb{R}^{n+k} \arrow[d, hook, "j"] & \mathrm{pt} \arrow[l, hook', "\iota"'] \arrow[dl, hook, "T\iota"] \\
TX \arrow[r, hook, "Tf"] & T\mathbb{R}^{n+k} &
\end{tikzcd}$$

Applying the index theorem for elliptic operators 5.35 to D_E^+, we obtain

$$\begin{aligned}
\mathrm{index}(D_E^+) &= \text{top-ind}(\boldsymbol{\sigma}(D_E^+)) = (j\iota)_!^{-1}(Tf)_!\boldsymbol{\sigma}(D_E^+) = \iota_!^{-1}j_!^{-1}(Tf)_!i_!([E]) \\
&= \iota_!^{-1}j_!^{-1}j_!f_!([E]) = \iota_!^{-1}f_!([E]) = p_!([E])
\end{aligned} \tag{5.37}$$

This seems to show that in fact $\mathrm{index}(D_E^+) = p_!([E])$ *without* the factor of $(-1)^m$ claimed in equation (5.36). However, we have been careless in the above calculation with regards to *which* K-orientation ($U_{\mathbb{C}}^K$ or U^K, see 5.33) we use for the normal bundles of the embeddings i, f, Tf, and $T\iota$. Let us take a close look at the normal bundles of these embeddings.

(1) The normal bundle of the embedding $i\colon X \hookrightarrow TX$ is the tangent bundle TX which is assumed to be even dimensional and equipped with a spin structure, but in general TX does not have a complex structure. Hence the K-orientation $U^K(TX)$ is the only option, and consequently, the relevant umkehr map is $i_!^{\mathrm{spin}}$.

(2) The normal bundle N of the embedding $f\colon X \hookrightarrow \mathbb{R}^{n+k}$ has dimension k, which is even by assumption. The bundle isomorphism $TX \oplus N \cong \underline{\mathbb{R}}^{n+k}$ and the spin structure on TX induces a spin structure on N, but in general N does not have complex structure. Hence we only have the K-orientation $U^K(N)$ and the umkehr map $f_!^{\mathrm{spin}}$ at our disposal.

(3) The normal bundle of the embedding $Tf\colon \hookrightarrow TX \hookrightarrow T\mathbb{R}^{n+k}$ as discussed above is the pullback of $N \oplus N \cong N \otimes_{\mathbb{R}} \mathbb{C}$ via the projection maps $\pi\colon TX \to X$. The spin structure on N determines a spin structure on $N \oplus N$. This bundle also has a complex structure (via the isomorphism $N \oplus N \cong N \otimes_{\mathbb{R}} \mathbb{C}$), and hence this vector bundle has both orientations $U^K(N \oplus N)$ (due to the spin structure) and $U^K_{\mathbb{C}}(N \oplus N)$ (due to the complex structure), and we have both umkehr maps, $Tf_!^{\mathrm{spin}}$ and $Tf_!^{\mathbb{C}}$, at our disposal.

(4) The same arguments as for Tf apply to see that for the embedding $T\iota$ both umkehr maps, $T\iota_!^{\mathrm{spin}}$ and $T\iota_!^{\mathbb{C}}$ are defined.

We note that we need to use *both* umkehr maps, $Tf_!^{\mathrm{spin}}$ *and* $Tf_!^{\mathbb{C}}$; the latter is used in the definition of top-ind, while the former is needed to make the argument that $Tf \circ i = j \circ f$ implies $Tf_! \circ i_! = j_! \circ f_!$! So we need to know how these two umkehr maps compare. This comparison is provided by the following result.

Lemma 5.38. *Let $N \to X$ be spin vector bundle of dimension 2ℓ. Let $U^K(N \oplus N) \in K_c(N \oplus N)$ be the K-orientation determined by the spin structure on $N \oplus N$, and let $U^K_{\mathbb{C}}(N \oplus N) \in K_c(N \oplus N)$ be the K-orientation determined by the complex structure on $N \oplus N \cong N \otimes_{\mathbb{R}} \mathbb{C}$. Then $U^K(N \oplus N) = (-1)^\ell U^K_{\mathbb{C}}(N \oplus N)$. In particular, if $f\colon X \hookrightarrow Y$ is an embedding with normal bundle N, then $f_!^{\mathrm{spin}} = (-1)^\ell f_!^{\mathbb{C}}$.*

Let us now repeat the calculation (5.37), but carefully differentiating between umkehr maps, adding the super script $\mathbb{C}$ to indicate umkehr maps using the complex structure. No superscript for an umkehr maps means that it is given using the spin structure. For $n = 2m$, $k = 2\ell$, and using the lemma above we obtain:

$$\begin{aligned}
\operatorname{index}(D_E^+) &= \text{top-ind}(\boldsymbol{\sigma}(D_E^+)) = ((T\iota)_!^{\mathbb{C}})^{-1} (Tf)_!^{\mathbb{C}} \boldsymbol{\sigma}(D_E^+) \\
&= (-1)^{m+\ell} (T\iota)_!^{-1}\ (-1)^\ell (Tf)_! \boldsymbol{\sigma}(D_E^+) \\
&= (-1)^m \iota_!^{-1} j_!^{-1} (Tf)_! i_!([E]) = (-1)^m \iota_!^{-1} j_!^{-1} j_! f_!([E]) \\
&= (-1)^m \iota_!^{-1} f_!([E]) = (-1)^m p_!([E])
\end{aligned}$$

□

Let $W \to X$ be a complex vector bundle of dimension n, and let $W_{\mathbb{R}}$ the real vector bundle of dimension $2n$ obtained by forgetting the complex structure. Then $W_{\mathbb{R}}$ has a *canonical orientation* determined by the complex structure on W constructed as follows. Let $w_1, \dots, w_n$ be a $\mathbb{C}$-basis of the fiber W_x for some point $x \in X$. Then

$$w_1, iw_1, w_2, iw_2, \dots, w_n, iw_n$$

is an ordered $\mathbb{R}$-basis of W_x and hence provides the real vector space W_x with an orientation. This orientation is independent of the choice of the ordered $\mathbb{C}$-basis $w_1, \dots, w_n$.

Let $V \to X$ be a real vector bundle of dimension n, and let $V \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Then the real vector bundle $(V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ is isomorphic to $V \oplus V$ via the vector bundle isomorphism

$$V \oplus V \xrightarrow{\cong} (V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}} \qquad \text{given by} \qquad (v, w) \mapsto v \otimes 1 + w \otimes i. \tag{5.39}$$

Lemma 5.40. *Let $V \to X$ be an oriented real vector bundle of dimension n. Then the orientation on $V \oplus V$ agrees with the canonical orientation on $(V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ if and only if $\frac{n(n-1)}{2}$ is even. In particular the associated orientation classes are related by*

$$U^H(V \oplus V) = (-1)^{\frac{n(n-1)}{2}} U^H((V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}) \in H_c^{2n}(V \oplus V).$$

Remark 5.41. Let X be an oriented n-manifold with tangent bundle $\pi \colon TX \to X$. Then the total space TX is a $2n$-manifold whose tangent bundle is isomorphic to $\pi^*(TX \oplus TX)$. Hence there are two natural orientations on the tangent bundle of TX (and hence on the manifold TX):

1. the orientation on $TX \oplus TX$ given by the orientation on each summand, and

2. the canonical orientation on the complex vector bundle $TX \otimes_{\mathbb{R}} \mathbb{C} \cong TX \oplus TX$.

By the lemma above, these two orientations differ by the factor $(-1)^{\frac{n(n-1)}{2}}$. This contrasts with a statement in the book *Spin geometry* where it is claimed on p. 256 that these orientations differ by the factor $(-1)^{\frac{n(n+1)}{2}}$, explaining why that factor shows up in their index formula (13.24) in Theorem 13.8.

The main case of interest to us is n even, in which case $(-1)^{\frac{n(n-1)}{2}}$ is equal to $(-1)^{\frac{n(n+1)}{2}}$. However, for n odd these expressions differ by a minus sign.

Proof of Lemma 5.40. Let $x \in X$ and let $v_1, \dots, v_n$ an ordered basis for the fiber V_x representing the orientation of the vector bundle V. Identifying $V_x \oplus V_x$ with $V_x \otimes_{\mathbb{R}} \mathbb{C}$ via the isomorphism (5.39),

1. the orientation on $V_x \otimes_{\mathbb{R}} \mathbb{C}$ induced by the orientation on V_x is given by the ordered basis
$$v_1, v_2 \dots, v_n, iv_1, iv_2, \dots, iv_n,$$

2. the canonical orientation on $V_x \otimes_{\mathbb{R}} \mathbb{C}$ is given by the ordered basis
$$v_1, iv_1, \dots, v_n, iv_n.$$

Rearranging the first sequence of symbols to obtain the second sequence requires $(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1}{2}$ transpositions. Hence the two orientations agree if and only if $\frac{n(n-1}{2}$ is even. $\square$

5.5 Outline of the proof of the Index Theorem for twisted Dirac operators

In this section we outline the proof the Index Theorem, basically following the arguments in section 13 of Chapter III of the book *Spin Geometry* by Lawson and Michelsohn [LM]. However, we found it easier to present the arguments if we restrict ourselves to twisted Dirac operators.

We recall that for an even dimensional compact spin manifold X equipped with a complex $\mathbb{Z}/2$-graded vector bundle $E \to X$, the associated Dirac operator $D(X,E)$ acts on the sections of the tensor product $S \otimes E$, where S is the spinor bundle on X. The $\mathbb{Z}/2$-grading on S and E induces a $\mathbb{Z}/2$-grading on $S \otimes E$ and hence the space of sections $\Gamma(S \otimes E)$. The Dirac operator $D(X,E)$ is an *odd* operator, i.e., its restriction to $\Gamma((S \otimes E)^+)$ maps to $\Gamma((S \otimes E)^-)$, resulting in an operator

$$D^+(X,E) \colon \Gamma((S \otimes E)^+) \longrightarrow \Gamma((S \otimes E)^-).$$

We recall that

$$\begin{aligned}
\operatorname{index}(D^+(X,E)) &= \dim\ker D^+(X,E) - \dim\operatorname{coker} D^+(X,E) \\
&= \dim\ker D^+(X,E) - \dim\ker D^-(X,E) \\
&= \dim(\ker D(X,E))^+ - \dim(\ker D(X,E))^- \\
&= \operatorname{sdim}\ker D(X,E) \\
&= \operatorname{sdim}\ker D^2(X,E)
\end{aligned}$$

This point of view will be convenient for us in this section, and we will use the notation

$$\operatorname{ind}(D(X,E)) := \operatorname{sdim}(\ker D^2(X,E)).$$

Throughout this section we will be working with the K-theory orientation class $U^K(V) \in K_c(V)$ of even dimensional real vector vector bundles equipped with spin structures. They will often occur as normal bundle of embeddings $f \colon X \hookrightarrow Y$, and so it will be convenient for us to make the following assumption.

Assumption. All manifolds and real vector bundles in this section are even dimensional.

Theorem 5.42. (Index theorem for twisted Dirac operators). *Let X be a closed spin manifold, and let $E \to X$ be a complex graded vector bundle. Then*

$$\operatorname{ind}(D(X,E)) = p_!([E]) \in K(\mathrm{pt}) = \mathbb{Z}.$$

Here $p_! \colon K(X) \to K(\mathrm{pt}) = \mathbb{Z}$ is the umkehr map whose construction we recall below.

The astute reader might notice a glaring contradiction between this statement and equation (5.36). This is due to the fact that up to this section, I've followed the conventions in the Spin Geometry book. In this section, where we'll deal extensively with the twisted Dirac operators and umkehr maps in K-theory I find lugging around the signs annoying. So I prefer to change the convention for the K-theory orientation class $U^K(V)$ for even dimensional vector bundles with spin structure. That makes the sign in this formula go away without changing the cohomology version of the index theorem.

Motivated by the fact that $p_!$ is constructed in purely topological terms, but it calculates the index of twisted Dirac operators, the homomorphism $p_!$ is also referred to as the *topological index*:

$$\text{top-ind} := p_! \colon K(X) \longrightarrow K(\mathrm{pt}) = \mathbb{Z}. \tag{5.43}$$

A a *spin embedding* is an embedding $f\colon X \hookrightarrow Y$ together with a spin structure on the normal bundle $N \to X$. Let $i\colon X \hookrightarrow N$ be the zero section, and let $e\colon N \hookrightarrow Y$ be the open embedding of N as a tubular neighboorhood of $f(X)$ in Y. Then the commutative triangle

$$\begin{tikzcd} X \arrow[rr, hook, "f"] \arrow[dr, hook, "i"'] & & Y \\ & N \arrow[ur, hook, "e"'] & \end{tikzcd}$$

induces a commutative triangle of umkehr maps between the corresponding K-theory groups

$$\begin{tikzcd} K_c(X) \arrow[rr, hook, "f_!"] \arrow[dr, hook, "i_!"', "\cong"] & & K_c(Y) \\ & K_c(N) \arrow[ur, hook, "e_!"'] & \end{tikzcd} \tag{5.44}$$

Here $i_!$ is the Thom isomorphism, $e_!$ is the extend-by-zero map, and $f_!$ is defined as the composition $f_! := e_! i_!$. A crucial property of the umkehr map is its compatibility with compositions, i.e., if $X \overset{f}{\hookrightarrow} Y \overset{g}{\hookrightarrow} Z$ are spin embeddings, then

$$(g \circ f)_! = g_! \circ f_!. \tag{5.45}$$

This is a consequence of the multiplicativity property

$$U^K(V \oplus W) = U^K(V) \otimes U^K(W)$$

of the K-theory oriention U^K applied to the normal bundles of the embeddings.

Let X be a closed spin manifold and let $f\colon X \hookrightarrow \mathbb{R}^{n+k}$ be an embedding. The spin structure on X induces a spin structure on the normal bundle. Then top-ind $= p_!$ is the composition

$$K(X) \xrightarrow{f_!} K_c(\mathbb{R}^{n+k}) \xleftarrow[\cong]{\iota_!} K(\mathrm{pt})$$

Now we begin with the proof of the index theorem. The first step is to show that the index of the twisted Dirac operator indeed only depends on $[E] \in K(X)$.

Lemma 5.46. *The map*

$$\mathrm{ind}\colon K(X) \longrightarrow \mathbb{Z} \qquad \text{given by} \qquad [E] \mapsto \mathrm{ind}(D(X,E))$$

is a welldefined homomorphism.

The map ind is called the *analytic index map*. The strategy to show that the two index homomorphisms ind and top-ind agree, is to contemplate the formal properties of top-ind, and in fact to *characterize* the homomorphism top-ind by these properties. Then the real work is to show that the analytic index map ind has the same properties.

Lemma 5.47. 1. *The topological index top-ind*$(X,u) \in K(\mathrm{pt})$ *for a compact spin manifold X and $u \in K(X)$ has the following two properties*

compatibility with embeddings: *top-ind*$(Y, f_! u)$ = *top-ind*(X,u) *for spin embedding of compact spin manifolds $f\colon X \hookrightarrow Y$ and $u \in K(X)$.*

normalization property: *top-ind*$(\mathrm{pt}, u) = u$ *for $u \in K(\mathrm{pt})$.*

2. *There is only one assignment $(X,u) \mapsto I(X,u) \in K(\mathrm{pt})$ satisfying the two conditions above.*

The normalization property obviously holds for top-ind; in fact, it might seem silly to mention it. However, the without it, part 2 no longer holds, since the assignment defined by $I(X,u) := 0 \in K(\mathrm{pt})$ for all closed spin manifolds X is compatible with embeddings.

Proof. To prove property (ii) for top-ind, let $g\colon Y \hookrightarrow \mathbb{R}^{n+k}$ be a spin embedding. Then the spin embedding $gf\colon X \hookrightarrow \mathbb{R}^{n+k}$ can be used to calculate top-ind(u) for $u \in K(X)$. As above, let $\iota\colon \mathrm{pt} \hookrightarrow \mathbb{R}^{n+k}$. Then

$$\text{top-ind}(Y, f_! u) = \iota_!^{-1} g_!(f_! u) = \iota_!^{-1}(g \circ f)_! u = \text{top-ind}(u).$$

To prove part 2 of the lemma, let $(X,u) \mapsto I(X,u) \in K(\mathrm{pt})$ for X a closed spin manifold and $u \in K(X)$ be an assignment satisfying properties (i), (ii) and (ii). Then for any spin embedding $f\colon X \hookrightarrow \mathbb{R}^{n+k}$

$$I(X,u) \overset{(ii)}{=} I(\mathbb{R}^{n+k}, f_! u) \overset{(ii)}{=} I(\mathrm{pt}, \iota_!^{-1} f_! u) \overset{(i)}{=} \iota_!^{-1} f_! u = \text{top-ind}(X,u)$$

$\square$

Let X be compact spin manifold, $E \to X$ a $\mathbb{Z}/2$-graded vector bundle, and $[E] \in K(X)$ the K-theory class it represents. Let

$$\operatorname{ind}(X, [E]) := \operatorname{ind}(D(X, E)) \in \mathbb{Z} = K(\mathrm{pt}).$$

Part 2 of the lemma above shows that for the proof of the Index Theorem 5.42, it suffices to show that the assignment $(X, u) \mapsto \operatorname{ind}(X, u)$ satisfies the two properties of lemma 5.47(1). The normalization property is obviously satisfied, and hence it suffice to show that $\operatorname{ind}(X, u)$ is compatible with embeddings, i.e., that

$$\operatorname{ind}(X, u) = \operatorname{ind}(Y, f_! u) \tag{5.48}$$

for every spin embedding $f \colon X \hookrightarrow Y$ of compact spin manifolds.

We recall from (5.44) that for a spin embedding

$$f \colon X \xrightarrow{i} N \overset{e}{\hookrightarrow} Y$$

of compact manifolds X, Y the map $f_! \colon K(X) \to K(Y)$ is the composition of the shriek maps induced by the zero section i and the embedding e of the normal bundle N as a tubular neighborhood of $f(X)$ in Y. This suggests to verify the compatibility condition $\operatorname{ind}(X, u) = \operatorname{ind}(Y, f_! u)$ of ind with $f_!$ by checking the compatibility condition of ind with $i_!$ and $e_!$. The problem is that N, the total space of the normal bundle is *not compact* and hence the envisioned compatibility condition for $i_!$, namely

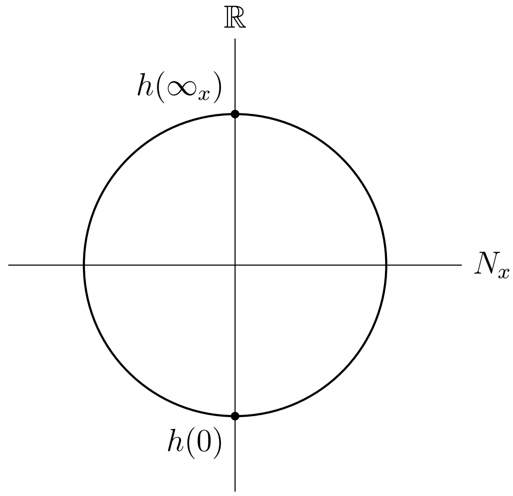
$$\operatorname{ind}(X, u) = \operatorname{ind}(N, i_! u) = \operatorname{ind}(D(N, i_! u))$$

does not make sense, since the Dirac operator $D(N, i_! u)$ on the non-compact manifold should be expected to have infinite dimensional kernel. It turns out that one *can* make sense of the index of the operator $D(N, i_! u)$, since $i_! u$ has *compact support* (recall that $i_! u \in K_c(N)$ is the orientation class). This was done after the original work of Atiyah and Singer and is known as *relative index theory* [GL].

In their proof Atiyah and Singer avoid the non-compact manifold N by compactifying it, by adding a point at infinity ∞_x to every fiber N_x for $x \in X$. To give a precise definition, we equip the vector bundle N with a bundle metric. Then there is a homeomorphism

$$h \colon N_x \cup \{\infty_x\} \xrightarrow{\approx} S(N_x \oplus \mathbb{R})$$

between the one-point-compactification $N_x \cup \{\infty_x\}$ of the vector space N_x and the sphere $S(N_x \oplus \mathbb{R})$ inside the vector space $N_x \oplus \mathbb{R}$, with $h(0) = (0, -1)$ and $h(\infty_x) = (0, 1)$. Here is a picture of $S(N_x \oplus \mathbb{R})$:



This construction provides an embedding $j\colon N \hookrightarrow S(N \oplus \underline{\mathbb{R}})$ from N to the sphere bundle of the direct sum of N and the trivial real line bundle.

The composition $s := j \circ i\colon X \to S(N \oplus \underline{\mathbb{R}})$ is a section of the sphere bundle $S(N \oplus \underline{\mathbb{R}})$. It is a spin embedding of compact spin manifolds.

Proposition 5.49. (Compatibility of the analytical index with sphere bundles). *Let X be compact spin manifold, $N \to X$ a spin vector bundle, and let $s\colon X \to S(N \oplus \underline{\mathbb{R}})$ be the section of the sphere bundle described above. Then*

$$\operatorname{ind}(X, u) = \operatorname{ind}(S(N \oplus \underline{\mathbb{R}}), s_! u) \qquad \textit{for all } u \in K(X).$$

In other words, this result shows that the desired compatibility of the index map with spin embeddings (5.48) holds for the spin embedding s from X to the sphere bundle

$$S(N \oplus \underline{\mathbb{R}}) \to X.$$

To address how this is related to the compatibility for the embedding $f\colon X \hookrightarrow Y$, consider the commutative diagram of spin embeddings:

$$\begin{tikzcd}
 & & Y \\
X \arrow[r, hook, "i"] \arrow[urr, hook, bend left, "f"] \arrow[drr, hook, bend right, "s"'] & N \arrow[ur, hook, "e"] \arrow[dr, hook, "j"] & \\
 & & S(N \oplus \underline{\mathbb{R}})
\end{tikzcd}$$

It follows that by the proposition above,

$$\operatorname{ind}(X,u) = \operatorname{ind}(S(N\oplus\underline{\mathbb{R}}), s_!u) = \operatorname{ind}(S(N\oplus\underline{\mathbb{R}}), j_!(i_!u)) \qquad \text{for all } u\in K(X),$$

while

$$\operatorname{ind}(Y, f_!u) = \operatorname{ind}(Y, e_!(i_!u)).$$

Hence the following result implies $\operatorname{ind}(X,u) = \operatorname{ind}(Y, f_!u)$ and hence the index theorem.

Proposition 5.50. *Let $\mathcal{O}$ be an open manifold, and let*

$$e\colon \mathcal{O}\hookrightarrow Y \qquad \textit{and} \qquad e'\colon \mathcal{O}\hookrightarrow Y'$$

be two open embeddings into compact spin manifolds. Then

$$\operatorname{ind}(Y, e_!u) = \operatorname{ind}(Y', e'_!u) \qquad \textit{for any } u\in K_c(\mathcal{O}).$$

This is what Lawson and Michelsohn call *The Excision Property* in their book [LM, p. 248]. They actually state a result for general elliptic operators rather than just twisted Dirac operators. Consequently, their K-theory elements live in $K_c(T^*X)$, rather than in $K(X)$; moreover, their K-theory elements correspond to the principal symbol class $\boldsymbol{\sigma}(P)\in K_c(T^*X)$ of elliptic operators P on X, rather than the element $[E]\in K(X)$ given by a vector bundle E that is used to form the twisted Dirac operator $D(X,E)$. Note that if X is spin, then the Thom isomorphism $K(X)\cong K_c(T^*X)$ sends $[E]\in K(X)$ to the principal symbol class $\boldsymbol{\sigma}(D(X,E))\in K_c(T^*X)$.

We refer to [LM] for a proof of the Excision Property, which uses the theory of pseudo differential operators.

The proof of Proposition 5.49 require us to show that the index a twisted Dirac operator on the total space of the sphere bundle $S(N\oplus\underline{\mathbb{R}})\to X$ is equal to the index of a twisted Dirac operator on X. In the case where the vector bundle N is trivial, the sphere bundle $S(N\oplus\underline{\mathbb{R}})$ is just the product of X and the sphere S^k, where $k=\dim N$. In that case the statement follows from the calculation of the index of the relevant twisted Dirac operator on S^k and the following *product formula* for the index of Dirac operators.

Proposition 5.51. (Product formula for the index of Dirac operators). *Let X_1, X_2 be compact spin manifold and $E_i\to X_i$ be $\mathbb{Z}/2$-graded vector bundles over X_i. Let $D(X_i,E_i)$ be the Dirac operator on X_i twisted by E_i, and let $D(X_1\times X_2, p_1^*E_1\otimes p_2^*E_2)$ be the Dirac operator on the product $X_1\times X_2$ twisted by the $\mathbb{Z}/2$-graded tensor product bundle $p_1^*E_1\otimes p_2^*E_2\to X_1\times X_2$, where $p_i\colon X_1\times X_2\to X_i$ is the projection map. Then*

$$\operatorname{ind} D(X_1\times X_2, p_1^*E_1\otimes p_2^*E_2) = \operatorname{ind} D(X_1,E_1))\cdot \operatorname{ind} D(X_2,E_2). \tag{5.52}$$

Proof. Let $S_i \to X_i$ be the $\mathbb{Z}/2$-graded spinor bundle on X_i and $F_i := S_i \otimes E_i$ the $\mathbb{Z}/2$-graded tensor product. The Dirac operator $D := D(X_1 \times X_2, p_1^* E_1 \otimes p_2^* E_2)$ acts on the section space $\Gamma(p_1^* F_1 \otimes p_2^* F_2)$. Unwinding the definition of the Dirac operator, it is not hard to check that the following diagram is commutative:

$$\begin{tikzcd}
\Gamma(F_1) \otimes \Gamma(F_2) \arrow[r, "D_1 \otimes \mathrm{id} + \mathrm{id} \otimes D_2"] \arrow[d] & \Gamma(F_1) \otimes \Gamma(F_2) \arrow[d] \\
\Gamma(p_1^* F_1 \otimes p_2^* F_2) \arrow[r, "D"] & \Gamma(p_1^* F_1 \otimes p_2^* F_2)
\end{tikzcd}$$

Here the vertical maps are given by multiplying sections. In more detail, if $\psi_i \colon X_i \to F_i$ is a section of F_i, then the section

$$\psi_1 \otimes \psi_2 \in \Gamma(p_1^* F_1 \otimes p_2^* F_2)$$

is defined by

$$(\psi_1 \otimes \psi_2)(x_1, x_2) := \psi_1(x_1) \otimes \psi_2(x_2) \in (F_1)_{x_1} \otimes (F_2)_{x_2}$$

The vertical maps are not literally isomorphisms, but they are injective maps with dense images, and hence the kernels of the horizontal maps can be identified.

$$\begin{aligned}
D^2 &= (D_1 \otimes \mathrm{id} + \mathrm{id} \otimes D_2)^2 \\
&= (D_1 \otimes \mathrm{id})^2 + (D_1 \otimes \mathrm{id})(\mathrm{id} \otimes D_2) + (\mathrm{id} \otimes D_2)(D_1 \otimes \mathrm{id}) + (\mathrm{id} \otimes D_2)^2 \\
&= D_1^2 \otimes \mathrm{id} + D_1 \otimes D_2 - D_1 \otimes D_2 + \mathrm{id} \otimes D_2^2 \\
&= D_1^2 \otimes \mathrm{id} + \mathrm{id} \otimes D_2^2
\end{aligned} \tag{5.53}$$

Here the crucial minus sign in the third line above is due to the Koszul sign rule in the definition of the tensor product $f \otimes g$ of maps $f \colon V \to V'$ and $g \colon W \to W'$ between $\mathbb{Z}/2$-graded vector spaces. If f, g, and $v \in V$, $w \in W$ are homogenous, then this is defined by

$$(f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w).$$

The general case is handled by decomposing all participants as sums of homogenous elements. This convention implies that if $f' \colon V' \to V''$ and $g' \colon W' \to W''$ are homogeneous maps, then the composition is given by

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|} (f' \circ f) \otimes (g' \otimes g).$$

In particular, since D_1, D_2 are *odd* operators, we have

$$(\mathrm{id} \otimes D_2)(D_1 \otimes \mathrm{id}) = -D_1 \otimes D_2.$$

To calculate the kernel of D^2 using (5.53), we note that the operators D_i are self-adjoint, and hence all its eigenvalues are real. In particular, the eigenvalues of D_i^2 are all ≥ 0. Hence by (5.53)

$$\ker D^2 = \ker(D_1^2 \otimes \mathrm{id}) \cap \ker(\mathrm{id} \otimes D_2^2) = (\ker D_1^2) \otimes (\ker D_2^2).$$

In particular,

$$\operatorname{sdim} \ker D^2 = (\operatorname{sdim} \ker D_1^2) \cdot (\operatorname{sdim} \ker D_2^2),$$

which proves the multiplicativity of the index of Dirac operators. $\square$

The proof of Proposition 5.49 in the general case, where the vector bundle $N \to X$ is non-trivial, requires a generalization of the product formula 5.52 from products $X \times Y$ to *twisted products*, i.e., fiber bundles over X with fiber Y. More precisely, we need fiber bundles with *compact structure group* G. This means that G acts on the fiber Y, and there is a principal G-bundle $P \to X$, such that the fiber bundle is *associated to the principal G-bundle* $P \to X$, i.e., it is of the form

$$\pi \colon P \times_G Y \to X.$$

As in the construction of a vector bundle associated to a principal bundle P and a G-action on a vector space V in section 2.2, the total space $P \times_G Y$ is the quotient $(P \times Y)/\sim$, where the equivalence relation is given by $(pg, y) \sim (p, gy)$ for $p \in P$, $g \in G$, $y \in Y$.

Furthermore, we require that:

- Y is a Riemannian manifold on which G acts by isometries; this is simple to accomplish by choosing an arbitrary Riemannian metric h on Y. While an element $g \in G$ might not leave the metric h fixed, i.e., the pullback g^*h of h via the diffeomorphism affected by g might not be equal to h, by averaging the metrics g^*h over the elements g of the compact group G, we can produce a G-equivariant metric.

- Y is equipped with a *G-equivariant spin structure*. We won't give the precise definition of this, but just mention that this assumption guarantees that the associated spinor bundle $S \to Y$ is G-equivariant, and that the Dirac operator $D \colon \Gamma(S) \to \Gamma(S)$ is G-equivariant. More generally, if $F \to Y$ is a G-equivariant $\mathbb{Z}/2$-graded vector bundle with equivariant connection, then the twisted Dirac operator $D(Y, F)$ is G-equivariant.

In particular, with these assumptions the kernel of $D(Y, F)$ is a representation of G. More precisely, the G-action preserves the splitting

$$\ker D(Y, F) = (\ker D(Y, F))^+ \oplus (\ker D(Y, F))^-$$

of the kernel into its even and odd subspace. We recall that the *representation ring of G*, denoted $R(G)$ is the group completion of the abelian monoid given by the isomorphism classes of finite dimensional representations of G under the direct sum.

Definition 5.54. The *equivariant index* of the Dirac operator $D(Y,F)$ is defined by

$$\operatorname{ind}_G D(Y,F) := (\ker D(Y,F))^+ - (\ker D(Y,F))^- \in R(G).$$

If G is trivial, then a G-representation is just a finite dimensional vector space, and the monoid of finite dimensional representations is isomorphic to $\mathbb{N}$, with the isomorphism determined by mapping a vector space V to $\dim V \in \mathbb{N}$. It follows that $R(G) \cong \mathbb{Z}$, and the definition of the equivariant index is precisely our description of

$$\operatorname{ind} D(Y,F) = \dim(\ker D(Y,F))^+ - \dim(\ker D(Y,F))^-.$$

The assumption that Y is equipped with a G-equivariant spin structure also guarantees that a spin structure on X induces a spin structure on the twisted product $P \times_G Y$, the total space of the fiber bundle $\pi\colon P \times_G Y \to X$. In particular, there is an associated Dirac operator $D(P \times_G Y)$.

In the case of the product $X \times Y$, we considered the Dirac operator on $X \times Y$ twisted by the tensor product of a bundle pulled back from X and another pulled back from Y. Similarly, in the case of the fiber bundle $\pi\colon P \times_G Y \to X$, we will twist the Dirac operator on $P \times_G Y$ by the tensor product of two types of (complex, $\mathbb{Z}/2$-graded) vector bundles over $P \times_G Y$:

- a vector bundle $E \to X$ can be pulled back via the projection map π to yield the vector bundle $\pi^* E$.
- A G-equivariant vector bundle $F \to Y$ produces via the associated bundle construction a vector bundle
$$P \times_G F \longrightarrow P \times_G X.$$

Proposition 5.55. (Twisted Product formula for the index of Dirac operators). *Let $D(P \times_G Y, \pi^*E \otimes (P \times_G F))$ be the Dirac operator on $P \times_G Y$ twisted by the tensor product of the vector bundles π^*E and $P \times_G F$ described above. Then the index of this twisted Dirac operator is given by the formula*

$$\operatorname{ind} D(P \times_G Y, \pi^*E \otimes (P \times_G F)) = \operatorname{ind} D(X, E \otimes (P \times_G \operatorname{ind}_G D(Y,F))).$$

The twisted product formula for the index is then applied to prove Proposition 5.49 by observing that the sphere bundle $S(N \oplus \underline{\mathbb{R}}) \longrightarrow X$ is of the form $P \times_G Y$. We recall the $N \to X$ is a real vector bundle of dimension $2k$ equipped with a spin structure. Let $P \to X$ be the principal $\mathrm{Spin}(2k)$-bundle determined by the spin structure on N. Let $Y = S(\mathbb{R}^{2k} \oplus \mathbb{R})$ be the sphere of dimension $2k$, equipped with the action of $\mathrm{Spin}(2k)$ given by the double covering map $\mathrm{Spin}(2k) \to \mathrm{SO}(2k)$ and the obvious action of $\mathrm{SO}(2k)$ on $Y = S(\mathbb{R}^{2k} \oplus \mathbb{R})$. Then is it easy to see that there is an isomorphism

$$P \times_{\mathrm{Spin}(2k} S(\mathbb{R}^{2k} \oplus \mathbb{R}) \cong S(N \oplus \underline{\mathbb{R}})$$

of fiber bundles over X. Moreover, the K-theory element $s_![E] \in K(S(N \oplus \underline{\mathbb{R}}))$ can be represented by the vector bundle

$$\pi^* E \otimes (P \times_{\mathrm{Spin}(2k)} S_Y),$$

where S_Y is the spinor bundle on the sphere $Y = S^{2k}$.

Then the crucial, but not difficult calculation is that of the G-equivariant index of the twisted Dirac operator $D(Y, S_Y)$ for the action of $G = \mathrm{Spin}(2k)$ on the sphere $Y = S^{2k}$. It turns out that $\mathrm{ind}_G\ D(Y, S_Y) = 1 \in R(G)$. Since the unit in the representation ring is the trivial representation of dimension 1, the associated vector bundle $P \times_G \mathrm{ind}_G(Y, S_Y)$ is just the trivial line bundle over X. Hence Proposition 5.55 implies

$$\mathrm{ind} D(S(N \oplus \mathbb{R}), s_![E]) = \mathrm{ind} D(X, E),$$

which proves Proposition 5.49.

6 The equivariant Index Theorem and the Witten genus

This section is a quick survey on equivariant index theory and the Witten genus, which should be thought of the equivariant index of the Dirac operator on the free loop space LX, the space of maps from S^1 to a manifold X.

6.1 The equivariant index theorem

Let X be closed Riemannian spin manifold of dimension $n = 2k$. We recall that the index of the Dirac operator D on X is given by

$$\mathrm{ind} D := \dim(\ker D)^+ - \dim(\ker D)^-,$$

where $(\ker D)^\pm$ is the even (resp. odd) part of the kernel of D, which is a $\mathbb{Z}/2$-graded vector space. Let G be a compact Lie group which acts on X by spin-structure preserving isometries (which means that the G-action on the oriented frame bundle $\mathrm{SO}(X)$ – given by the differential – comes equipped with a lift to the double covering $\mathrm{Spin}(X) \to \mathrm{SO}(X)$). This in turn implies that G acts on the spinor bundle S and its space of sections $\Gamma(S)$. Moreover, this G-action on $\Gamma(S)$ commutes with D, and hence $\ker D$ and its subspaces $(\ker D)^\pm$ are representations of G.

Definition 6.1. The *G-equivariant index* of the Dirac operator on X is given by

$$\mathrm{ind}_G(g, D(X)) := \mathrm{tr}(g, (\ker D)^+) - \mathrm{tr}(g, \mathrm{tr}(\ker D)^-) \in \mathbb{C} \qquad \text{for } g \in G$$

Here we denote by $\mathrm{tr}(g, V)$ for a representation $\rho\colon G \to \mathrm{Aut}(V)$ the trace of $\rho(g)$ for $g \in G$. We note that the equivariant index of D is a generalization of the index of D, since $\mathrm{tr}(e, V) = \dim V$ for the identity element $e \in G$, and hence $\mathrm{ind}_G(e, D(X)) = \mathrm{ind}\, D$.

The *equivariant index theorem*, due to Atiyah and Singer [ASIII, Lefschetz Theorem 3.9] expresses the equivariant index $\mathrm{ind}_G(g, D)$ in terms of data associated to the fixed point set

$$X^g = \{x \in X \mid gx = x\}.$$

Their theorem is actually much more general, not just for the Dirac operator, but for any G-equivariant elliptic operator (in fact, even more generally, for generalizations of elliptic operators called "elliptic complexes"). A reference for the explicit index formula for the Dirac operator is [AH, section 1.4(5), 1.4(8)]. We will more closely follow the presentation in Witten's paper [Wi], and restrict to $G = S^1$, which is the case of interest in the next section on the Witten genus.

In this case, the equivariant index $\mathrm{ind}_{S^1}(q, D)$ for any $q \in S^1$ can be calculated in terms of data associated to the full fixed point set

$$X^{S^1} = \{x \in X \mid qx = x \text{ for all } q \in S^1\}.$$

The simplest situation is where X^{S^1} is discrete, i.e., consists of finitely many fixed points. In that case, the equivariant index theorem takes the form

$$\mathrm{ind}_{S^1}(q, D) = \sum_{x \in X^{S^1}} F_x(q), \tag{6.2}$$

where $F_x(q) \in \mathbb{C}$ is a number which is determined by the action of S^1 on the tangent space T_xX. We observe that no tangent vector $v \in T_xX$ is fixed under the action of S^1. Otherwise the geodesic through x with tangent vector v would also have to be point-wise fixed under the S^1-action, contradicting our assumption that x is an isolated fixed point.

6.1.1 Representations of S^1

To describe $F_x(q)$ for $q \in S^1$, it will be useful to recall a little bit of the representation theory of S^1. We recall that there are three equivalent ways to think about a (finite dimensional) *complex representation* of S^1:

(1) a homomorphism $\rho\colon S^1 \to \mathrm{Aut}(V)$ to the automorphism group of a complex vector space V;

(2) a complex vector space V equipped with a $\mathbb{Z}$-grading, i.e., a direct sum decomposition $V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell$ of V as a sum of subspaces $V_\ell \subset V$;

(3) a Laurent polynomial $\sum_{\ell\in\mathbb{Z}} V_\ell q^\ell$ in a formal variable q whose coefficients V_ℓ are finite dimensional complex vector spaces, which are trivial for almost all ℓ (i.e., this is a finite sum). Generalizing the notation $\mathbb{Z}[q, q^{-1}]$ for Laurent polynomials with coefficients in $\mathbb{Z}$, we will write $\mathrm{Vect}_{\mathbb{C}}[q, q^{-1}]$ for the collection of Laurent series with values in vector spaces.

To pass from a homomorphism $\rho\colon S^1 \to \mathrm{Aut}(V)$ to a $\mathbb{Z}$-grading on V, let $V_\ell \subset V$ be the subspace of V on which $\rho(q)$ acts by scalar multiplication by q^ℓ for $q \in S^1$. Conversely, if $V = \bigoplus_{\ell\in\mathbb{Z}} V_\ell$ is a $\mathbb{Z}$-graded complex vector space, a homomorphism $\rho\colon S^1 \to \mathrm{Aut}(V)$ is determined by defining $\rho(q)(v) := q^\ell v$ for $v \in V_\ell$. Clearly the descriptions (2) and (3) are equivalent, both use just a slightly different syntax to write down a $\mathbb{Z}$-graded vector space.

For a complex S^1-representation V, its *character* is the function $S^1 \to \mathbb{C}$ given by $q \mapsto \mathrm{tr}(q, V)$. Using the decomposition $V = \bigoplus_{\ell\in\mathbb{Z}} V_\ell$,

$$\mathrm{tr}(q, V) = \sum_{\ell\in\mathbb{Z}} \mathrm{tr}(q, V_\ell) = \sum_{\ell\in\mathbb{Z}} \dim V_\ell q^\ell.$$

Let $R(S^1)$ be the complex representation ring of S^1, i.e., the group completion of the abelian monoid given by complex representations of S^1 with respect the direct sum of representations. In other words, an element of $R(S^1)$ is represented as a formal difference between representations. The ring structure on $R(S^1)$ is induced by the tensor product of representations. Writing a representation V as in (3) above as $V = \sum_{\ell\in\mathbb{Z}} V_\ell q^\ell$, the map

$$\begin{aligned} \dim_*\colon R(S^1) &\longrightarrow \mathbb{Z}[q, q^{-1}] \\ V = \sum_{\ell\in\mathbb{Z}} V_\ell q^\ell &\mapsto \mathrm{tr}(q, V) = \sum_{\ell\in\mathbb{Z}} \dim V_\ell q^\ell. \end{aligned} \tag{6.3}$$

is called the *graded dimension map* or *character map*.

If S^1 acts on a real vector space V, for example the tangent space T_xX of an isolated fixed point, then V can be decomposed in a similar way into subspaces V_ℓ as in the complex case, but more care is needed. For a representation $\rho\colon S^1 \to \mathrm{Aut}(V)$ let

$$P\colon V \longrightarrow V \qquad \text{be the operator defined by} \qquad P := \frac{d}{d\theta}_{|\theta=0} \rho(e^{i\theta})$$

For example, if $V = \mathbb{C}_\ell$, then

$$P = \frac{d}{d\theta}_{|\theta=0} \rho(e^{i\theta}) = \frac{d}{d\theta}_{|\theta=0} e^{i\ell\theta} = i\ell$$

acts by multiplication by $i\ell$. If we regard $\mathbb{C}$ as a real vector space with the standard basis, then P is given by the matrix

$$\rho(e^{i\theta}) = \begin{pmatrix} \cos\ell\theta & -\sin\ell\theta \\ \sin\ell\theta & \cos\ell\theta \end{pmatrix} \qquad \text{and hence} \qquad P = \frac{d}{d\theta}_{|\theta=0} \rho(e^{i\theta}) = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix}$$

In particular, P has complex eigenvalues $\pm i\ell$. For a general real representation V, the possible complex eigenvalues of P are 0 and pairs $\pm i\ell$ of complex conjugate numbers with $\ell \in \mathbb{Z}_+ = \{1, 2, \dots\}$. Moreover, if P does not have eigenvalue 0, which is the case of interest to us, then the following holds.

(i) the real dimension $\dim_{\mathbb{R}} V$ is even, and there is a basis $e_1, f_1, \dots, e_k, f_k$ of V such that the matrix P has block diagonal form with 2×2 blocks

$$\begin{pmatrix} 0 & -\ell_i \\ \ell_i & 0 \end{pmatrix}$$

for $i = 1, \dots, k$ and $\ell_i \in \mathbb{Z}_+$. In particular, V has a complex structure $J\colon V \to V$ given by $Je_i := f_i$ and $Jf_i = -e_i$. Then $e_1, \dots, e_k$ is a basis of V as a complex vector space, i.e., we have a decomposition of complex vector spaces

$$V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k, \tag{6.4}$$

and $\rho(q)$ acts on the complex subspace $\mathbb{C}e_i \subset V$ by multiplication by q^{ℓ_i}.

(ii) More invariantly, there is a complex structure J on the real vector space V, such that V as complex vector space of complex dimension k decomposes as the direct sum

$$V = \bigoplus_{\ell > 0} V_\ell \tag{6.5}$$

of complex subspaces V_ℓ where $\rho(q)$ acts by multiplication by q^ℓ. The subspace V_ℓ is equal to the sum $\bigoplus \mathbb{C}e_i$, where we sum over those indices i for which $\ell_i = \ell$.

6.1.2 The equivariant index theorem in the case of isolated fixed points

Theorem 6.6. (The equivariant index theorem for isolated fixed points). *Let X be a closed Riemannian spin manifold of dimension $n = 2k$. Let S^1 act on X by spin structure preserving isometries with fixed point set X^{S^1} consisting of finitely many points. Then the equivariant index of the Dirac operator $D(X)$ is given by*

$$\operatorname{ind}_{S^1}(q, D(X)) = \sum_{x \in X^{S^1}} F_x(q), \qquad \textit{with} \qquad F_x(q) = \lambda_x \prod_{i=1}^{k} \frac{q^{\ell_i/2}}{1 - q^{\ell_i}} \in \mathbb{C}.$$

Here $\ell_1, \dots, \ell_k \in \mathbb{Z}_+$ are the positive integers such that $\pm i\ell_i$ are the complex eigenvalues of the operator P for the S^1-action on T_xX (see (i) above). Moreover, $\lambda_x \in \{\pm 1\}$, where $\lambda_x = 1$ if and only if the orientation on T_xX given by the spin structure on X agrees with the orientation provided by the complex structure $-J$ on T_xX. Here J is the complex structure on $V = T_xX$ determined by the S^1-action on V as described above.

This is the Fixed Point Theorem that Witten in his paper [Wi] states as equation (4), with the definition of $F_x(q)$ in equation (7). Superficially, his definition of the sign λ_x looks different than ours since he compares with the orientation given by the complex structure J rather than $-J$. However, his matrix form of P differs from ours by a minus sign. This has the effect that $\rho(q)$ on the subspace spanned by e_i acts by multiplication by $q^{-\ell_i}$, a convention I find awkward.

The fixed point contribution $F_x(q)$ can be rewritten in terms of the canonical decomposition (6.5) of $V = T_xX$, using the fact that $\dim V_\ell$ is the cardinality of the index set $I_\ell := \#\{i \mid \ell_i = \ell\}$:

$$\begin{aligned}\prod_{i=1}^{k} \frac{q^{\ell_i/2}}{1-q^{\ell_i}} &= \prod_{\ell>0}\left(\prod_{i\in I_\ell} \frac{q^{\ell_i/2}}{1-q^{\ell_i}}\right)\\ &= \prod_{\ell>0} \frac{q^{(\ell \dim V_\ell)/2}}{(1-q^\ell)^{\dim V_\ell}} \\ &= \left(\prod_{\ell>0} q^{\frac{1}{2}\ell \dim V_\ell}\right)\left(\prod_{\ell>0}\frac{1}{(1-q^\ell)^{\dim V_\ell}}\right)\end{aligned} \tag{6.7}$$

In general, the fixed point set X^{S^1} consists not just of a finite number of points, but it can be decomposed as disjoint union

$$X^{S^1} = \coprod_\alpha X_\alpha^{S^1}$$

of its connected components $X_\alpha^{S^1}$ which are all manifold, but possibly of varying dimension. The equivariant Index Theorem is the statement

$$\operatorname{ind}_{S^1}(q, D) = \sum_\alpha F_\alpha(q),$$

where the summand $F_\alpha(q)$ can be expressed in terms of the component $X_\alpha^{S^1}$ of the fixed point set and its equivariant normal bundle N. In order to write down the explicit formula for $F_\alpha(q)$ we will again proceed by first decomposing the equivariant normal bundle N.

The S^1-action on X induces an action of S^1 on the normal bundle N. In particular, the fiber N_x is a representation of S^1 for each $x \in X_\alpha^{S^1}$. The same argument as in the isolated fixed point case shows that no non-zero vector $v \in N_x$ is fixed by S^1, and hence N_x has a complex structure and a decomposition (6.5) into eigenspaces of the operator P. This can be done simultaneously in all fibers of N and leads to a complex structure on N, as well as a decomposition

$$N = \bigoplus_{\ell>0} N_\ell \tag{6.8}$$

of N as a sum of complex subbundles N_ℓ where $q \in S^1$ acts by multiplication by q^ℓ.

The definition of $F_\alpha(q)$ is a generalization of the expression $F_x(q)$ for the case of an isolated fixed point x. This involves replacing the complex *vector spaces* V_ℓ by the complex *vector bundles* N_ℓ. Writing down the generalization of formula (6.7) requires to recognize the two factors in that formula as *graded dimensions* of suitable $\mathbb{Z}$-graded vector spaces from the representation V, which are constructed as *determinant* and *total symmetric power*, respectively.

6.1.3 The determinant construction

Let V be a vector space of dimension n. Then the *determinant line* is the 1-dimensional vector space

$$\mathrm{Det}(V) := \Lambda^n(V).$$

This is a functorial construction, and if $f\colon V \to V$ is a linear map, the induced map $\mathrm{Det}(V) \to \mathrm{Det}(V)$ is multiplication by the scalar $\det(f)$, the usual determinant of f. This construction is *exponential* in the sense that for vector spaces V, W, there is a natural isomorphism

$$\mathrm{Det}(V \oplus W) \cong \mathrm{Det}(V) \otimes \mathrm{Det}(W).$$

By functoriality, a S^1-action on V induces an S^1-action on $\mathrm{Det}(V)$, and hence the determinant line construction induces a map

$$\mathrm{Det}\colon R(S^1) \longrightarrow R(S^1) \qquad V \mapsto \mathrm{Det}(V).$$

For example, let V be a complex vector space, and let $Vq^\ell \in R(S^1)$ be the representation where $q \in S^1$ acts on V by multiplication by q^ℓ. Then $q \in S^1$ acts on $\mathrm{Det}(V) = \Lambda^n(V)$ by multiplication by $q^{\ell n}$. In other words,

$$\mathrm{Det}(Vq^\ell) = \mathrm{Det}(V)q^{\ell \dim V}$$

More generally, for a representation $V = \sum_{\ell>0} V_\ell q^\ell \in R(S^1)$,

$$\mathrm{Det}(V) = \bigotimes_\ell \mathrm{Det}(V_\ell q^\ell) = \bigotimes_\ell \left(\mathrm{Det}(V_\ell) q^{\ell \dim V_\ell}\right) = \left(\bigotimes_\ell \mathrm{Det}(V_\ell)\right) \prod_{\ell>0} q^{\ell \dim V_\ell}. \tag{6.9}$$

Since $\dim \mathrm{Det}(V_\ell) = 1$, the graded dimension of this representation (see (6.3)) is given by

$$\dim_*(\mathrm{Det}(V)) = \prod_\ell q^{\ell \dim V_\ell}. \tag{6.10}$$

We notice that this is the *square* of the first factor of the expression (6.7) for $F_x(q)$. Hence the first factor should be thought of as $\sqrt{\mathrm{Det}(V)}$; we will discuss later how to make sense of this.

6.1.4 The symmetric power construction

For a complex vector space V, let S^kV be the k-th symmetric power of V. For a finite dimensional vector space of dimension ≥ 1, the total symmetric power

$$S(V) := S^0V \oplus S^1V \oplus S^2V \oplus \dots$$

is infinite dimensional (unlike the total exterior power of V). So for many purposes, it it more useful to assemble the symmetric powers of V in a formal power series

$$S_t(V) := S^0V + S^1Vt + S^2Vt^2 + \dots, \tag{6.11}$$

in a formal variable t, all of whose coefficients are are finite dimensional vector spaces. This is an *exponential* construction in the sense that for vector spaces V, W

$$S_t(V \oplus W) = S_t(V) \otimes S_t(W),$$

where the tensor product on the right is defined by the usual multiplication of powerseries, using the tensor product to multiply the vector space coefficients.

The construction S_t is useful for symmetric powers of S^1-representations. Let $V = V_\ell q^\ell$ be the S^1-representation given by multiplication by q^ℓ on the complex vector space V_ℓ. Then by functoriality, S^1 acts on the symmetric power $S^k(V_\ell q^\ell)$; as a vector space this is just S^kV_ℓ with $q \in S^1$ acting on it by multiplication by $q^{\ell k}$. In other words,

$$S^kV = S^k(V_\ell q^\ell) = S^k(V_\ell)q^{\ell k},$$

and hence as a representation, the total symmetric power $S(V)$ has the form

$$\begin{aligned} S(V) &= \bigoplus_{k\geq 0} S^kV = \bigoplus_{k\geq 0} S^k(V_\ell)(q^\ell)^k \\ &= S^0(V_\ell) + S^1(V_\ell)q^\ell + S^2(V_\ell)(q^\ell)^2 + S^3(V_\ell)(q^\ell)^3 + \dots \\ &= S_{q^\ell}(V_\ell), \end{aligned}$$

where $S_{q^\ell}(V_\ell)$ is the formal power series (6.11), with q^ℓ substituted for t. This shows that although the total symmetric power is infinite dimensional, each isotypical component of the represenation $S(V)$ is finite dimensional provided $\ell \neq 0$. Moreover, for $\ell > 0$, the total symmetric power is a powerseries in q. More generally, this is the case for any representation V of the form $V = \bigoplus_{\ell>0} V_\ell q^\ell$:

$$S(V) = \bigotimes_{\ell>0} S(V_\ell q^\ell) = \bigotimes_{\ell>0} S_{q^\ell}(V_\ell). \tag{6.12}$$

Our next goal is to calculate the graded dimension $\dim_* S(V)$ of the total symmetric power of an S^1-representation V of the form $V = \bigoplus_{\ell>0} V_\ell q^\ell$. In the simple case $V = \mathbb{C}^\ell$

$$\begin{aligned}\dim_*(S(\mathbb{C}q^\ell)) &= \dim_* \left(S^0(\mathbb{C}) + S^1(\mathbb{C})q^\ell + S^2(\mathbb{C})q^{2\ell} + S^3(\mathbb{C})q^{3\ell} + \dots\right)\\ &= 1 + q^\ell + q^{2\ell} + q^{3\ell} + \dots = \frac{1}{1-q^\ell}.\end{aligned}$$

and hence if $\dim V_\ell = n_\ell$

$$\dim_*(S(V_\ell q^\ell)) = \dim_*(S(\underbrace{\mathbb{C}q^\ell \oplus \dots \oplus \mathbb{C}q^\ell}_{n_\ell})) = \frac{1}{(1-q^\ell)^{n_\ell}}$$

It follows that for $V = \bigoplus_{\ell>0} V_\ell q^\ell$

$$\dim_*(S(V)) = \dim_*(\bigotimes_{\ell>0} S(V_\ell q^\ell)) = \prod_{\ell>0} \frac{1}{(1-q^\ell)^{n_\ell}} \tag{6.13}$$

which we recognize as the second factor in the expression (6.7) for $F_x(q)$.

6.1.5 The equivariant index theorem

Putting all of this together, we end up with a more conceptual way of writing the contribution $F_x(q)$ of an isolated fixed point $x \in X^{S^1}$ to the equivariant index in the Fixed Point Formula 6.6:

$$F_x(q) = \lambda_x \dim_*(\sqrt{\operatorname{Det}(T_x X)}) \dim_*(S(T_x X)). \tag{6.14}$$

The point of this rewriting is that $T_x X$ can be thought of as the normal bundle of the fixed point x considered as a submanifold of X. So the analog of $T_x X$ for a fixed point component $X^{S^1}_\alpha \subset X$ is its normal bundle N. The S^1-action on N gives N the structure of a complex vector bundle, and a decomposition $N = \bigoplus_{\ell>0} N_\ell q^\ell$ of N as S^1-equivariant vector bundle as described in (6.8). The determinant line construction and the symmetric power construction works for equivariant vector bundles the same way as for representations. In more detail, the latter is given by

$$S(N) := \bigotimes_{\ell>0} S(N_\ell q^\ell),$$

where $S(N_\ell q^\ell) := S_{q^\ell}(N_\ell)$, and $S_t(V)$ for a vector bundle V is the formal power series

$$S_t(V) = S^0 V + S^1 V t + S^2 V t^2 + \dots.$$

While the definition of $\operatorname{Det}(N)$ as equivariant line bundle over X is clear, it is less clear how to make sense of the square root $\sqrt{\operatorname{Det}(N)}$. It is helpful to separate the construction of the line bundle from the question of how S^1 acts on it, as in (6.9) for representations by writing

$$\operatorname{Det}(N) = \operatorname{Det}(\bigoplus_{\ell>0} N_\ell q^{n_\ell}) = \operatorname{Det}(\underline{N}) \prod_{\ell>0} q^{\ell n_\ell}$$

where we write $\underline{N}$ for the vector bundle N obtained by forgetting about the S^1-action. This shows that if the complex line bundle $\mathrm{Det}(\underline{N})$ has a square root $\sqrt{\mathrm{Det}(\underline{N})}$, then we can define

$$\sqrt{\mathrm{Det}(N)} := \sqrt{\mathrm{Det}(\underline{N})} \prod_{\ell>0} q^{\ell n_\ell/2}$$

as an element of $\mathrm{Vect}_{\mathbb{C}}(X)[q^{1/2}]$, i.e., as a polynomial in $q^{1/2}$ whose coefficients are complex vector bundles over X (geometrically, this can be interpreted as an action of the double cover of S^1). While $\mathrm{Det}(\underline{N})$ might not have a square root, the first Chern class of such a square root is just half of the first Chern class $c := c_1(\mathrm{Det}(\underline{N})) \in H^2(X;\mathbb{Q})$ and hence we can define

$$\mathrm{ch}(\mathrm{Det}(\underline{N})) := e^{c/2} \in H^*(X;\mathbb{Q}).$$

Theorem 6.15. (Equivariant index theorem). *Let X be a closed Riemannian spin manifold of dimension $n = 2k$. Let S^1 act on X by spin structure preserving isometries with fixed point set $X^{S^1} = \coprod_\alpha X^{S^1}_\alpha$. Then the equivariant index of the Dirac operator $D(X)$ is given by*

$$\mathrm{ind}_{S^1}(q, D(X)) = \sum_\alpha F_\alpha(q).$$

Here the summand $F_\alpha(q)$ is determined by the equivariant normal bundle N of the component $X^{S^1}_\alpha$. In terms of the equivariant decomposition $N = \bigoplus_{\ell>0} N_\ell q^\ell$ it is given by

$$F_\alpha(q) = \lambda_\alpha \langle \widehat{A}(TX)\,\mathrm{ch}\left(\sqrt{\mathrm{Det}(\underline{N})} \prod_{\ell>0} q^{\ell n_\ell/2} S(N)\right), [X^{S^1}_\alpha]\rangle$$

If $X^{S^1}_\alpha$ has a spin structure, and the square root of the complex line bundle $\Delta(\underline{N})$ exists, $F_\alpha(q)$ can be alternatively expressed as

$$F_\alpha(q) = \lambda_\alpha \mathrm{ind}(D(X^{S^1}_\alpha), \sqrt{\mathrm{Det}(\underline{N})} \prod_{\ell>0} q^{\ell n_\ell/2} S(N)). \tag{6.16}$$

Remark 6.17. Atiyah and Hirzebruch have shown that the equivariant index of the Dirac operator on a closed spin $2k$-manifold with a non-trivial S^1-action vanishes [AH]. This does not mean that explaining the complicated right hand side of the equivariant index theorem is pointless since it is 0 anyway, since

(i) The proof of the Atiyah-Hirzebruch result *starts* with this index formula, and is based on analyzing the behavior of both sides as functions of q. Using the fact that the two sides represent quite different flavors of functions, they are able to conclude that these functions must be identically zero.

(ii) Formally applying the equivariant theorem to the Dirac operator on the free loop space (see next section), that index is often not zero.

(iii) There is a slightly more general equivariant index theorem for twisted Dirac operators (see e.g. [Wi]); their S^1-equivariant indices are generally non-zero.

6.2 The Witten genus

Let $LX = \mathrm{map}(S^1, X)$ be the *free loops space* of X consisting of the smooth maps $\gamma\colon S^1 \to X$. This is an infinite dimensional manifold, with a natural S^1-action given by reparametrizing the loops. The fixed point set LX^{S^1} consists of a the constant loops, which can be identified with the manifold X itself:

$$LX^{S^1} = X \subset LX.$$

In order to formally apply the equivariant index theorem 6.15 to the Dirac operator on LX, we need to determine the equivariant normal bundle $N \to X$ of X in LX.

The tangent space $T_\gamma LX$ at a loop γ is given by the space of *vector fields along the loop* γ which is defined to be the space $\Gamma(\gamma^*TX)$ of smooth sections of the pullback bundle $\gamma^*TX \to S^1$. In particular, if γ is the constant map with image $x \in X$, then

$$\Gamma(\gamma^*TX) = C^\infty(S^1, T_xX)$$

is the space of smooth maps from S^1 to T_xX. Fourier decomposition of maps $S^1 \to T_xX$ gives an injective linear map with dense image

$$\bigoplus_{\ell\in\mathbb{Z}} (T_xX)_\ell \longrightarrow C^\infty(S^1, T_xX)$$

which sends $(a_\ell)_{\ell\in\mathbb{Z}}$ to the smooth map $f\colon S^1 \to T_xX$ given by

$$f(\theta) = \sum_{\ell>0} a_\ell \cos i\ell\theta + a_0 + \sum_{\ell<0} a_\ell \sin i\ell\theta \qquad \text{for } a_\ell \in T_xX.$$

Alternatively, using the embedding $T_xX \hookrightarrow T_xX \otimes \mathbb{C}$ and rewriting

$$a_\ell \cos \ell\theta + a_{-\ell} \sin \ell\theta = \mathrm{Re}((a_\ell - ia_{-\ell})(\cos \ell\theta + i \sin \ell\theta)) = \mathrm{Re}((a_\ell - ia_{-\ell})e^{\ell\theta})$$

we obtain the injective linear map with dense image

$$T_xX \oplus \bigoplus_{\ell>0} (T_xX \otimes \mathbb{C})_\ell \longrightarrow C^\infty(S^1, T_xX) = T_xLX.$$

This maps $(v_\ell)_{\ell>0} \in \bigoplus_{\ell>0}(T_xX \otimes \mathbb{C})_\ell$ to the map $f(\theta) = \mathrm{Re}(\sum_{\ell>0}(v_\ell e^{\ell\theta})$. This map is S^1-equivariant, where $q \in S^1$ acts trivially on T_xX, by multiplication by q^ℓ on the summand $(T_xX \otimes \mathbb{C})_\ell$, and by reparametrization on the mapping space $C^\infty(S^1, X)$. The summand T_xX on the left corresponds to the tangent space of the fixed point set $X = LX^{S^1}$, and hence the fiber N_x of the normal bundle N corresponds to $\bigoplus_{\ell>0}(T_xX \otimes \mathbb{C})_\ell$. It follows that this procedure produces an isomorphism of equivariant vector bundles

$$\bigoplus_{\ell>0} N_\ell q^\ell \cong N \qquad \text{with } N_\ell = TX \otimes \mathbb{C} \text{ for all } \ell > 0.$$

Formally applying the equivariant index theorem 6.15 to the Dirac operator $D(LX)$ on the free loop space LX, we analyse the terms appearing on the right hand side of (6.16)

1. The sign λ_α will be ignored here, since the fixed point set has only one component, and so it only leads to an overall sign (more than one component would require a careful consideration of the relative sign).

2. $\mathrm{Det}(N_\ell) = \mathrm{Det}(TX \otimes \mathbb{C}) = \mathrm{Det}(TX) \otimes \mathbb{C}$, where $\mathrm{Det}(TX)$ is the determinant line bundle of the real vector bundle TX. This real line bundle is the orientation line bundle of X, and hence is trivial since X has a spin structure. It follows that
$$\mathrm{Det}(\underline{N}) = \bigotimes_{\ell>0} \mathrm{Det}(N_\ell)$$
is trivial.

3. $n_\ell = \dim_{\mathbb{C}} N_\ell = \dim X = n$, and hence
$$\prod_{\ell>0} q^{\ell n_\ell/2} = \prod_{\ell>0} q^{\ell n/2} = q^{\frac{n}{2}\sum_{\ell>0}\ell} = q^{-\frac{n}{24}}$$
Here the last equality comes from interpreting the obviously diverging sum $\sum_{\ell>0}\ell$ as $-\frac{1}{12}$ via "zeta regularization". We recall that the ζ-function $\zeta(s)$ is defined for $s \in \mathbb{C}$ with sufficiently large real part by the convergent series
$$\zeta(s) := \sum_{\ell>0} \frac{1}{\ell^s}.$$
It can be analytically continued to give the value $-\frac{1}{12}$ at $s = -1$. *Formally*, the value of $\zeta(s)$ at $s = -1$ is the divergent sum $\sum_{\ell>0}\ell$.

4. $S(N) = \bigotimes_{\ell>0} S_{q^\ell}(N_\ell) = \bigotimes_{\ell>0} S_{q^\ell}(TX \otimes \mathbb{C})$.

Putting all these terms together, we obtain the following formal expression for the equivariant index:
$$\mathrm{ind}_{S^1}(q, D(LX)) = \mathrm{ind}(D(X), q^{-n/24} \bigotimes_{\ell>0} S_{q^\ell}(TX \otimes \mathbb{C})). \tag{6.18}$$
It is useful to rewrite this expression in a different way. To explain why, let us look at the map
$$S\colon K(X) \to K(X)[[q]] \qquad \text{given by} \qquad V \mapsto \bigotimes_{\ell>0} S_{q^\ell}(V).$$
This map is *exponentional* in the obvious sense; topologists would call it an exponential characteristic class with values in the generalized cohomology theory $K(\ )[[q]]$, known as *Tate K-theory*. However, unlike the exponential characteristic classes we looked at (e.g., the

$\widehat{A}$-class, the Todd class or the L-class), this class is *not stable*, i.e., applied to a trivial bundle it does not give the unit $1 \in K(X)[[q]]$. For the trivial line bundle $\underline{\mathbb{C}}$ we obtain

$$S_{q^\ell}(\underline{\mathbb{C}}) = S^0\underline{\mathbb{C}} + S^1\underline{\mathbb{C}}q^\ell + S^2\underline{\mathbb{C}}q^{2\ell} + \cdots = 1 + q^\ell + q^{2\ell} + \cdots = \frac{1}{1-q^\ell} \in K(X)[[q]]$$

and hence

$$S(\underline{\mathbb{C}}) = \bigotimes_{\ell>0} S_{q^\ell}(\underline{\mathbb{C}}) = \prod_{\ell>0} \frac{1}{1-q^\ell}$$

A stable operation can be manufactured by replacing $S(V)$ for a vector bundle of dimension n by $S(V - \underline{\mathbb{C}}^n) := S(V)S(\underline{\mathbb{C}}^n)^{-1}$. Rewriting the right hand side of (6.18) in this way we obtain

$$\begin{aligned}
\operatorname{ind}_{S^1}(q, D(LX)) &= q^{-n/24}(\prod_{\ell>0} \frac{1}{(1-q^\ell)})^n \operatorname{ind}(D(X), \bigotimes_{\ell>0} S_{q^\ell}(TX \otimes \mathbb{C} - \underline{\mathbb{C}}^n)) \\
&= \frac{\operatorname{ind}(D(X), \bigotimes_{\ell>0} S_{q^\ell}(TX \otimes \mathbb{C} - \underline{\mathbb{C}}^n))}{\eta(q)^n}.
\end{aligned}$$

Here $\eta(q) := q^{1/24}\prod_{\ell>0}(1-q^\ell)$ is the *Dedekind η-function.*

Definition 6.19. For a closed spin n-manifold X the power series

$$\operatorname{Wit}(X) := \operatorname{ind}(D(X), \bigotimes_{\ell>0} S_{q^\ell}(TX \otimes \mathbb{C} - \underline{\mathbb{C}}^n)) \in \mathbb{Z}[[q]]$$

is the *Witten genus of X.*

We note that the expression $\bigotimes_{\ell>0} S_{q^\ell}(TX \otimes \mathbb{C} - \underline{\mathbb{C}}^n)$ is a power series

$$\bigotimes_{\ell>0} S_{q^\ell}(TX \otimes \mathbb{C} - \underline{\mathbb{C}}^n) = V_0 + V_1 q + V_2 q^2 + \ldots$$

in q whose coefficients V_i are complex vector bundles over X which are built from the complexified tangent bundle $TX \otimes \mathbb{C}$ and its symmetric powers. More accurately, the coefficients are *differences of vector bundles*, i.e., elements of $K(X)$. Explicitly,

$$\begin{aligned}
&\bigotimes_{\ell>0} S_{q^\ell}(TX \otimes \mathbb{C} - \underline{\mathbb{C}}^n) \\
&\qquad = \underline{\mathbb{C}} + (TX \otimes \mathbb{C} - \underline{\mathbb{C}}^n)q + \left(S^2 TX_{\mathbb{C}} - (n-1)TX_{\mathbb{C}} + \frac{n(n-3)}{2}\underline{\mathbb{C}}\right) q^2 + \ldots
\end{aligned}$$

Exercise 6.20. Prove the above statement.

The Witten genus is then the power series

$$\mathrm{Wit}(X) = \sum_k \mathrm{ind}(D(X, V_k))q^k$$

whose coefficient of q^k is the index of the Dirac operator on X twisted by V_k; alternatively, using the Index Theorem 3.2,

$$\mathrm{ind}(D(X, V_k)) = \langle \widehat{A}(TX)\,\mathrm{ch}(V_k), [X]\rangle.$$

In particular, the constant term of the Witten genus of X is $\widehat{A}(X)$, the $\widehat{A}$-genus of X, and the coefficient of q is $\langle \widehat{A}(TX)\,\mathrm{ch}(TX \otimes \mathbb{C} - \underline{\mathbb{C}}^n), [X]\rangle$.

This shows that the definition of the Witten genus does not necessarily require a spin structure on X. An orientation on X is sufficient to have a fundamental homology class $[X]$, and hence to be able to define the Witten genus as the power series

$$\mathrm{Wit}(X) := \sum_{k=0}^{\infty} \langle \widehat{A}(TX)\,\mathrm{ch}(V_k), [X]\rangle\; q^k.$$

However, if X is not spin, the coefficients of this power series in general will only be *rational numbers*, since $\widehat{A}(TX)\,\mathrm{ch}(V_k)$ is a rational cohomology class whose evaluation on the fundamental class in general only yields a rational number, not an integer; e.g., $\widehat{A}(\mathbb{CP}^2) = -1/8$. If X is spin, this number by the index theorem is equal to the index of the Dirac operator twisted by V_k, which then *forces* it to be an integer.

The Witten genus $\mathrm{Wit}(X) \in \mathbb{Z}[[q]]$ has a very interesting property, it is a *modular form of weight* $\frac{1}{2}\dim X$, provided that the tangent bundle of X restricted to the 4-skeleton is trivial. This property is very unexpected from the way it was defined above, but it was conjectured by Witten based on arguments based on the physics interpretation of the Witten genus as the "partition function of a 2-dimensional field theory", and proved by Zagier [Za].

Definition 6.21. Let $\mathfrak{h}$ be the upper half plane consisting of all points $\tau \in \mathbb{C}$ with positive imaginary part. A function $f\colon \mathfrak{h} \to \mathbb{C}$ is a *modular form of weight* n if the following conditions hold.

holomorphicity: f is holomorphic;

equivariance:

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^n f(\tau) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{6.22}$$

In particular $f(\tau + 1) = f(\tau)$ for all $\tau \in \mathfrak{h}$, and hence f factors in the form

$$\mathfrak{h} \twoheadrightarrow \mathfrak{h}/\mathbb{Z} \xrightarrow[\cong]{g} B^{\times} \xrightarrow{F} \mathbb{C}.$$

Here $B^\times$ is the open punctured 2-disk $\{q \in \mathbb{C} \mid 0 < |q| < 1\}$, and the biholomorphic map g sends $[\tau] \in \mathfrak{h}/\mathbb{Z}$ to $e^{2\pi i \tau} \in B^\times$ (biholomorphic means that g and its inverse are holomorphic).

holomorphicity at ∞: Via the biholomorphic map g, the "point at ∞" of $\mathfrak{h}$ corresponds to the origin of the punctured disk $B^\times$, and consequently, holomorphicity at ∞ for f amounts to the requirement that the holomorphic function $F\colon B^\times \to \mathbb{C}$ on the punctured disk $B^\times$ extends to a holomorphic function on the disk. More explicitly, in terms of the Laurent expansion

$$F(q) = \sum_{k \in \mathbb{Z}} a_k q^k, \tag{6.23}$$

of F, this is the requirement $a_k = 0$ for $k < 0$.

The expansion (6.23) is called the *q-expansion of f*.

Remark 6.24. The above is the classical definition of modular forms. It is ok for calculations, but it does a great job to hide any conceptual context for this definition! (Where does $\mathfrak{h}$ come from, why do we consider that specific action of $SL_2(\mathbb{Z})$ on $\mathfrak{h}$, and maybe most mystifying, where does the funny factor $(c\tau + d)^n$ in the equivariance requirement come from?). Here is an attempt to give a little bit of conceptual backdrop. A point $\tau \in \mathbb{C}$ determines a lattice in $\mathbb{C}$ given by $\mathbb{Z}\tau + \mathbb{Z}1 \subset \mathbb{C}$. Modding out by the lattice we obtain a torus $T_\tau := \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}1$. This is a complex 1-manifold via the complex structure on T_τ induced from $\mathbb{C}$. In fact, *every* compact complex 1-manifold with Euler characteristic 0 is biholomophically equivalent to T_τ for some $\tau \in \mathfrak{h}$. In other words, the map

$$\mathfrak{h} \longrightarrow \mathcal{M} := \left\{ \begin{matrix} \text{compact complex 1-manifolds} \\ \text{of Euler characteristic 0} \end{matrix} \right\} / \text{biholomorphic equivalence}$$

is surjective. It is not injective; rather for $\tau, \tau' \in \mathfrak{h}$ the corresponding tori T_τ, $T_{\tau'}$ are biholomorphically equivalent if and only if there is some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $\tau' = \frac{a\tau + b}{c\tau + d}$. In other words, the induced map

$$SL_2(\mathbb{Z}) \backslash \mathfrak{h} \longrightarrow \mathcal{M}$$

is a bijection, and so the quotient $SL_2(\mathbb{Z})\backslash\mathfrak{h}$ is the *moduli space of compact complex* 1*-manifolds with Euler characteristic* 0.

What about the funny factor $(c\tau + d)^n$? Every compact complex 1-manifold Σ has a canonical complex line $\mathrm{Det}(\Sigma)$ associated to it, defined by

$$\mathrm{Det}(\Sigma) := \mathrm{Det}(\Omega^1_{\mathrm{hol}}(\Sigma)^*),$$

where $\Omega^1_{\mathrm{hol}}(\Sigma)$ is the space of holomorphic 1-forms on Σ, $\Omega^1_{\mathrm{hol}}(\Sigma)^*$ is its dual space, and $\mathrm{Det}(\Omega^1_{\mathrm{hol}}(\Sigma)^*)$ is as in section 6.1.3 the top dimensional exterior power of the finite dimensional complex vector space $\Omega^1_{\mathrm{hol}}(\Sigma)^*$. If Σ is a torus, then $\Omega^1_{\mathrm{hol}}(\Sigma)^*$ is 1-dimensional, and so

$$\mathrm{Det}(\Sigma) = \Omega^1_{\mathrm{hol}}(\Sigma)^* \qquad \text{if } \Sigma \text{ has Euler characteristic } 0$$

The determinant lines $\mathrm{Det}(T_\tau)$ for $\tau \in \mathfrak{h}$ fit together to form a line bundle $\mathrm{Det} \to \mathfrak{h}$ whose fiber over $\tau \in \mathfrak{h}$ is $\mathrm{Det}(T_\tau)$. In fact, this is an $SL_2(\mathbb{Z})$-equivariant line bundle (a matrix $A \in SL_2(\mathbb{Z})$ determines functorially a biholomorphic equivalence $T_\tau \cong T_{A\tau}$ and hence an isomorphism $\mathrm{Det}(T_\tau) \cong \mathrm{Det}(T_{\tau'})$. From a conceptual point of view, a modular form of weight n is a section of $\mathrm{Det}^{\otimes n} \to \mathfrak{h}$ which is holomorphic, $SL_2(\mathbb{Z})$-equivariant and holomorphic at ∞.

How does that relate to the definition of a modular form as a *function* on $\mathfrak{h}$? The line bundle $\mathrm{Det} \to \mathfrak{h}$ has a nowhere vanishing section s, given at $\tau \in \mathfrak{h}$ by evaluation on the holomorphic 1-form on $T_\tau = \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}1$ whose pullback to $\mathbb{C}$ is the holomorphic 1-form $dz \in \Omega^1_{\mathrm{hol}}(\mathbb{C})$. Using this section s, functions on $\mathfrak{h}$ can be interpreted as sections of $\mathrm{Det}^{\otimes n}$ (by multiplying with s). Since s is *not an equivariant section* $SL_n(\mathbb{Z})$-equivariant sections of $\mathrm{Det}^{\otimes n}$ do not correspond to equivariant functions on $\mathfrak{h}$, but rather to functions with the funny transformation property (6.22).

7 Solutions to some exercises

Proof of Lemma 1.32. Let X be a riemannian manifold of dimension $n = 2\ell$. The homomorphism $\tau\colon \Omega^*(X) \to \Omega^*(X)$ is defined by

$$\tau\alpha := i^{k(k-1)+\ell} \star \alpha \qquad \text{for } \alpha \in \Omega^k(X).$$

To prove that τ is an involution, we calculate:

$$\begin{aligned}
\tau^2(\alpha) =&\tau\left(i^{k(k-1)+\ell} \star \alpha\right)\\
=&i^{(n-k)(n-k-1)+\ell} i^{k(k-1)+\ell} \star^2 \alpha\\
=&i^{(n-k)(n-k-1)+k(k-1)+2\ell}(-1)^{k(n-k)}\alpha\\
=&i^{(n-k)(n-k-1)+k(k-1)+n+2k(n-k)}\alpha
\end{aligned}$$

So it suffice to calculate the exponent modulo 4:

$$\begin{aligned}
&(n-k)(n-k-1)+k(k-1)+n+2k(n-k)\\
=&(n-k)^2-(n-k)+k^2-k+n+2kn-2k\\
=&n^2-2nk+k^2-n+k+k^2-k+n+2kn-2k^2\\
=&n^2
\end{aligned}$$

This is congruent 0 modulo 4 since n is even.

To show that τ anti-commutes with $D = d + d^*$, we note that for n even the formulas for $d^*\alpha$ and $\star^2\alpha$ simplify to

$$d^*\alpha = -\star d \star \qquad \star^2\alpha = (-1)^k\alpha \qquad \text{for } \alpha \in \Omega^k(X).$$

Then

$$\begin{aligned}
\tau(d+d^*)\alpha =& \tau d\alpha - \tau \star d \star \alpha \\
=& i^{(k+1)k+\ell} \star d\alpha - i^{(k-1)(k-2)+\ell} \star \star d \star \alpha \\
=& i^{k^2+k+\ell} \star d\alpha - i^{k^2-3k+2+\ell}(-1)^{n-k+1} d \star \alpha \\
=& i^{k^2+k+\ell} \star d\alpha - i^{k^2+k+\ell} i^2 (-1)^{n-k+1} d \star \alpha \\
=& i^{k^2+k+\ell} \left(\star d\alpha - (-1)^k d \star \alpha\right)
\end{aligned}$$

$$\begin{aligned}
(d+d^*)\tau\alpha =& (d - \star d \star)(i^{k(k-1)+\ell} \star \alpha \\
=& i^{k^2-k+\ell} \left(d \star \alpha - (-1)^k \star d\alpha\right) \\
=& i^{k^2+k+\ell}(-1)^k \left(d \star \alpha - (-1)^k \star d\alpha\right) \\
=& i^{k^2+k+\ell} \left((-1)^k d \star \alpha - \star d\alpha\right)
\end{aligned}$$

This proves that τ and $d + d^*$ anti-commute. $\square$

To describe the map $i_!$ in the middle row, we note that for any compact subset $K \subset V$, the inclusion $i \colon V \hookrightarrow \mathbb{R}^{n+k}$ gives a map of pairs $i \colon (V, V \subset K) \to (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \subset K)$. The induced map in cohomology

$$i^* \colon H^*(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus K) \longrightarrow H^*(V, V \setminus K)$$

is an isomorphism by excision. Then $i_!$ is the composition

$$H_c^*(V) = \varinjlim_{K \subset V \text{ compact}} H^*(V, V \setminus K) \xrightarrow{(i^*)^{-1}} \varinjlim_{K \subset V \text{ compact}} H^*(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} \setminus K)$$

$$\xrightarrow{\qquad\qquad\qquad} \varinjlim_{L \subset \mathbb{R}^{n+k} \text{ compact}} H^*(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} \setminus L) = H_c^*(\mathbb{R}^{n+k})$$

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